

A Unique Common Fixed Point Theorems in generalized D*-Metric Spaces

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Abstract
In this paper we establish some common fixed point theorem for contractive type mapping in cone metric spaces and prove some generalized complete *D**- metric spaces.
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1 Introduction and Preliminaries

Fixed point theorems play a major role in mathematics such as optimization, mathematical models, economy, military and medicine. So, the metric fixed point theory has been investigated extensively in the past two decades by numerous mathematicians. Some generalizations of a metric space concept have been studied by several authors. These different generalizations have been improved by Gahler [7, 10], by introducing 2-metric spaces, and Dhage [1] by studying the theory of *D*- metric spaces.

In 2005, Mustafa and Sims [11] introduced a new structure of generalized metric spaces which are called *G*-metric spaces as a generalization of metric spaces. Later, Mustafa et al. [12–14] obtained several fixed point theorems for mappings satisfying different contractive conditions in *G*-metric spaces. Later in 2007 Shaban Sedghi et.al [8] modified the *D*-metric space and defined *D**-metric spaces and then C.T.Aage and J.N.Salunke [3] generalized the *D**-metric spaces by replacing the real numbers by an ordered Banach space and defined *D**-cone metric spaces and prove the topological properties.

Further, Huang and Zhang [6] generalized the notion of metric spaces by replacing the real numbers by ordered Banach space and defined the cone metric spaces. They have investigated the convergence in cone metric spaces, introduced the completeness of cone metric spaces and have proved Banach contraction mapping theorem, some other fixed point theorems of contractive type mappings in cone metric spaces using the normality condition. Afterwards, Rezapour and Hamlbarani [9], Ilic and Rakocevic [5], contributed some fixed point theorems for contractive type mappings in cone metric spaces.

In this paper, we obtain a unique common fixed point theorem for generalized *D**-metric spaces.

First, we present some known definitions and propositions in *D**-metric spaces.

Let E be a real Banach space and P a subset of E. P is called a cone if and only if:

(i) P is closed, non-empty and $P \neq \{0\}$,

(ii) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,

(iii) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ if $x \leq y$ and $x \neq y$; we shall write $x \ll y$ if $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|. \quad (1)$$

The least positive number K satisfying the above is called the normal constant of P [12]. The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}_{n\geq 1}$ is a sequence such that $x_1 \leq x_2 \leq \cdots \leq y$ for some $y \in E$, then there is $x \in E$ such that $\lim_{n\to\infty}\|x_n - x\| = 0$.

The cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. Rezapour and Hamlbarani [9] proved every regular cone is normal and there are normal cone with normal constant $M \geq 1$.

Definition (1.1) [4]. Let X be a non empty set. A generalized D^*-metric on X is a function, $D^*: X^3 \to E$ that satisfies the following conditions for all $x, y, z, a \in X$:

(1) $D^*(x,y,z) \geq 0$,

(2) $D^*(x,y,z) = 0$ if and only if $x = y = z$,

(3) $D^*(x,y,z) = D^*(p\{x,y,z\})$, (Symmetry) where p is a permutation function,

(4) $D^*(x,y,z) \leq D^*(x,y,a) + D^*(a,z,z)$,

Then the function D^* is called a generalized D^*-metric and the pair (X, D^*) is called a generalized D^*-metric space

Example (1.2) [4]. Let $E = R^2, P = \{(x,y) \in E : x,y \geq 0\}, X = R$ and $D^*: X \times X \times X \to E$ defined by

$D^*(x,y,z) = (|x-y| + |y-z| + |x-z|, \alpha(|x-y| + |y-z| + |x-z|))$, where $\alpha \geq 0$ is a constant. Then (X, D^*) is a generalized D^*- metric space.

Proposition (1.3) [4]. If (X, D^*) be generalized D^*- metric space, then for all $x, y, z \in X$, we have $D^*(x,x,y) = D^*(x,y,y)$.

Proof. Let $D^*(x,x,y) \leq D^*(x,x,x) + D^*(x,y,y) = D^*(x,y,y)$ and similarly $D^*(y,y,x) \leq D^*(y,y,y) + D^*(y,x,x) = D^*(y,x,x)$. Hence we have

$$D^*(x,x,y) = D^*(x,y,y).$$

Definition (1.4) [4]. Let (X, D^*) be a generalized D^*- metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in E$ with $0 \ll c$ there is N such that for all $m,n > N, D^*(x_m, x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x, and x is the limit of $\{x_n\}$. We denote this by $x_n \to x (n \to \infty)$.

Lemma (1.5) [4]. Let (X, D^*) be a generalized D^*-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $D^*(x_m, x_n, x)(m,n \to \infty)$.

