# The Dynamics of an SIS Epidemic Disease with Contact and External Source

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## Abstract

In this paper, we discuss the dynamical behavior of eco-epidemiological mathematical model consisting of prey-predator model involving SIS infectious disease in prey population, is proposed and analyzed. This disease passed from a prey to predator through attacking of predator to prey. It is assumed that the disease transmitted within the same species by contact between susceptible individuals and infected individuals, in additional to the external sources from the environment. The existence, uniqueness and boundedness of the solution of the system are studied. The local and global stability conditions of all possible equilibrium points are established. Finally, some numerical simulations are given to illustrate the analytical results.

**Keywords:** prey-predator model, SIS epidemics disease, stability analysis, Lyapunov function.

## 1. Introduction:

Eco-epidemiological is a new branch in mathematical biology which considers both the ecological and epidemiological issues simultaneously. The first breakthrough in modern mathematical ecology was done by Lotka (1924) for a predator – prey competing species. On the other hand, most of the models for the transmission of infectious diseases originated from the classic work of Kermack and Mckendrik (1927) . After these pioneering works in two different fields, lots of research works have been done both in theoretical ecology and epidemiology. Anderson and May [1] where the first who merged the above two fields and formulated a prey – predator model where prey species infected by some diseases. Later on many researchers, especially in the last two decades , have proposed and studied different prey – predator models in the presence of disease in one of the species see for example [ 2,3,4,5,6,7,8,9,10,11 ] and the references there in .In most previous studies , the only means of transmission of disease is the direct contact between individuals. However, many diseases are transmitted in the species not only by contact, but also directly from environment, such as influenza (birds flu) and others for example see [12,13,14]. Hethcote [14], May and Leonard [15] constructed a model to study the effect of infectious diseases in predator – prey systems, in this model, the basic epidemic model was combined with Lotika – Volterra model. Das et al [16] proposed prey – predator model with disease in prey spread by contact and external sources, including Holling type II as a functional response and linear disease incidence. Hsich and Hsiao [17] proposed a prey – predator model with disease in both populations. They observed that ecological threshold number for the prey – predator ecosystem always determine the coexistence of predator and prey where as disease basic reproduction number dictates whether the disease would become endemic in the ecosystem or not . R. Latief Tayeh and R. Kamel Naji [18] had previously studied a prey- predator model involving SI infection disease in both the prey and predator species and the disease passed from a prey to predator through predation process. While the disease transmitted within the same species by contact**.** In this paper an eco-epidemiological mathematical model consisting of prey-predator model involving SIS epidemic disease in both the prey and predator species has been proposed and analyzed. Further, in this model, linear type of functional response as well as linear incidence rate for describing the transition of disease are used.

## 2. Model Formulation:

In this section an eco-epidemiological model is proposed for study. The model consists of a prey, whose total population density at time T is denoted by N(T), interacting with predator whose total population density at time T is denoted by P(T). The following assumptions are made in formulating the basic eco - epidemiological model**:**

1. There is an SIS epidemic disease in prey population divides the prey population into two classes namely $S(T)$ that represents the density of susceptible prey species at time T and $I(T)$ which represents the density of infected prey species at time T. Therefore at any time T, we have $N(T) = S(T) + I(T)$.
2. The disease is transmitted from a prey to predator during attacking of predator to prey, which divides the predator population into two classes namely $P_1(T)$ that represents the density of susceptible predator species at time T and $P_2(T)$ which represents the density of infected predator species at time T. Therefore at any time T, we have $P(T) = P_1(T) + P_2(T)$.
3. The susceptible prey is capable of reproducing in logistic fashion with carrying capacity $k > 0$ and intrinsic growth rate $r > 0$.
4. The disease transmitted within the same species by contact with an infected individual at infection rates $\lambda_1 > 0$ and $\lambda_2 > 0$ for the prey and predator respectively. In addition, there is an external source of disease causes incidence with the disease within the specific population at an external infection rates $\Theta_1 > 0$ and $\Theta_2 > 0$ for the prey and predator respectively.
5. The disease disappears and infected individuals become susceptible again at the recovery rates $\alpha > 0$ and $\beta > 0$ for the prey and predator respectively.
6. In the absence of the prey, the susceptible and infected predator decay exponentially with natural death rate $\gamma_2 > 0$.
7. The disease may causes mortality with a constant mortality rates $\gamma_1 > 0$ and $\gamma_3 > 0$ for the prey and predator respectively.
8. The susceptible predator consumes the susceptible and infected prey according to Lotka-Volterratype of functional response at constant consumption rates $c_1 > 0$ and $c_2 > 0$
for susceptible and infected respectively, while the infected predator can't attack the prey directly
due to the its weakness**.**

Therefore, by using the above assumptions, the dynamic of prey-predator model can be represented in the following set of the first order nonlinear differential equations.

