Common coupled fixed point theorems for contractive mappings in fuzzy metric spaces

[1]V.K. Agrawal, [2]Geeta Modi, [3]Ajay Soni*

[1]Department of Mathematics, Govt. H. S. College, H. bad, India, E-mail: vinkagar@yahoo.co.in

[2]Department of Mathematics, Govt. M.V.M. Bhopal, India, E-mail: modi.geeta@gmail.com

[3]Research Scholar Barkatullah University, Bhopal, India, E-mail: ajay.mvm01@yahoo.com

Abstract: The purpose of this paper is to give some new fixed point theorems for contractive type mappings in fuzzy metric spaces. The results presented improve and generalize some recent result. The result is a genuine generalization of the corresponding result of S.Sedghi et al. (2010).

Keywords: Fuzzy metric space, t-norm, g-convergent, coupled common fixed point.
2010 MSC: Primary 54E70; Secondary 54H25.

Introduction:

Since Zadeh [1] introduced the concept of fuzzy sets, many authors have extensively developed the theory of fuzzy sets and applications. George and Veeramani [2, 3] gave the concept of fuzzy metric space and defined a Hausdorff topology on this fuzzy metric space which have very important applications in quantum particle physics particularly in connection with both string and infinity theory.

Bhaskar and Lakshmikantham [4], Lakshmikantham and Ćirić [5] discussed the mixed monotone mappings and gave some coupled fixed point theorems which can be used to discuss the existence and uniqueness of solution for a periodic boundary value problem. Sedghi et al. [6] gave a coupled fixed point theorem for contractions in fuzzy metric spaces, and Fang [7] gave some common fixed point theorems under contractions for compatible and weakly compatible mappings in Menger probabilistic metric spaces. Many authors [8–11] have proved fixed point theorems in (intuitionistic) fuzzy metric spaces or probabilistic metric spaces.

In this paper, using similar proof as in [7], we give a new common fixed point theorem under weaker conditions than in [6] and give an example which shows that the result is a genuine generalization of the corresponding result in [6].

Preliminaries:

We start by recalling two definitions.

Definition 2.1. Let X be a non empty set The mappings $F: X \times X \to X$ and $g: X \to X$ are said to commute if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.

Definition 2.2. An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F: X \times X \to X$ and $g: X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

The mappings F and g have a common fixed if there exists $x \in X$ such that $x = gx = F(x, x)$.

Our main theorem states as follows.

Main Results:

Theorem 3.1: Let (X, M, T) be a complete fuzzy metric space satisfying (FM-6) with T a g-convergent t-norm . Let F : X $\times$ X $\to$ X and g : X $\to$ X be two mappings such that, for some k $\in$ (0,1),

$M(F(x, y), F(u, v), Kt) \geq Min\{M(gx, gu, t), M(gy, gv, t), M(gx, gv, t), M(gy, gu, t)\}.$

for all x, y, u ,v $\in$ X , t $\geq$ 0. Suppose that F (X $\times$ X) $\subset$ g(x) and that g is continuous and commutes with F. If there exists a $>$ 0 and x_0, y_0 $\in$ X such that

$\sup_{t>0} t^a \left(1 - M(gx_0, F(x_0, y_0), t)\right) < \infty$ and

$\sup_{t>0} t^a \left(1 - M(gy_0, F(y_0, x_0), t)\right) < \infty$

then F and g have a unique common fixed points in X .

We note that if (x_0, y_0) is a coupled coincidence point of F and g, then the conditions

$\sup_{t>0} t^a \left(1 - M(gx_0, F(x_0, y_0), t)\right) < \infty$ and

$\sup_{t>0} t^a \left(1 - M(gy_0, F(y_0, x_0), t)\right) < \infty$ are satisfied.

Proof: Let x_0, y_0 be as in the statement of the theorem . Since F(X $\times$ X) $\subset$ g(x), we can choose $x_1, y_1 \in$ X such that $gx_1 = F(x_0, y_0)$ and $gy_1 = F(y_0, x_0)$. Continuing in this way one can construct two sequences $\{x_n\}_{n \in N}$ and $\{y_n\}_{n \in N}$ in X with the properties

$gx_{n+1} = F(x_n, y_n), gy_{n+1} = F(y_n, x_n)$ for all $n \in N$

We divide the proof into five steps.

Step 1. We show that $\{gx_n\}_{n \in N}$ and $\{gy_n\}_{n \in N}$ are Cauchy sequences.