Proof. Let $\{x_n\}$ be a sequence in generalized D^*-metric space X converge to $x \in X$ and $\varepsilon > 0$ be any number. Then for any $c \in E$, with $0 \ll c$ there is a positive integer N such that $m,n > N$ implies

$$D^*(x_m, x_n, x) \ll c \Rightarrow \|D^*(x_m, x_n, x)\| \leq K\|c\|,$$

since $0 \leq D^*(x_m, x_n, x) < c$ and K is normal constant. Choose c such that $K\|c\| < \varepsilon$. Then $\|D^*(x_m, x_n, x)\| < \varepsilon$ for all $m,n > N$ and hence $D^*(x_m, x_n, x) \to 0$ as $m,n \to \infty$.

Conversely let $D^*(x_m,x_n,x)\to 0$ as $m,n\to\infty$ (in $P\subset E$). So for any $c\in E$ with $0\ll c$ (i.e. $c\in \operatorname{int} P$), we have $\|c\|>0$, let $r=dist(c,\partial P)=\inf\{\|c-t\|:t\in\partial P\}$, where ∂P denotes the boundary of P, for this given $r, 0<r\le\|c\|$, there exist a positive integer N such that $m,n>N$ implies that $\|G(x_m,x,x)\|\frac{r}{2}<\|c\|$ and for any $t\in\partial P$,

$$\|(c-D^*(x_m,x_n,x))-t\|\ge\|c-t\|-\|D^*(x_m,x_n,x)\|>r-\frac{r}{2}=\frac{r}{2}$$

which proves that $c-D^*(x_m,x_n,x)\in \operatorname{int} P$ i.e. $G(x_m,x_n,x)\ll c.$

Remark. If $\{u_n\}$ is a sequence in $P\subset E$ and $u_n\to u$ in E the $u\in P$ as P is a closed subset of E. From this $u_n\ge 0\Rightarrow u\ge 0$. Thus if $u_n\le v_n$ in P then $\lim u_n\le\lim v_n$, provided limit exist.

Lemma (1.6) [4]. Let (X,D^*) be a generalized D^*-metric space then the following are equivalent.

(i) $\{x_n\}$ is D^**–convergent* to x.

(ii) $D^*(x_n,x_n,x)\to 0$ as $n\to\infty$

(iii) $D^*(x_n,x,x)\to 0$ as $n\to\infty$.

Proof. $(i)\Rightarrow(ii)$ by Lemma (1.5).

$(ii)\to(i)$. Assume (ii), i.e. $D^*(x_n,x,x)\to 0$ as $n\to\infty$ i.e. for every $c\in E$ with $0\ll c$ there is N such that for all $n>N, D^*(x_n,x_n,x)\ll c/2$,

$$\begin{aligned}D^*(x_m,x_n,x)&\le D^*(x_m,x,x)+D^*(x,x_n,x_n)\\&\le D^*(x,x_m,x_m)+D^*(x,x_n,x_n)\\&\ll c \text{ for all } m,n>N.\end{aligned}$$

Hence $\{x_n\}$ is D^*-convergent to x.

$(ii)\Leftrightarrow(iii)$. Assume (ii), i.e. $D^*(x_n,x_n,x)\to 0$ as $n\to\infty$ i.e. for every $c\in E$ with $0\ll c$ there is N such that for all $n>N$, $D^*(x_n,x_n,x)\ll c$,

$$D^*(x_n,x,x)=D^*(x,x_n,x_n)=D^*(x_n,x_n,x)\ll c.$$

Hence $D^*(x_n,x,x)\to 0$ as $n\to\infty$, since c is arbitrary.

$(iii)\Rightarrow(ii)$. Assume that (iii) i.e. $D^*(x_n,x,x)\to 0$ as $n\to\infty$. Then for any $c\in E$ with $0\ll c$, there is an N such that $n>N$ implies $D^*(x_n,x,x)\ll\frac{c}{2}$. Hence $m,n>N$ gives $D^*(x_m,x_n,x)\le D^*(x_m,x,x)+D^*(x,x_n,x_n)\ll c$. Thus $\{x_n\}$ is D^*- convergent to x.

Lemma (1.7) [4]. Let (X,D^*) be a generalized D^*- metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x=y$. That is the limit of $\{x_n\}$, if exists, is unique.