$$\left.\begin{aligned}
\frac{dS}{dT} &= rS\left(1 - \frac{S+I}{K}\right) - c_1SP_1 - \lambda_1SI - \Theta_1S + \alpha I \\
\frac{dI}{dT} &= \lambda_1SI + \Theta_1S - c_2IP_1 - \gamma_1I - \alpha I \\
\frac{dP_1}{dT} &= -\lambda_1P_1P_2 - \Theta_2P_1 + e_1c_1SP_1 + (1-m)\,e_2c_2IP_1 - \gamma_2P_1 + \beta P_2 \\
\frac{dP_2}{dT} &= \lambda_1P_1P_2 + \Theta_2P_1 + m\,e_2c_2IP_1 - \gamma_2P_2 - \gamma_3P_2 - \beta P_2
\end{aligned}\right\} \quad \textbf{(1)}$$

With initial condition $S(0) \geq 0$, $I(0) \geq 0$, $P_1(0) \geq 0$, $P_2(0) \geq 0$, $0 < e_i < 1$; $i = 1,2$ represent the conversion rate constants and $0 < m < 1$ represents the infection rate of susceptible predator that predation the infected prey. Cleary, system (**1**) included (**16**) parameters, which make the analysis difficult.
So, in order to simplify the system the number of parameters is reduced by using the following dimensionless variables.

$$t = r\,T,\ \ x = \frac{S}{k},\ \ y = \frac{I}{k},\ \ z = \frac{c_1}{r}P_1,\ \ w = \frac{c_1}{r}P_2\,.$$

Thus we obtain:

$$\left.\begin{aligned}
\frac{dx}{dt} &= x\left(1 - x - (1 + u_1)y - z - u_2\right) + u_3y = f_1(x, y, z, w) \\
\frac{dy}{dt} &= y\left(u_1x - u_4z - (u_3 + u_5)\right) + u_2x = f_2(x, y, z, w) \\
\frac{dz}{dt} &= z\left(-u_6w + u_8x + u_9(1 - m)y - (u_7 + u_{10})\right) + u_{11}w = f_3(x, y, z, w)
\end{aligned}\right\} \quad \textbf{(2)}$$

$$\frac{dw}{dt} = u_6 zw + ( u_7 + u_9 my )z – ( u_{10} + u_{11} + u_{12} )w = f_4(x, y, z, w)$$

Where:

$u_1= \frac{\lambda_1 k}{r}$ , $u_2= \frac{\theta_1}{r}$ , $u_3= \frac{\alpha}{r}$ , $u_4= \frac{c_2}{c_1}$ , $u_5= \frac{\gamma_1}{r}$ , $u_6= \frac{\gamma_2}{c_1}$ , $u_7= \frac{\theta_2}{r}$ , $u_8= \frac{e_1 c_1 k}{r}$ , $u_9= \frac{e_2 c_2 k}{r}$ , $u_{10}= \frac{\gamma_2}{r}$ , $u_{11}= \frac{\beta}{r}$ , $u_{12}= \frac{\gamma_3}{r}$ .

represent the dimensionless parameters of the system (**2**). Moreover the initial condition of system (**2**) may be taken as any point in the region $R_+^4$ . The interaction functions in the right hand side of system (**2**) are continuously differentiable function on $R_+^4$ , hence they are Lipschitizian. Therefore the solution of system (**2**) exists and is unique. Further, all the solutions of system (**2**) with non negative initial condition are uniformly bounded as shown in the following theorem.

**Theorem(1):** All the solutions of system (**2**) which initiate in the $R_+^4$ are uniformly bounded.

**Proof:** Let x(t), y(t), z(t), w(t) be any solution of the system (**2**).

Define the function $M(t) = x(t) + y(t) + z(t) + w(t)$, then take the time derivative of M(t) along the solution of the system (**2**), gives

$$\frac{dM}{dt} \le 1- nM \text{ , where } n = \min \left\{ 1 , u_5 , u_{10} , ( u_{10} + u_{12} ) \right\}. \text{ Then}$$

$$\frac{dM}{dt} + nM \le 1$$

Again by solving this differential inequality for the initial value $M(0) = M_0$ , we get :

$$M(t) \le \frac{1}{n} + \left(M_0 - \frac{1}{n}\right) e^{-nt} .$$

Then $M(t) \le \frac{1}{n}$ as $t \to \infty$ . So $0 \le M(t) \le \frac{1}{n}$ , hence all the solutions of system (**2**) are uniformly bounded and the proof is complete. ■

## 3. Existence of equilibrium points:

It is observed that, system (**2**) has at most three biologically feasible equilibrium points, $E_i =(x, y, z, w)$; i = 0,1,2 . The existence conditions for each of these equilibrium points are discussed in the following:

1 - **The vanishing equilibrium point** $E_0 = (0,0,0,0)$ always exists.

