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The Product of Finite Numbers of Automorphic Composition Operators on Hardy Space $\mathbb{H}^2$

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Abstract

Throughout this paper we study the properties of the composition operator $C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}}$ induced by the composition of finite numbers of special automorphisms of U,

$$\alpha_{p_i}(z) = \frac{p_i - z}{1 - \overline{p_i} z}$$

such that $p_i \in U$, $i = 1, 2, \dots, n$, and discuss the relation between the product of finite numbers of automorphic composition operators on Hardy space $\mathbb{H}^2$ and some classes of operators.

Keywords: Composition operator, Normal composition operator, Compact operator

1. Introduction

Let U denote the unit ball in the complex plane, the Hardy space $\mathbb{H}^2$ is the collection of functions $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$, which holomorphic on U such that $\sum_{n=0}^{\infty} |\hat{f}(n)|^2 < \infty$ with $\hat{f}(n)$ denoting the n-th Taylor coefficient of f, and the norm of f is defined by:

$$\|f\|^2 = \sum_{n=0}^{\infty} |\hat{f}(n)|^2 .$$

The particular importance of $\mathbb{H}^2$ is due to the fact that it is a Hilbert space. Let φ be a holomorphic self-map of U, the composition operator C_φ induced by φ is defined on $\mathbb{H}^2$ by the equation $C_\varphi f = f \circ \varphi$, for every $f \in \mathbb{H}^2$, (see [9]).

A conformal automorphism of U is a univalent holomorphic mapping of U onto itself. Each such map is a linear fractional, and can be represented as product w.α_p, where:

$$\alpha_p(z) = \frac{p-z}{1-\overline{p}z}, \quad (z \in U)$$

for some $p \in U$ and $w \in \partial U$, (see [7]).

The map α_p is called special automorphism of U interchanges the point p and the origin and it is self-inverse map.

Let φ holomorphic self-map of U, φ is called an inner function if $|\varphi(z)| = 1$ almost everywhere on ∂U.

Clearly every conformal automorphism of U is an inner function.

It is well known that these are all linear fractional transformations, and they come in three flavors (see, e.g., [9, Chapter 0]):

- **Elliptic**: If it has one interior fixed point in U and one outside $\overline{U}$. These automorphisms having derivative < 1 at the interior fixed points.
- **Hyperbolic**: If it has two distinct fixed points on ∂U. These automorphisms having derivative < 1 at the boundary fixed points.
- **Parabolic**: If it has one fixed point of multiplicity 2 on ∂U. These automorphisms having derivative = 1 at a boundary fixed point.

Recall that for $\alpha \in U$, the reproducing kernel:

$$K_\alpha(z) = \frac{1}{1-\overline{\alpha}z} = \sum_{n=0}^{\infty} \overline{\alpha}^n z^n .$$

Although, there is no good description of the adjoint that works for all composition operators on all $\mathbb{H}^2$ functions, Shapiro [9, p.38] was computed its action on an important special family $\{K_\alpha\}_{\alpha \in U}$, which its span forms dense subset in $\mathbb{H}^2$.

Lemma 0:

Let φ be a holomorphic self-map of U, then for all $\alpha \in U$

$$C_\varphi^* K_\alpha = K_{\varphi(\alpha)}.$$

The study of composition operators on Hardy space $\mathbb{H}^2$ provides a rich arena in which to explore the connection between operator theory and classical function theory. In this chapter, we discuss the composition of finite numbers of special automorphisms of U. Throughout this study, we look at the properties of the composition operator $C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}}$ induced by the composition of finite numbers of special automorphisms of U,

$$\alpha_{p_i}(z) = \frac{p_i - z}{1 - \overline{p_i} z}.$$

such that $p_i \in U$, i = 1, 2, …, n and n is a fixed positive integer number and discuss how the change of $p_1, p_2, \ldots, p_n$ affects on the properties of the operator $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$.

2. The Relation Between the Product of Finite Numbers of Automorphic Composition Operators on Hardy Space $\mathbb{H}^2$ and Some Classes of Operators.

In this section, we will try to study the product of finite numbers of automorphic composition operators. We start this section by the following main theorem.

Theorem 1:

The composition of finite numbers of special automorphisms of U is a conformal automorphism of U.