Indeed, let $\alpha > 0$ be such that $t^\alpha \left(1 - M(gy_{0,}F(y_0, x_0), t)\right) \leq \alpha$

and $t^\alpha (1 - M(gx_0, F(x_0, y_0), t)) \leq \alpha$

For all $t > 0$. Then $M\left(gx_{0,} \ gx_1, \frac{1}{t^n}\right) \geq 1 - \alpha(t^\alpha)^n$ and $M\left(gy_{0,} \ gy_1, \frac{1}{t^n}\right) \geq 1 - \alpha(t^\alpha)^n$ for every $t > 0$ and $n \in N$.

If $t > 0$ and $\varepsilon \in (0,1)$ are given, we choose μ in the interval $(k,1)$ such that

$T_{i=n+1}^\infty (1 - \mu^q)^i) > 1 - \varepsilon$ and $\delta = \frac{K}{\mu}$. As $\delta \varepsilon (0,1)$, we can find $n_1(= n_1(t))$ such that $\sum_{n=n1}^\infty \delta^n < t$.

condition (2.1) implies that, for all $s > 0$,

$M(gx_{1,} \ gx_{2,}ks) = M(F(x_0, y_0), F(x_1, y_1), s)$
$$\geq Min\{M(gx_{0,} \ gx_1, S), M(gy_{0,} \ gy_1, s)\}$$

$M(gy_{1,} \ gy_{2,}ks) = M(F(y_0, x_0), F(y_1, \ x_1), s)$
$$\geq Min\{M(gy_0, gy_1, s), M(gx_0, gx_1, s)\}$$

$M(gx_{2,} \ gx_{3,}ks) = M\left(F(x_1, y_1 \ ), F(x_2, y_2), s\right)$

$$\geq Min\{M(gx_{1,} \ gx_2), M(gy_1, gy_2), s\}$$

$M(gy_{2,} \ gy_{3,}ks) = M(F(y_1, x_1), F(y_2, x_2), s)$

$$\geq Min\{M(gy_{1,} \ gy_2, S), M(gx_{1,}gx_2, S)\}$$

It follows by including that

$M(gx_{n,} \ gx_{n+1}, k^n S) \geq Min\{M(gx_0, gx_1, S), M(gy_0, gy_1, S), M(gx_1, gx_2, S), M(gy_1, gy_2, S)\}$
$M(gy_{n,} \ gy_{n+1}, k^n S) \geq$
$Min\{M(gy_0, gy_1, S), M(gx_1, gx_2, S), M(gy_1, gy_2, S), M(gx_1, gx_2, S)\}$

for all n ∈ N. Then for all x ∈ X and n ∈ N we obtain

$$M(gx_n, gx_{n+m}, t) \geq M(gx_n, gx_{n+,m}, \Sigma_{x=n1}^{\infty} \delta^i)$$

$$\geq M(gx_n, gx_{n+m} \Sigma_{i=n}^{n+m-1} \delta^i)$$

$$\geq T_{i=n}^{n+m-1} M(gx_i, gx_{i+1}, S^i)$$

$$\geq T_{i=n}^{n+m-1} \left(Min \left\{ \begin{array}{l} M\left(gx_0, gx_1, \frac{1}{\mu^i}\right), M\left(gy_0, gy_1, \frac{1}{\mu^i}\right) \\ M\left(gx_1, gx_2, \frac{1}{\mu^i}\right), M\left(gy_1, gy_2, \frac{1}{\mu^i}\right) \end{array} \right\} \right)$$

$$\geq T_{i=n}^{n+m-1}(1 - \alpha\mu^{\alpha i})$$

If we choose $l_0 \in N$ such that $\alpha\mu^{\alpha l_0} \leq \mu^a,$ then

$$1 - \alpha(\mu^a)^{n+l_0} \geq 1 - (\mu^a)^{n+1} \quad \text{for all n,}$$

Thus $M(gx_{n+l_0}, gx_{n+l_0+m}, t) \geq T_{l=n+1}^{\infty} (1 - (\mu^a)^i > 1 - \varepsilon,$

for every n ≥ n_1 and m ∈ N, hence {gx_n} is a Cauchy sequence.

Similarly one can show that {gy_n} is a Cauchy sequence.