Proof. For any $c\in E$ with $0\ll c$, there is N such that for all $m,n>N$, $D^*(x_m,x_n,x)\ll c.$ We have

$$0 \le D^*(x,x,y) \le D^*(x,x,x_n) + D^*(x_n,y,y)$$
$$= D^*(x_n,x_n,x) + D^*(x_n,x_n,y)$$
$$\ll c. \text{ for all } n > N.$$

Hence $\|D^*(x,x,y)\| \le 2K\|c\|$. Since c is arbitrary, $D^*(x,x,y)=0$, therefore $x=y$.

Definition (1.8) [4]. Let (X,D^*) be a generalized D^*- metric space, $\{x_n\}$ be a sequence in X. If for any $c \in E$ with $0 \ll c$, there is N such that for all $m,n,l > N$, $D^*(x_m,x_n,x_l) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

Definition (1.9) [4]. Let (X,D^*) be a generalized D^*-metric space. If every Cauchy sequence in X is convergent in X, then X is called a complete generalized D^*-metric space.

Lemma (1.10) [4]. Let (X,D^*) be a generalized D^*-metric space, $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x, then $\{x_n\}$ is a Cauchy sequence.

Proof. For any $c \in E$ with $0 \ll c$, there is N such that for all $m,n,l > N$, $D^*(x_m,x_n,x) \ll c/2$ and $D^*(x_l,x_l,x) \ll c/2$. Hence

$$D^*(x_m,x_n,x_l) \le D^*(x_m,x_n,x) + D^*(x,x_l,x_l) \le c.$$

Therefore $\{x_n\}$ is a Cauchy sequence.

Lemma (1.11) [4]. Let (X,D^*) be a generalized D^*-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $D^*(x_m,x_n,x_l) \to 0 (m,n,l \to \infty)$.

Proof. Let $\{x_n\}$ be a Cauchy sequence in generalized D^*-metric space (X,D^*) and $\varepsilon > 0$ be any real number. Then for any $c \in E$ with $0 \ll c$, there exist a positive integer N such that $m,n,l > N$ implies $D^*(x_m,x_n,x_l) \ll c \Rightarrow 0 \le D^*(x_m,x_n,x_l) < c$ i.e. $\|D^*(x_m,x_n,x_l)\| \le K\|c\|$, where K is a normal constant of P in E. Choose c such that $K\|c\| < \varepsilon$. Then $\|D^*(x_m,x_n,x_l)\| < \varepsilon$ for all $m,n,l > N$, showing that $D^*(x_m,x_n,x_l) \to 0$ as $m,n,l \to \infty$.

Conversely let $D^*(x_m,x_n,x_l) \to 0$ as $m,n,l \to \infty$. For any $c \in E$ with $0 \ll c$ we have $K\|c\| > 0 (\|c\| > 0$, as $c = c - 0 \in IntP$ and $K \ge 1)$. For given $K\|c\|$ there is a positive integer N such that $m,n,l > N \Rightarrow \|D^*(x_m,x_n,x_l)\| < K\|c\|$. This proves that $D^*(x_m,x_n,x_l) \ll c$ for all $m,n,l > N$ and hence $\{x_n\}$ is a Cauchy sequence.

Definition (1.12) [4]. Let (X,D^*), (X',D'^*) be generalized D^*-metric spaces, then a function $f: X \to X'$ is said to be D^*-continuous at a point $x \in X$ if and only if it is D^*-sequentially continuous at x, that is, whenever $\{x_n\}$ is D^*-convergent to x we have $\{fx_n\}$ is D^*-convergent to fx.

Lemma (1.13) [4]. Let (X,D^*) be a generalized D^*-metric space, P be a normal cone with normal constant K. Let $\{x_n\}$, $\{y_n\}$, and $\{z_n\}$ be three sequences in X and $x_n \to x, y_n \to y$, $z_n \to z$ $(n \to \infty)$. Then $D^*(x_n,y_n,z_n) \to D^*(x,y,z) (n \to \infty)$.

Proof. For every $\varepsilon > 0$, choose $c \in E$ with $0 \ll c$ and $\|c\| < \frac{\varepsilon}{6K+3}$. From $x_n \to x, y_n \to y$ and $z_n \to z$, there is N such that for all $n > N, D^*(x_n, x_n, x) \ll c, D^*(y_n, y_n, y) \ll c$ and $D^*(z_n, z_n, z) \ll c.$ We have

$$\begin{aligned} D^*(x_n, y_n, z_n) &\le D^*(x_n, y_n, z) + D^*(z, z_n, z_n) = D^*(z, x_n, y_n) + D^*(z_n, z_n, z) \\ &\le D^*(z, x_n, y) + D^*(y, y_n, y_n) + D^*(z_n, z_n, z) \\ &= D^*(y, z, x_n) + D^*(y_n, y_n, y) + D^*(z_n, z_n, z) \\ &\le D^*(y, z, x) + D^*(x, x_n, x_n) + D^*(y_n, y_n, y) + D^*(z_n, z_n, z) \\ &\le 3c + D^*(x, y, z) \end{aligned}$$