Proof:

We will prove by induction that for all $n \in Z^+$

$$\alpha_{p_1} o\, \alpha_{p_2} o \ldots o\, \alpha_{p_n}(z) = w_m\ \alpha_{h_m}(z) \qquad \ldots\ldots\ldots\ldots(1)$$

where $p_i \in U$, $h_m \in U$ and $w_m \in \partial U$, for all i = 1, 2, …, n.

Base case: When n = 1, the left hand side of (1.2) is $\alpha_{p_1}(z)$ and the right hand side is $w_1 \alpha_{h_1}(z)$, where $w_1 = 1$ and $p_1 = h_1$, so both sides are equal and (1.2) is true for n = 1.

Induction step: Let $k \in Z^+$ be given and suppose (1.2) is true for n = k. That is $\alpha_{p_1} o\, \alpha_{p_2} o \ldots o\, \alpha_{p_k}(z) = w_k \alpha_{h_k}(z)$

where $w_k \in \partial U$ and $h_k \in U$. Then,

$$\alpha_{p_1} o\, \alpha_{p_2} o \ldots o\, \alpha_{p_{k+1}}(z) = \alpha_{p_1} o\, \alpha_{p_2} o \ldots o\, \alpha_{p_k}(\alpha_{p_{k+1}}(z))$$

$$= w_k \alpha_{h_k}(\alpha_{p_{k+1}}(z))$$

$$= w_k \left[\frac{h_k - \alpha_{p_{k+1}}(z)}{1 - \overline{h}_k \alpha_{p_{k+1}}(z)} \right]$$

$$= w_k\left[\frac{h_k - \left(\dfrac{p_{k+1}-z}{1-\overline{p}_{k+1}z}\right)}{1-\overline{h}_k\left(\dfrac{p_{k+1}-z}{1-\overline{p}_{k+1}z}\right)}\right]$$

$$= w_k\left[\frac{h_k(1-\overline{p}_{k+1}z)-(p_{k+1}-z)}{(1-\overline{p}_{k+1}z)-\overline{h}_k(p_{k+1}-z)}\right]$$

$$= w_k\left[\frac{(h_k-p_{k+1})-(\overline{p}_{k+1}h_k-1)z}{(1-\overline{h}_k p_{k+1})-(\overline{p}_{k+1}-\overline{h}_k)z}\right]$$

$$= w_k\left(\frac{\overline{p}_{k+1}h_k-1}{1-\overline{h}_k p_{k+1}}\right)\left[\frac{\left(\dfrac{h_k-p_{k+1}}{\overline{p}_{k+1}h_k-1}\right)-z}{1-\left(\dfrac{\overline{p}_{k+1}-\overline{h}_k}{1-\overline{h}_k p_{k+1}}\right)z}\right]$$

$$= w_{k+1}\left(\frac{h_{k+1}-z}{1-\overline{h_{k+1}}z}\right)$$

$$= w_{k+1}\,\alpha_{h_{k+1}}(z).$$

where $w_{k+1} = w_k\left(\dfrac{\overline{p}_{k+1}h_k-1}{1-\overline{h}_k p_{k+1}}\right)$ and $h_{k+1} = \dfrac{h_k-p_{k+1}}{\overline{p}_{k+1}h_k-1}$.

Now, it is enough to prove that $w_{k+1} \in \partial U$

$$|w_{k+1}|^2 = \left|w_k\left(\frac{\overline{p}_{k+1}h_k-1}{1-\overline{h}_k p_{k+1}}\right)\right|^2$$

$$= |w_k|^2\left|\frac{\overline{p}_{k+1}h_k-1}{\overline{1-h_k\overline{p}_{k+1}}}\right|^2$$

$$= 1.$$

Moreover, we must show that $h_{k+1} \in U$. Note that, since $h_k \in U$ and $p_{k+1} \in U$, then $(1-|h_k|^2)(1-|p_{k+1}|^2) > 0$. Hence:

$1-|p_{k+1}|^2-|h_k|^2+|h_k|^2|p_{k+1}|^2 > 0$. Therefore,

$|p_{k+1}|^2+|h_k|^2 < 1+|h_k|^2|p_{k+1}|^2$. This inequality implies that,

$$|h_{k+1}|^2 = \left|\frac{h_k - p_{k+1}}{\overline{p_{k+1}}h_k - 1}\right|^2 = \frac{(h_k - p_{k+1})(\overline{h_k} - \overline{p_{k+1}})}{(\overline{p_{k+1}}h_k - 1)(p_{k+1}\overline{h_k} - 1)} < 1.$$

Thus, $h_{k+1} \in U$.