Step 2. We prove that g and F have a coupled coincidence point. Since X is complete, thus exist x, y ∈ x such that $\lim_{n\to\infty} F(x_{n,}y_n) = \lim_{n\to\infty} g(x_n) = x, \lim_{n\to\infty} F(y_{n,}x_n) = \lim_{n\to\infty} g(y_n) = y$

Since F & g are compatible.

$$\lim_{n\to\infty}(gF(x_{n,}y_n), F(g(x_n), g(y_n)), t) = 1,$$

$$\lim_{n\to\infty}(gF(y_{n,}x_n), F(g(y_n), g(x_n)), t) = 1.$$

for all t > 0. Next we prove that

$$g(x) = F(x, y), g(y) = F(y, x)$$

from the continuity of g it follows that $\lim_{n\to\infty} ggx_n = gx$ and $\lim_{n\to\infty} ggy_n` = gy$ as F and g comments.

$$ggx_{n+1} = gF(x_n, y_n) = F(gx_n, gy_n) \text{ and } ggy_{n+1} = gF(y_n, x_n) = F(gy_n, gx_n)$$

Consequently, for all t > 0 and n ϵ N,

$$M(gx, F(x,y), kt) \geq M(ggx_{n+1}, F(x,y), kt), M(gx, ggx_{n+1}, k\,t)$$

$$= M(gF(x_n, y_n), F(x,y), kt), M(gx, ggx_{n+1}, kt)$$

$$\geq Min \begin{Bmatrix} M(gF(x_n, y_n), F(gx_n, gy_n), \underline{k}t), M(ggx_n, gx, kt) \\ M(ggy_n, gy, kt), M(gx, ggx_{n+1}, k\,t) \end{Bmatrix}$$

Since g & F are compatible with the continuity of g, we get $M(gx, F(x,y), \underline{k}t) \geq 1$ which implies that $gx = F(x,y)$.

Similarly we can get $gy = F(y,x)$.

Step 3. We prove that $gx = y$ and $gy = x$.

Obtained from

$$M(gx, gy_{n+1}, kt) = M(F(x,y), F(y_n, x_n)),$$

Letting n→∞ in the inequality

$$M(gx, gy_{n+1}, kt) \geq Min\{(gx, gy_n, t), M(gy, gx_n, t)$$

We get, $M(gx, y, kt) \geq Min\{M(gx, y, t), M(gy, x, t)\}$

and similarly we can get,

$$M(gy, x, kt) \geq Min\{M(gx, y, t), M(gy, x, t)\}$$

Thus, $Min\{M(gx, y, t), M(gy, x, t)\} \geq Min\left\{M\left(gx, y, \frac{t}{k^n}\right), M\left(gy, x, \frac{1}{k^n}\right)\right\}.$

By this way, we can get for all n ϵ N

$Min\{M(gx, y, t), M(gy, x, t)\} = 1$ for all t > 0.

It follows that $M(gx, y, t) = M(gy, x, t) = 1$ for t > 0.

We can get that $gx = y \; and \; gy = x$, as claimed.

Step 4. We prove that $x = y$.

Indeed from $\qquad M\big(gx_{n+1}, gy_{n+1,} \, kt\big) = M(F\big(x_{n,}y_n\big), F\big(y_{n,}x_n\big), kt$

$$\geq Min\{M\big(gx_{n,}gy_{n,}t\big), M\big(gy_{n,}gx_{n,} \, t\big), M\big(gy_{n,}gx_{n,}t\big)$$

$$M(x, y, kt) \geq Min\{M(x, y, t), M(y, x, t) \qquad\qquad (t > 0)$$

It follows that $M(x, y, kt) \geq M(x, y, t)$ for all t > 0 and so x = y

Step 5. We prove that the fixed point is unique. Let z , w be common fixed points for F and g. Then from (2.1) we obtain

$$M(F(z, z), F(w, w), kt) \geq Min\{M(gz, gw, t), M(gz, gw, t), M(gz, gw, t), M(gz, gw, t) \, (t$$
$$> 0)$$

That is $M(z, w, kt) \geq M(z, w, t)$ for all t > 0, implying $z = w$.

Our next theorem shows that ,if the t-norm T is of Hadzic- type, then the conditions

$$\sup_{t>0} t^\alpha \, (1 - M\big(gx_0, F\big(x_{0,} \, y_0\big), t\big)) < \infty \;\; \text{and} \; \sup_{t>0} t^\alpha \, (1 - M\big(gy_0, F\big(y_{0,} \, x_0\big), t\big)) < \infty$$

Can be dropped.

Theorem 3.2: Let (X, M, T) be a complete fuzzy metric space satisfying (FM-6) with $T \in H$. Let F : X × X → X and g : X → X be two mappings such that, for some k ϵ (0,1),

$$M(F(x, y), F(u, v), Kt) \geq Min\{M(gx, gu, t), M(gy, gv, t), M(gx, gv, t), M(gy, gu, t)\}.$$

for all x, y, u ,v ϵ X , t > 0. Suppose that F (X × X) ⊂ g(x) and g is continuous and commutes with F. Then F and g have a unique common fixed point in X.