Similarly, we infer $D^*(x, y, z) \le D^*(x_n, y_n, z_n) + 3c.$ Hence

$$0 \le D^*(x, y, z) + 3c - D^*(x_n, y_n, z_n) \le 6c$$

and

$$\begin{aligned} \|D^*(x_n, y_n, z_n) - D^*(x, y, z)\| &\le \|D^*(x, y, z) + 3c - D^*(x_n, y_n, z_n)\| + \|3c\| \\ &\le (6K+3)\|c\| < \varepsilon, \text{ for all } n > N. \end{aligned}$$

Therefore $D^*(x_n, y_n, z_n) \to D^*(x, y, z) (n \to \infty).$

Remark. If $x_n \to x$ in generalized D^*-metric space X, then every subsequence of $\{x_n\}$ converges to x in X. Let $\{x_{k_n}\}$ be any subsequence of $\{x_n\}$ and $x_n \to x$ in X then $D^*(x_m, x_n, x) \to 0$ as $m, n \to \infty$

and also $D^*(x_{k_m}, x_{k_n}, x) \to 0$ as $m, n \to \infty$ since $k_n \ge n$ for all n.

Definition (1.14) [4]. Let f and g be self maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

Proposition (1.15) [4]. Let f and g be weakly compatible self maps of a set X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

2. Main Results

The first main result is

Theorem (2.1). Let (X, D^*) be complete generalized D^*-metric spaces, P be a normal cone with normal constant K and let $T : X \to X$, be a mapping satisfies the following conditions

$$\begin{aligned} D^*(Tx, Ty, Tz) \le{}& aD^*(x, y, z) + bD^*(x, Tx, Tx) \\ &+ cD^*(y, Ty, Ty) + dD^*(z, Tz, Tz) \qquad (2) \end{aligned}$$

for all $x, y, z \in X$, where $a, b, c, d \ge 0, a + b + c + d < 1$. Then T have a unique fixed point in X .

Proof. Let $x_0 \in X$ be arbitrary, there exist $x_1 \in X$ such that $Tx_0 = x_1$, in this way we have a sequence $\{x_n\}$ with $Tx_{n-1} = x_n$. Then from the above inequality we have

$$D^*(x_n, x_{n+1}, x_{n+1}) = D^*(Tx_{n-1}, Tx_n, Tx_n)$$

$$\begin{aligned}
&\le aD*(x_{n-1},x_n,x_n)+bD*(x_{n-1},Tx_{n-1},Tx_{n-1})\\
&\quad +cD*(x_n,Tx_n,Tx_n)+dD*(x_n,Tx_n,Tx_n)\\
&= aD*(x_{n-1},x_n,x_n)+bD*(x_{n-1},x_n,x_n)\\
&\quad +cD*(x_n,x_{n+1},x_{n+1})+dD*(x_n,x_{n+1},x_{n+1})\\
&=(a+b)D*(x_{n-1},x_n,x_n)+(c+d)D*(x_n,x_{n+1},x_{n+1})
\end{aligned}$$

This implies

$$(1-(c+d))D*(x_n,x_{n+1},x_{n+1})\le(a+b)D*(x_{n-1},x_n,x_n)$$

$$D*(x_n,x_{n+1},x_{n+1})\le\frac{a+b}{1-(c+d)}D*(x_{n-1},x_n,x_n)$$

$$D*(x_n,x_{n+1},x_{n+1})\le qD*(x_{n-1},x_n,x_n)$$

where $q=\dfrac{a+b}{1-(c+d)}$, then $0\le q<1$. By repeated the application of the above inequality we have

$$D*(x_n,x_{n+1},x_{n+1})\le q^nD*(x_0,x_1,x_1) \tag{3}$$

Then for all $n,m\in\mathbb{N}, n<m$ we have by repeated use the triangle inequality and equality (3) that

$$\begin{aligned}
D*(x_n,x_m,x_m)&\le D*(x_n,x_n,x_{n+1})+D*(x_{n+1},x_{n+1},x_{n+2})\\
&\quad +D*(x_{n+2},x_{n+2},x_{n+3})+\cdots+D*(x_{m-1},x_{m-1},x_m)\\
&\le D*(x_n,x_{n+1},x_{n+1})+D*(x_{n+1},x_{n+2},x_{n+2})\\
&\quad +D*(x_{n+2},x_{n+3},x_{n+3})+\cdots+D*(x_{m-1},x_m,x_m)\\
&\le(q^n+q^{n+1}+\cdots+q^{m-1})D*(x_0,x_1,x_1)\\
&\le\frac{q^n}{1-q}D*(x_0,x_1,x_1).
\end{aligned}$$