Thus, (1) holds for n=k+1 and the proof of the induction step is complete.

Conclusion: By the principle of induction, the equation (1.2) is true for all $n \in Z^+$. That is $\alpha_{p_1} o\, \alpha_{p_2}\, o \ldots o\, \alpha_{p_n}(z) = w_m\ \alpha_{h_m}(z)$

where $$h_m = \left(\frac{h_{m-1} - p_m}{\overline{p_m}h_{m-1} - 1}\right) \in U, \quad \text{for} \quad m \in Z^+,$$

and $$w_m = w_{m-1}\left(\frac{\overline{p_m}h_{m-1} - 1}{1 - p_m\overline{h_{m-1}}}\right) \in \partial U, \quad \text{for } m \in Z^+.$$

Thus the map $\alpha_{p_1} o\, \alpha_{p_2}\, o \ldots o\, \alpha_{p_n}$ is a conformal automorphism of U. ■

Note that:

$$C\alpha_{p_1} o\, \alpha_{p_2}\, o \ldots o\, \alpha_{p_n} = C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$$

$$= C_{w_m}\ \alpha_{h_m}.$$

We remark that ψ in this thesis stands for the function

$$\psi(z) = \alpha_{p_1} o\, \alpha_{p_2}\, o \ldots o\, \alpha_{p_n}(z).$$

unless otherwise stated.

We will study the normality of the composition operator induced by $\alpha_{p_1} o\, \alpha_{p_2}\, o \ldots o\, \alpha_{p_n}$. Recall that an operator T on a Hilbert space H is said to be normal if TT* = T*T, T is isometric operator if T*T = I and an operator T is called unitary if TT* = T*T = I, where I is the identity operator.

H. J. Schwartz, in his thesis [8] investigated the relationship between properties of the symbol φ and the normality of the operator C_φ.

Theorem 2:

Let φ be a holomorphic self-map of U. Then C_φ is normal if and only if $\varphi(z) = \lambda z$, for some $|\lambda| \le 1$.

Now, we are ready to characterize the normality of the product of finite numbers of automorphic composition operators on Hardy space $\mathbb{H}^2$.

Theorem 3:

The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is normal if and only if $p_1 = p_2 = \dots = p_n = 0$.

Proof:

Assume that $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ is normal. Thus by theorem (2.1) C_ψ is normal operator, such that

$$\psi(z) = \alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n} = w_m \ \alpha_{h_m} \ (z)$$

where:

$$h_m = \left(\frac{h_{m-1} - p_m}{\overline{p_m} h_{m-1} - 1} \right), \text{ for } \quad m \in Z^+ \qquad \dots\dots\dots\dots\dots\dots(2)$$

This it is clear by theorem (1) that

$$h_m = 0, \quad \text{for} \quad m \in Z^+ \qquad \dots\dots\dots\dots\dots\dots(3)$$

Thus by (2), we have:

$$h_{m-1} = p_m, \quad \text{for} \quad m \in Z^+ \qquad \dots\dots\dots\dots\dots\dots(4)$$

(where $h_0 = p_0 = 0$ and $w_0 = -1$).

Therefore, it is easily seen that by (4) and (3):

$$p_n = 0, \qquad \text{for all } n = 1, 2, \dots.$$

That is, $p_1 = p_2 = \dots = p_n = 0$.

The converse is clear by reverse the inclusion. ■

Theorem 4:

The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is isometric if and only if $p_1 = p_2 = \dots = p_n = 0$.