Proof: We only have to verify step 1 in Theorem 2.3, that is to prove that $\{gx_n\} \; and \; \{gy_n\}$are Cauchy sequence.

Let $t > 0 \; and \; \varepsilon \in (0,1)$ be given. Since T is a t-norm of Hadzic- type then exists $\mu > 0$ such that $T^k(1 - \mu) > 1 - \varepsilon \; for \; all \; k \in N.$

By (FM-6), we can find $s > 0$ such that

$$M(gx_0, gx_1, s) > 1 - \mu \quad and \quad M(gy_0, gy_1, s) > 1 - \mu$$

Let $n_0 \in N$ be such that $t > \sum_{i=n_0}^{\infty} k^i s$.

As in Step 1 in the proof of theorem 2.3 it can be proved that

$$M\big(gx_n, \ gx_{n+1}, k^n S\big) \geq Min \begin{Bmatrix} M(gx_0, gx_1, S), M\big(gy_0, gy_1, S\big) \\ M(gx_1, gx_2, S), M\big(gy_1, gy_2, S\big) \end{Bmatrix} > 1 - \mu$$

and $\qquad M\big(gy_n, \ gy_{n+1}, k^n S\big) \geq Min \begin{Bmatrix} M(gy_0, gy_1, S), M(gx_0, gx_1, S) \\ M(gy_1, gy_2, S), M(gx_1, gx_2, S) \end{Bmatrix} > 1 - \mu$

for all $n \in N$. Therefore, for all $n \geq n_0$ and all $m \in N$ the following inequalities hold:

$$M\big(gx_n, \ gx_{n+1}, t\big) \geq M\left(gx_n, \ gx_{n+1}, \sum_{i=n_1}^{\infty} k^i S\right) \geq M\left(gx_n, \ gx_{n+m}, \sum_{i=n}^{n+m-1} k^i S\right)$$

$$\geq T_{i=n}^{n+m-1} M\big(gx_i, gx_{i+1}, k^i s\big) \geq T_{i=n}^{n+m-1}(1 - \mu) > 1 - \varepsilon.$$

Acknowledgement:

References:

1. Zadeh, LA: Fuzzy sets. Information and Computation. 8, 338–353 (1965).

2. George, A, Veeramani, P: On some results in fuzzy metric spaces. Fuzzy Sets and Systems. 64(3), 395–399 (1994).

3. George, A, Veeramani, P: On some results of analysis for fuzzy metric spaces. Fuzzy Sets and Systems. 90(3), 365–368 (1997).

4. Bhaskar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Analysis. Theory, Methods & Applications. 65(7), 1379–1393 (2006).

5. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Analysis. Theory, Methods & Applications. 70(12), 4341–4349 (2009).

6. Sedghi, S, Altun, I, Shobe, N: Coupled fixed point theorems for contractions in fuzzy metric spaces. Nonlinear Analysis. Theory, Methods & Applications. 72(34), 1298–1304 (2010).

7. Fang, JX:Common fixed point theorems of compatible and weakly compatible maps in Menger spaces. Nonlinear Analysis. Theory, Methods & Applications. 71(56), 1833–1843 (2009).

8. Ćirić, LB, Miheţ, D, Saadati, R: Monotone generalized contractions in partially ordered probabilistic metric spaces. Topology and its Applications. 156(17), 2838–2844 (2009).

9. O'Regan, D, Saadati, R: Nonlinear contraction theorems in probabilistic spaces. Applied Mathematics and Computation. 195(1), 86–93 (2008).

10. Jain, S, Jain, S, Bahadur Jain, L: Compatibility of type (P) in modified intuitionistic fuzzy metric space. Journal of Nonlinear Science and its Applications. 3(2), 96–109 (2010)

11. Ćirić, LB, Ješić, SN, Ume, JS: The existence theorems for fixed and periodic points of nonexpansive mappings in intuitionistic fuzzy metric spaces. Chaos, Solitons and Fractals. 37(3), 781–791 (2008).

The IISTE is a pioneer in the Open-Access hosting service and academic event management. The aim of the firm is Accelerating Global Knowledge Sharing.

More information about the firm can be found on the homepage:
http://www.iiste.org

CALL FOR JOURNAL PAPERS

There are more than 30 peer-reviewed academic journals hosted under the hosting platform.

Prospective authors of journals can find the submission instruction on the following page: http://www.iiste.org/journals/ All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Paper version of the journals is also available upon request of readers and authors.

MORE RESOURCES

Book publication information: http://www.iiste.org/book/

Academic conference: http://www.iiste.org/conference/upcoming-conferences-call-for-paper/

IISTE Knowledge Sharing Partners

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digtial Library , NewJour, Google Scholar