From (1) we infer

$$\|D*(x_n,x_m,x_m)\|\le\frac{q^n}{1-q}K\|D*(x_0,x_1,x_1)\|$$

which implies that $D*(x_n,x_m,x_m)\to 0$ as $n,m\to\infty$, since $\frac{q^n}{1-q}K\|D*(x_0,x_1,x_1)\|\to\infty$ as $n,m\to\infty$.

For $n,m,l\in\mathbb{N}$, and

$$D*(Tx_n,x_m,x_l)\le D*(x_n,x_m,x_m)+D*(x_m,x_l,x_l),$$

from (1)

$$\|D*(Tx_n,x_m,x_l)\|\le K\left[\|D*(x_n,x_m,x_m)\|+\|D*(x_m,x_l,x_l)\|\right]$$

taking limit as $n,m,l\to\infty$, we get $D*(x_n,x_m,x_l)\to 0$. So $\{x_n\}$ is $D*$-Cauchy sequence, since X is $D*$-complete, there exists $u\in X$ such that $\{x_n\}\to u$ as $n\to\infty$, there exist $p\in X$ such that $p=u$. If $T(X)$ is complete, then there exist $u\in T(X)$ such that $x_n\to u$, as $T(X)\subset X$, we have $u\in X$. Then there exist $p\in X$ such that $p=u$. We claim that $Tp=u$,

$$\begin{aligned}D^*(Tp,u,u) &= D^*(Tp,Tp,u)\\ &\le D^*(Tp,Tp,Tx_n)+D^*(Tx_n,u,u)\\ &\le aD^*(p,p,x_n)+bD^*(p,Tp,Tp)+cD^*(p,Tp,Tp)\\ &\quad +dD^*(x_n,Tx_n,Tx_n)+D^*(x_{n+1},u,u)\\ &\le aD^*(u,u,x_n)+bD^*(u,Tp,Tp)+cD^*(u,Tp,Tp)\\ &\quad +dD^*(x_n,x_{n+1},x_{n+1})+D^*(x_{n+1},u,u)\end{aligned}$$

This implies that

$$D^*(Tp,Tp,u)\le\frac{1}{1-(b+c)}\{aD^*(u,u,x_n)+dD^*(x_n,x_{n+1},x_{n+1})+D^*(x_{n+1},u,u)\}$$

from (1)

$$\|D^*(Tp,Tp,u)\|\le K\frac{1}{1-(b+c)}\{a\|D^*(u,u,x_n)\|+d\|D^*(x_n,x_{n+1},x_{n+1})\|+\|D^*(x_{n+1},u,u)\|\}$$

as $n\to\infty$, right hand side approaches to zero. Hence $\|D^*(Tp,Tp,u)\|=0$ and $Tp=u$. i.e. $Tp=p$. Now we show T has a unique fixed point. For this, assume that there exists a point q in X such that $q=Tq$. Now

$$\begin{aligned}D^*(Tp,Tp,Tq) &\le aD^*(p,p,q)+bD^*(p,Tp,Tp)+cD^*(p,Tp,Tp)\\ &\quad +dD^*(q,Tq,Tq)\\ &= aD^*(Tp,Tp,Tq)+bD^*(Tp,Tp,Tp)+cD^*(Tp,Tp,Tp)\\ &\quad +dD^*(Tq,Tq,Tq)\\ &= aD^*(Tp,Tp,Tq)\end{aligned}$$

we have $D^*(Tp,Tp,Tq)\le aD^*(Tp,Tp,Tq)$, i.e. $(a-1)D^*(Tp,Tp,Tq)\in P$, but $(a-1)D^*(Tp,Tp,Tq)\in -P$, since $k-1<0$. Thus p is a unique common fixed point of T.

Corollary (2.2). Let (X,D^*) be a complete generalized D^*-metric space, P be a normal cone with normal constant K and let $T:X\to X$ be a mappings satisfy the condition

$$\begin{aligned}D^*(Tx,Ty,Tzq) &\le a[D^*(x,Ty,Ty)+D^*(y,Tx,Tx)]\\ &\quad +b[D^*(y,Tz,Tz)+D^*(z,Ty,Ty)]\\ &\quad +c[D^*(x,Tz,Tz)+D^*(z,Tx,Tx)]\end{aligned} \tag{4}$$

for all $x,y,z\in X$, where $a,b,c\ge 0, 2a+2b+2c<1$. Then T has a unique fixed point in X.