Proof:

Assume that $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is isometric, then:

$$(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}})^*(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}) = I$$

Hence,

$$C^*_{\alpha_{p_1}} C^*_{\alpha_{p_2}} \ldots C^*_{\alpha_{p_n}} C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}} K_0 = K_0$$

$$C^*_{\alpha_{p_1}} C^*_{\alpha_{p_2}} \ldots C^*_{\alpha_{p_n}} C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_2}} (C_{\alpha_{p_1}} K_0) = K_0$$

Since ($C_{\alpha_{p_1}} K_0 = K_0 o\, \alpha_{p_1}(z) = K_0$), then after n-times we get:

$$C^*_{\alpha_{p_1}} C^*_{\alpha_{p_2}} \ldots C^*_{\alpha_{p_n}} (K_0) = K_0 \qquad \text{.....................(5)}$$

But by lemma (0), we have:

$$C^*_{\alpha_{p_n}} K_0 = K\alpha_{p_n}(0)$$

Thus by (5), we have:

$$C^*_{\alpha_{p_1}} C^*_{\alpha_{p_2}} \ldots C^*_{\alpha_{p_{n-1}}} (K\alpha_{p_n}(0)) = K_0$$

Again, by lemma (0), we have:

$$C^*_{\alpha_{p_1}} C^*_{\alpha_{p_2}} \ldots C^*_{\alpha_{p_{n-2}}} K\alpha_{p_{n-1}}(\alpha_{p_n}(0)) = K_0$$

Therefore, after n-times , we have $K\,\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}(0) = K_0$

Thus, $K\, w_m \alpha_{h_m}(0) = K_0$. This implies that, $\alpha_{h_m}(0) = 0$. Hence, for $m \in Z^+$, $h_m = 0$.

Thus by the same argument of theorem (2.1), we have $p_1 = p_2 = \ldots = p_n = 0$.

Conversely, if $p_1 = p_2 = \ldots = p_n = 0$. Therefore:

$$C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}} = (-I)(-I)\ldots(-I)$$
$$= (-1)^n I$$

Thus $(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}})^* = (-1)^n I$.

Hence, it is clear that $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is isometric. ■

From the fact that every operator is unitary if and only if it is normal and isometric operator. One can get the straightforward consequence.

Corollary 5:

The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is unitary if and only if $p_1 = p_2 = \dots = p_n = 0$.

An operator T on a Hilbert space H is said to be subnormal if there exists a normal operator S on a Hilbert space K such that H is a closed invariant subspace f K under S and the restriction of S to H coincides with the operator T. Moreover, an operator T on a Hilbert space H is called hyponormal if and only if $TT^* \leq T^*T$. It is well known [4, p.108] that every subnormal operator is hyponormal, but the converse is not necessarily true. We will try to study the subnormality and hyponormality of the product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$.

Corollary 6:

The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is hyponormal if and only if it is normal.

Proof:

Assume that $C_\psi = C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}}$ is hyponormal operator. Therefore by [2, Theorem 2] we have $\psi(0) = w_m \alpha_{h_m}(0) = 0$. But $w_n \in \partial U$, then $\alpha_{h_m}(0) = 0$. This implies that $h_m = 0$. Thus one can show that $p_1 = p_2 = \dots = p_n = 0$. Therefore by theorem (2) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ is normal operator. We can get the converse inclusion by the fact that every normal operator is hyponormal. ■

We can get the next straightforward conclusion from corollary (6) and the fact that every normal and hyponormal operator is subnormal.

Corollary 7:

The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ on

Hardy space $\mathbb{H}^2$ is subnormal if and only if it is normal.

Now, recall that an operator T is called reductive if every invariant subspace for the operator T is reducing. Mahvidi [5] gave the characterization of the reductive composition operator.

Proposition 8:

Let φ be a holomorphic self-map of U. Then C_φ is reductive operator if and only if $\varphi(z) = \alpha z$, for some α, $|\alpha| \leq 1$.

Therefore by proposition (8), one can get the following result:

Corollary 9:

The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is reductive if and only if $p_1 = p_2 = \ldots = p_n = 0$.

Recall that an operator T on a Hilbert space H is said to be compact if it maps every bounded set into a relatively compact one (whose closure in H is compact set). We start this section by the following result, which is proved by Shapiro [9].

Theorem 10:

Let φ be a linear-fractional self-map of U. Then C_φ is not compact if φ maps a point of the unit circle ∂U to a point of ∂U.

Then we can get the following consequences:

Corollary 11:

Let $p_i \in U$, i = 1, 2, …, n; then The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is not compact.