Proof. Let $x_0\in X$ be arbitrary, there exist $x_1\in X$ such that $Tx_0=x_1$, in this way we have sequence $\{Tx_n\}$ with $Tx_n=x_{n+1}$. Then from inequality (4), we have

$$\begin{aligned}D^*(x_n,x_{n+1},x_{n+1}) &= D^*(Tx_{n-1},Tx_n,Tx_n)\\ &\le a[D^*(x_{n-1},Tx_n,Tx_n)+D^*(x_n,Tx_{n-1},Tx_{n-1})]\\ &\quad +b[D^*(x_n,Tx_n,Tx_n)+D^*(x_n,Tx_n,Tx_n)]\\ &\quad +c[D^*(x_{n-1},Tx_n,Tx_n)+D^*(x_n,Tx_{n-1},Tx_{n-1})]\end{aligned}$$

$$\begin{aligned}
&= a\left[D^*(x_{n-1},x_{n+1},x_{n+1})+D^*(x_n,x_n,x_n)\right]\\
&\quad +b\left[D^*(x_n,x_{n+1},x_{n+1})+D^*(x_n,x_{n+1},x_{n+1})\right]\\
&\quad +c\left[D^*(x_{n-1},x_{n+1},x_{n+1})+D^*(x_n,x_n,x_n)\right]\\
&= (a+c)\left(D^*(x_{n-1},x_n,x_n)+D^*(x_n,x_{n+1},x_{n+1})\right)\\
&\quad +2b\,D^*(x_n,x_{n+1},x_{n+1})\}.
\end{aligned}$$

This implies that

$$D^*(x_n,x_{n+1},x_{n+1}) \le q^n D^*(x_{n-1},x_n,x_n)$$

where $q=\dfrac{(a+c)}{1-(a+2b+c)}$, then $0\le q<1$ and by repeated application of above inequality,

we have,

$$D^*(x_n,x_{n+1},x_{n+1}) \le q^n D^*(x_0,x_1,x_1) \tag{5}$$

Then, for all $n,m\in\mathbb{N}, m<n,$ we have, by repeated use of the rectangle inequality,

$$\begin{aligned}
D^*(x_m,x_n,x_n) &\le D^*(x_m,x_{m+1},x_{m+1})+D^*(x_{m+1},x_n,x_n)\\
&\le D^*(x_m,x_{m+1},x_{m+1})+D^*(x_{m+1},x_{m+2},x_{m+2})\\
&\quad +D^*(x_{m+2},x_n,x_n)\\
&\le D^*(x_m,x_{m+1},x_{m+1})+D^*(x_{m+1},x_{m+2},x_{m+2})\\
&\quad +\cdots+D^*(x_{n-1},x_n,x_n)\\
&\le \left(q^m+q^{m+1}+\cdots+q^{n-1}\right)D^*(x_0,x_1,x_1)\\
&\le \frac{q^m}{1-q}D^*(x_0,x_1,x_1).
\end{aligned}$$

from (1)

$$\left\|D^*(x_m,x_n,x_n)\right\| \le \frac{q^m}{1-q}K\left\|D^*(x_0,x_1,x_1)\right\| \to 0 \text{ as } m,n\to\infty.$$

since $0\le q<1$. So $\{x_n\}$ is D^*-Cauchy sequence. By the completeness of X, there exists $u\in X$ such that $\{x_n\}$ is D^*-convergent to u. Then there is $p\in X$, such that $p=u$. If $T(X)$ is complete, then there exist $u\in T(X)$ such that $x_n\to u,$ as $T(X)\subset X,$ we have $u\in X$ .Then there exist $p\in X$ such that $p=u.$ We claim that $Tp=u$,

$$\begin{aligned}
D^*(Tp,Tp,u) &\le D^*(Tp,Tp,Tx_n)+D^*(Tx_n,u,u)\\
&\le a\left[D^*(p,Tp,Tp)+D^*(p,Tp,Tp)\right]\\
&\quad +b\left[D^*(p,Tx_n,Tx_n)+D^*(x_n,Tp,Tp)\right]\\
&\quad +c\left[D^*(p,Tx_n,Tx_n)+D^*(x_n,Tp,Tp)\right]+D^*(Tx_n,u,u)\\
&\le a\left[D^*(u,Tp,Tp)+D^*(u,Tp,Tu)\right]\\
&\quad +b\left[D^*(u,x_{n+1},x_{n+1})+D^*(Tp,Tp,u)+D^*(u,x_n,x_n)\right]\\
&\quad +c\left[D^*(u,x_{n+1},x_{n+1})+D^*(Tp,Tp,u)+D^*(u,x_n,x_n)\right]
\end{aligned}$$

$$+D^*(x_n,u,u)$$

This implies that

$$D^*(Tp,Tp,u)\le\frac{1}{1-(2a+b+c)}\{(b+c)[D^*(u,x_{n+1},x_{n+1})+D^*(u,x_n,x_n)]$$
$$+D^*(x_{n+1},u,u)\}$$

from (1)

$$\|D^*(Tp,Tp,u)\|\le\frac{1}{1-(2a+b+c)}\{(b+c)[\|D^*(u,x_{n+1},x_{n+1})\|+\|D^*(u,x_n,x_n)\|]$$
$$+\|D^*(x_{n+1},u,u)\|\}$$

the right hand side approaches to zero as $n\to\infty$. Hence $\|D^*(Tp,Tp,u)\|=0$ and $Tp=u$. Hence $Tp=p$. Now we show that T has a unique fixed point. For this, assume that there exists a point q in X such that $q=Tq$. Now

$$\begin{aligned}D^*(Tp,Tp,Tq)&\le a[D^*(p,Tp,Tp)+D^*(p,Tp,Tp)]\\&+b[D^*(p,Tq,Tq)+D^*(q,Tp,Tp)]\\&+c[D^*(p,Tq,Tq)+D^*(q,Tp,Tp)]\\&=a[D^*(Tp,Tp,Tp)+D^*(Tp,Tp,Tp)]\\&+b[D^*(Tp,Tq,Tq)+D^*(Tq,Tp,Tp)]\\&+c[D^*(Tp,Tq,Tq)+D^*(Tq,Tp,Tp)]\\&=b[D^*(Tp,Tp,Tq)+D^*(Tp,Tp,Tq)]\\&+c[D^*(Tp,Tp,Tq)+D^*(Tp,Tp,Tq)]\\&=(2b+2c)D^*(Tp,Tp,Tq)\\D^*(Tp,Tp,Tq)&\le(2b+2c)D^*(Tp,Tp,Tq).\end{aligned}$$

This implies $((2b+2c)-1)D^*(Tp,Tp,Tq)\in P$ and $((2b+2c)-1)D^*(Tp,Tp,Tq)\in -P$, since $(2b+2c)-1<0$. As $P\cap -P=\{0\}$, we have $((2b+2c)-1)D^*(Tp,Tp,Tq)=0,$ i.e. $D^*(Tp,Tp,Tq)=0$. Hence $Tp=Tq$. Also $p=q$, since $Tp=p$. Hence p is a unique fixed point of T in X .

Corollary (2.3). Let (X,D^*) be a complete generalized D^*-metric space, P be a normal cone with normal constant K and let $T:X\to X$ be a mapping which satisfies the following condition,

$$D^*(Tx,Ty,Ty)\le a[D^*(y,Ty,Ty)+D^*(x,Ty,Ty)]$$
$$+bD^*(y,Tx,Tx) \qquad (6)$$

for all $x,y\in X$, where $a,b\ge 0,3a+b<1,$ Then T has a unique fixed point in X .

Proof. Let $x_0\in X$ be arbitrary, there exist $x_1\in X$ such that $Tx_0=x_1$, in this way we have sequence $\{Tx_n\}$ with $Tx_n=x_{n+1}$. Then from inequality (6), we have

$$\begin{aligned}D^*(x_n,x_{n+1},x_{n+1})&=D^*(Tx_{n-1},Tx_n,Tx_n)\\&\le a[D^*(x_n,Tx_n,Tx_n)+D^*(x_{n-1},Tx_n,Tx_n)]\end{aligned}$$

$$+bD^*(x_n, Tx_{n-1}, Tx_{n-1})$$

$$\leq a\left[D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_{n-1}, x_{n+1}, x_{n+1})\right]$$
$$+bD^*(x_n, x_n, x_n)$$
$$\leq a\left[D^*(x_n, x_{n+1}, x_{n+1}) + D^*(x_{n-1}, x_n, x_n)\right.$$
$$\left.+bD^*(x_n, x_n, x_n)\right]$$