Proof:

Since $\psi = \alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}$ is an automorphism of U that maps ∂U into itself, then by theorem (10),

$C_\psi = C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}}$ is not compact. ∎

Let B(H) be an algebra of all bounded linear operators on H and K(H) the ideal of all compact operators in B(H). An operator T is essentially normal if $T^*T - TT^* = 0$ in Calkin algebra B(H)/K(H), or equivalently, if $T^*T = TT^*$ is compact. It is clear that every compact and normal operator is essentially normal.

Recall that [4] the spectrum of an operator T, denoted by $\sigma(T)$, is defined as follows $\sigma(T) = \{\lambda \in C : T - \lambda I \text{ is not invertible}\}$, and the spectral radius of T is $r(T) = \sup \{|\lambda| : \lambda \in \sigma(T)\}$. Recall that an operator T is called normaloid if $\|T\| = r(T)$. Nina Zorbosca [11, p.289] discussed the essential normality induced by conformal automorphism of $\mathbb{U}$.

Theorem 11:

If φ is a conformal automorphism of $\mathbb{U}$, then C_φ is essentially normal if and only if $\varphi(0) = 0$.

Therefore, by theorem (11), one get the direct consequence:

Corollary 12:

Let $p_i \in \mathbb{U}$, $i = 1, 2, \ldots, n$, then the product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is essentially normal if and only if $p_1 = p_2 = \ldots = p_n = 0$.

Theorem 13:

The following statements are equivalent:

(1) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is normal operator on $\mathbb{H}^2$.

(2) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is isometric operator on $\mathbb{H}^2$.

(3) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is unitary operator on $\mathbb{H}^2$.

(4) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is hyponormal operator on $\mathbb{H}^2$.

(5) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is subnormal operator on $\mathbb{H}^2$.

(6) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is reductive operator on $\mathbb{H}^2$.

(7) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is essentially normal operator on $\mathbb{H}^2$.

The norms induced by inner functions were computed by Nordgren [6].

Theorem 14:

If φ is an inner holomorphic self-map of $\mathbb{U}$, then:

$$\| C_\varphi \| = \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

Now, we give the following result:

Corollary 15:

(1) $\|C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}}\| = 1$ if and only if $p_1 = p_2 = \ldots = p_n = 0$.

(2) $\|C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}}\| > 1$ if and only if $p_j \neq 0$, for some $j = 1, 2, \ldots, n$.

Proof:

(1) By theorem (1.4), we have:

$$\|C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}}\|^2 = \| C_\psi \|^2$$

$$= \frac{1+|\psi(0)|}{1-|\psi(0)|}$$

$$= \frac{1+|h_n|}{1-|h_n|}$$

Thus it is clear that $\|C_\psi\| = 1$ if and only if $\psi(0) = 0$ if and only if $h_m = 0$ if and only if $p_1 = p_2 = \ldots = p_n = 0$.

(2) The proof is straightforward by negation (1). ■

C. C. Cowen [1] gave an easy estimate for the spectral radius of a composition operator on Hardy space $\mathbb{H}^2$.

Theorem 16:

Let φ be a holomorphic self-map of U and suppose φ has the Denjoy-Wolff point c. Then $r(C_\varphi) = 1$, when $|c| < 1$ and $r(C_\varphi) = |\varphi'(c)|^{-1/2}$ when $|c| = 1$.

Corollary 17:

Let $p_i \in U$, i = 1, 2, … n, then:

(1) If $\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}$ is either elliptic or parabolic, then $r(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}) = 1$.

(2) If $\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}$ is hyperbolic, then $r(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}) > 1$.

Proof:

(1) If $\psi = \alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}$ is elliptic, then ψ has a fixed point $c \in U$, such that $|\psi'(c)| < 1$. Thus by theorem (16) $r(C_\psi) = 1$.

Now, if ψ is parabolic then ψ has only one fixed point $c \in \partial U$, such that $|\psi'(c)| = 1$. Hence by theorem (16) $r(C_\psi) = |\psi'(c)|^{-1/2} = 1$, as desired.

(2) If ψ is hyperbolic, then ψ has a fixed point $c \in \partial U$, such that $|\psi'(c)| < 1$. Therefore, by theorem (16), $r(C_\psi) = |\psi'(c)|^{-1/2} > 1$. ■

In what follows we aim to give the characterization of normaloidity of the product of finite numbers of automorphic composition operators on Hardy space $\mathbb{H}^2$.

Corollary 18:

If $\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}$ is either elliptic or parabolic conformal automorphism of U. Then $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is normaloid composition operator if and only if $p_1 = p_2 = \ldots = p_n = 0$.

Proof:

Assume that $C_\psi = C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}}$ is normaloid operator, then $\| C_\psi \| = r(C_\psi)$. Therefore by corollary (17)(1); $\| C_\psi \| = r(C_\psi) = 1$. Hence by corollary (15)(1) $p_1 = p_2 = \ldots = p_n = 0$.

The converse holds directly by the fact that theorem (2) and the fact that every normal operator is normaloid. ■

Corollary 19:

Let $\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}$ is hyperbolic conformal automorphism of U. If $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is normaloid operator on $\mathbb{H}^2$, then $p_j \neq 0$, for some j = 1, 2, …, n.

Proof:

Assume that $C_\psi = C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}}$ is normaloid operator on $\mathbb{H}^2$, then $\| C_\psi \| = r(C_\psi)$. Since $\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ is hyperbolic, then by corollary (17)(2), $\|C_\psi\| = r(C_\varphi) > 1$. Hence by corollary (15)(2), $p_j \neq 0$ for some j = 1, 2, …, n. ■

The converse of corollary (19) is not necessary true in general. See the following example.

Example 20:

Let $p_i \in U$, i = 1, 2, …, n; such that $\alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}$ is hyperbolic with boundary Denjoy-Wolff point $c \in R^+$. If $p_i \in R^+$, for all i = 1, 2, …, n such that $p_j \neq 0$, for some j = 1, 2, …, n, then $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \dots C_{\alpha_{p_1}}$ is not normaloid.

Proof:

Note that, $\psi(z) = \alpha_{p_1} \circ \alpha_{p_2} \circ \dots \circ \alpha_{p_n}(z)$

$$= w_m \, \alpha_{h_m}(z)$$

$$= w_m \left(\frac{h_m - z}{1 - \overline{h_m} z} \right).$$

Therefore, by theorem (16) $r(C_\psi) = |\psi'(c)|^{-1/2}$ such that $c \in R^+$ is a boundary Denjoy-Walff point of ψ (i.e., $|c| = c = 1$). Therefore:

$$r(C_\psi) = |\psi'(1)|^{-1/2}$$

$$= \left| \frac{(h_m - 1) + h_m (h_m - 1)}{(1 - h_m)^2} \right|^{-1/2}$$

$$= \left| \frac{(h_m{}^2 - 1)}{(1 - h_m)^2} \right|^{-1/2}$$

$$= \left| \frac{(1 - h_m)(1 + h_m)}{(1 - h_m)^2} \right|^{-1/2}$$

$$= \left| \frac{(1 - h_m)}{(1 + h_m)} \right|^{-1/2}$$

$$= \left|\frac{(1-|h_m|)}{(1+|h_m|)}\right|^{-1/2}$$

$$= \frac{1}{\|C_\psi\|}. \qquad ...(6)$$

Since $p_j \neq 0$, for some $j = 1, 2, \ldots, n$, then by theorem (15)(2), $\|C_\psi\| > 1$. Thus by equation (6) $r(C_\psi) \neq \|C_\psi\|$. This implies that $C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}}$ is not normaloid. ■

5. Conclusion

In this paper, we prove the following

Theorem 1: The composition of finite numbers of special automorphisms of $\mathbb{U}$ is a conformal automorphism of $\mathbb{U}$.

Theorem 2: Let φ be a holomorphic self-map of $\mathbb{U}$. Then C_φ is normal if and only if $\varphi(z) = \lambda z$, for some $|\lambda| \leq 1$.

Theorem 3: The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is normal if and only if $p_1 = p_2 = \ldots = p_n = 0$.

Theorem 4: The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is isometric if and only if $p_1 = p_2 = \ldots = p_n = 0$.

Corollary 5: The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is unitary if and only if $p_1 = p_2 = \ldots = p_n = 0$.