This implies that

$$D^*(x_n, x_{n+1}, x_{n+1}) \leq rD^*(x_{n-1}, x_n, x_n) \qquad (7)$$

where $r = \dfrac{a}{1-2a}$, then $0 \leq r < 1$. Then repeating application of (7), we get

$$D^*(x_n, x_{n+1}, x_{n+1}) \leq r^n D^*(x_0, x_1, x_1)$$

Then, for all $n, m \in \mathbb{N}, n > m$ we have, by repeated use of the rectangle inequality,

$$D^*(x_m, x_n, x_n) \leq D^*(x_m, x_{m+1}, x_{m+1}) + D^*(x_{m+1}, x_{m+2}, x_{m+2})$$
$$+\cdots+D^*(x_{n-1}, x_n, x_n)$$
$$\leq (r^m + r^{m+1} + \cdots + r^{n-1}) D^*(x_0, x_1, x_1)$$
$$\leq \frac{r^m}{1-r} D^*(x_0, x_1, x_1).$$

From (1)

$$\|D^*(x_m, x_n, x_n)\| \leq \frac{r^m}{1-r} K \|D^*(x_0, x_1, x_1)\| \to 0 \text{ as } m, n \to \infty.$$

since $0 \leq r < 1$. So $\{x_n\}$ is D^*-Cauchy sequence. By the completeness of X, there exists $u \in X$ such that $\{x_n\}$ is D^*-convergent to u. Then there is $p \in X$, such that $p = u$. If $T(X)$ is complete, then there exist $u \in T(X)$ such that $x_n \to u$, as $T(X) \subset X$, we have $u \in X$ .Then there exists $p \in X$ such that $p = u$. We claim that $Tp = u$,

$$D^*(Tp, Tu, u) \leq D^*(Tp, Tp, x_{n-1}) + D^*(x_{n-1}, u, u)$$
$$\leq a\left[D^*(p, Tp, Tp) + D^*(p, Tp, Tp)\right]$$
$$+bD^*(p, Tp, Tp) + D^*(x_{n-1}, u, u)$$
$$= a\left[D^*(u, Tp, Tp) + D^*(u, Tp, Tp)\right]$$
$$+bD^*(u, Tp, Tp) + D^*(x_{n-1}, u, u)$$
$$= a\left[D^*(Tp, Tp, u) + D^*(Tp, Tp, u)\right]$$
$$+bD^*(Tp, Tp, u) + D^*(x_{n-1}, u, u)$$

This implies that

$$D^*(Tp, u, u) \leq \frac{1}{1-(2a+b)} D^*(x_{n-1}, u, u)$$

from (1)

$$\|D^*(Tp, u, u)\| \leq K \frac{1}{1-(2a+b)} \|D^*(x_{n-1}, u, u)\|$$

right hand side approaches to zero as $n \to \infty$. Hence $\|D^*(Tp,u,u)\| = 0$ and $Tp = u$ and $p = Tp$. Now we show that T has a unique fixed point. For this, assume that there exists a point q in X such that $q = Tq$. Now

$$\begin{aligned} D^*(Tp,Tq,Tq) &\le a\left[D^*(q,Tq,Tq) + D^*(p,Tq,Tq)\right] + bD^*(q,Tp,Tp) \\ &= a\left[D^*(Tq,Tq,Tq) + D^*(Tp,Tq,Tq)\right] + bD^*(Tq,Tp,Tp) \\ &= aD^*(Tp,Tq,Tq) + bD^*(Tp,Tq,Tq) = (a+b)D^*(Tp,Tq,Tq). \end{aligned}$$

This implies $((a+b)-1)D^*(Tp,Tq,Tq) \in P$ and $((a+b)-1)D^*(Tp,Tq,Tq) \in -P$, since $D^*(Tp,Tq,Tq) \in P$ and $(a+b)-1 < 0$. As $P \cap -P = \{0\}$, we have $((a+b)-1)D^*(Tp,Tq,Tq) = 0$, i.e. $D^*(Tp,Tq,Tq) = 0$. Hence $Tp = Tq$. Also $p = q$, since $p = Tp$. Hence p is a unique fixed point of T in X.

Theorem (2.4). Let (X, D^*) be a generalized D^*-metric space, P be a normal cone with normal constant K and let $T: X \to X$, be a mappings which satisfies the following condition

$$\begin{aligned} D^*(Tx,Ty,Tz) \le\ & a\left[D^*(z,Tx,Tx) + D^*(y,Tx,Tx)\right] \\ & + b\left[D^*(y,Tz,Tz) + D^*(x,Tz,Tz)\right] \\ & + c\left[D^*(x,Ty,Ty) + D^*(z,Ty,Ty)\right] \qquad (8) \end{aligned}$$

for all $x, y, z \in X$, where $a, b, c \ge 0, 3a + 2b + 3c < 1$. Then T has a unique common fixed point in X.

Proof. Setting $z = y$ in condition (8), reduces it to condition (6), and the proof follows from Corollary (2.3).

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