Corollary 6: The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is hyponormal if and only if it is normal.

Corollary 7: The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is subnormal if and only if it is normal.

Proposition 8: Let φ be a holomorphic self-map of $\mathbb{U}$. Then C_φ is reductive operator if and only if $\varphi(z) = \alpha z$, for some α, $|\alpha| \leq 1$.

Corollary 9: The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$

on Hardy space $\mathbb{H}^2$ is reductive if and only if $p_1 = p_2 = \ldots = p_n = 0$.

Theorem 10: Let φ be a linear-fractional self-map of U. Then C_φ is not compact if φ maps a point of the unit circle ∂U to a point of ∂U.

Corollary 11: Let $p_i \in U$, i = 1, 2, …, n; then The product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is not compact.

Theorem 11: If φ is a conformal automorphism of U, then C_φ is essentially normal if and only if φ(0) = 0.

Corollary 12: Let $p_i \in U$, i = 1, 2, …, n, then the product of finite numbers of automorphic composition operators $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ on Hardy space $\mathbb{H}^2$ is essentially normal if and only if $p_1 = p_2 = \ldots = p_n = 0$.

Theorem 13: The following statements are equivalent:

(8) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is normal operator on $\mathbb{H}^2$.

(9) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is isometric operator on $\mathbb{H}^2$.

(10) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is unitary operator on $\mathbb{H}^2$.

(11) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is hyponormal operator on $\mathbb{H}^2$.

(12) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is subnormal operator on $\mathbb{H}^2$.

(13) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is reductive operator on $\mathbb{H}^2$.

(14) $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is essentially normal operator on $\mathbb{H}^2$.

Theorem 14: If φ is an inner holomorphic self-map of U, then:

$$\| C_\varphi \| = \sqrt{\frac{1+|\varphi(0)|}{1-|\varphi(0)|}}.$$

Corollary 15:

(3) $\|C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}}\| = 1$ if and only if $p_1 = p_2 = \ldots = p_n = 0$.

(4) $\|C_{\alpha_{p_1} \circ \alpha_{p_2} \circ \ldots \circ \alpha_{p_n}}\| > 1$ if and only if $p_j \neq 0$, for some j = 1, 2, …, n.

Theorem 16: Let φ be a holomorphic self-map of U and suppose φ has the Denjoy-Wolff point c. Then $r(C_\varphi) = 1$, when $|c| < 1$ and $r(C_\varphi) = |\varphi'(c)|^{-1/2}$ when $|c| = 1$.

Corollary 17: Let $p_i \in \mathbb{U}$, $i = 1, 2, \ldots n$, then:

(3) If $\alpha_{p_1} \circ \alpha_{p_2} \circ\ldots\circ \alpha_{p_n}$ is either elliptic or parabolic, then $r(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}) = 1$.

(4) If $\alpha_{p_1} \circ \alpha_{p_2} \circ\ldots\circ \alpha_{p_n}$ is hyperbolic, then $r(C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}) > 1$.

Corollary 18: If $\alpha_{p_1} \circ \alpha_{p_2} \circ\ldots\circ \alpha_{p_n}$ is either elliptic or parabolic conformal automorphism of $\mathbb{U}$. Then $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is normaloid composition operator if and only if $p_1 = p_2 = \ldots = p_n = 0$.

Corollary 19: Let $\alpha_{p_1} \circ \alpha_{p_2} \circ\ldots\circ \alpha_{p_n}$ is hyperbolic conformal automorphism of $\mathbb{U}$. If $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is normaloid operator on $\mathbb{H}^2$, then $p_j \neq 0$, for some $j = 1, 2, \ldots, n$.

Example 20: Let $p_i \in \mathbb{U}$, $i = 1, 2, \ldots, n$; such that $\alpha_{p_1} \circ \alpha_{p_2} \circ\ldots\circ \alpha_{p_n}$ is hyperbolic with boundary Denjoy-Wolff point $c \in R^+$. If $p_i \in R^+$, for all $i = 1, 2, \ldots, n$ such that $p_j \neq 0$, for some $j = 1, 2, \ldots, n$, then $C_{\alpha_{p_n}} C_{\alpha_{p_{n-1}}} \ldots C_{\alpha_{p_1}}$ is not normaloid.

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