2 - **The predator free equilibrium point** $E_1 = ( \hat{x} ,\hat{y} , 0, 0 )$, where :

$$\hat{y} = \frac{1-( \hat{x}+ u_2 )}{( 1+ u_1 ) - (\frac{u_3}{\hat{x}})} ,\ ( 1+ u_1 ) \ne \frac{u_3}{\hat{x}} \qquad \textbf{(3.1)}$$

While $\hat{x}$ represents a positive root of the following second order polynomial equation

$$A_1 x^2 + A_2 x + A_3 = 0 \qquad \textbf{(3.2)}$$

where:

$$A_1 = u_1 > 0$$
$$A_2 = - ( u_1 +u_2 + u_3 + u_5 ) < 0$$
$$A_3 = ( u_3 + u_5 ) – ( u_2 u_5 )$$

Consequently, straightforward computation shows that $E_1$ exists uniquely in the Int.$R_+^4$ if and only if the following conditions are hold.

$$u_3 + u_5 < u_2 u_5 \qquad \textbf{(3.3)}$$

$$\frac{u_3}{1+ u_1} < \hat{x} < 1 - u_2 \qquad \textbf{(3.4)}$$

3 - T**he positive (coexistence) equilibrium point** $E_2 = ( x^*, y^*, z^*, w^* )$ exists in the Int.$R_+^4$ if and only if there is a positive solution of the following set of algebraic equations

$$f_1(x,y,z,w) = 1 - x - (1+u_1)y - z - u_2 + \frac{u_3 y}{x} = 0 \quad \textbf{(3.5)}$$

$$f_2(x,y,z,w) = u_1x - u_4z - (u_3 + u_5) + \frac{u_2 x}{y} = 0 \quad \textbf{(3.6)}$$

$$f_3(x,y,z,w) = - u_6w + u_8x + u_9(1-m)y - (u_7 + u_{10}) + \frac{u_{11} w}{z} = 0 \quad \textbf{(3.7)}$$

$$f_4(x,y,z,w) = u_6z + (u_7 + mu_9 y)\frac{z}{w} - (u_{10} + u_{11} + u_{12}) = 0 \quad \textbf{(3.8)}$$

By solving **(3.7)** and **(3.8)**, we obtain that

$$z(x,y) = \frac{A}{u_6 B} \ ; B \neq 0 \quad \textbf{(3.9)}$$

$$w(x,y) = \frac{A}{u_6(u_{10} + u_{12})} \quad \textbf{(3.10)}$$

Where:

$$A = (u_{10} + u_{11} + u_{12})\left[u_8 x + u_9(1-m)y - (u_7 + u_{10})\right] + u_{11}(u_7 + mu_9 y)$$

$$B = (u_8 x + u_9 y - u_{10})$$

Then by using **(3.9)** and **(3.10)** in **(3.5)** and **(3.6)** yield the following two isoclines.

$$g_1(x,y) = x\left(1 - x - (1+u_1)y - \frac{A}{u_6 B} - u_2\right) + u_3 y = 0 \quad \textbf{(3.11)}$$

$$g_2(x,y) = y\left(u_1 x - u_4 \frac{A}{u_6 B} - (u_3 + u_5)\right) + u_2 x = 0 \quad \textbf{(3.12)}$$

Now from **(3.11)** we notice that, when $y \to \infty$, then either $x = 0$ c! or $x$ represents a positive root of the following second order polynomial equation.

$$B_1 x^2 + B_2 x + B_3 = 0 \quad \textbf{(3.13)}$$

Where:

$$B_1 = u_6 u_8$$

$$B_2 = u_8(u_{10} + u_{11} + u_{12} + u_2 u_6) - u_6(u_8 + u_{10})$$

$$B_3 = -\left(u_7(u_{10} + u_{12}) + u_{10}(u_{10} + u_{11} + u_{12} - u_6)\right)$$

Straightforward computation shows that Eq. **(3.13)** has a unique positive root namely $x_1$ if and only if the following condition hold.

$$u_6 < (u_{10} + u_{11} + u_{12}) \quad \textbf{(3.14)}$$

Moreover from Eq. **(3.11)** we have $\frac{dx}{dy} = \left(\frac{\partial g_1}{\partial y}\right) / \left(\frac{\partial g_1}{\partial x}\right)$. So, $\frac{dx}{dy} < 0$ if one set of the following sets of conditions holds.

$$\frac{\partial g_1}{\partial y} > 0, \frac{\partial g_1}{\partial x} < 0 \quad \text{OR} \quad \frac{\partial g_1}{\partial y} < 0, \frac{\partial g_1}{\partial x} > 0 \quad \textbf{(3.15)}$$

Further, from Eq. **(3.12) w**e notice that, when $y \to \infty$ then $x = 0$, in addition since we have $\frac{dx}{dy} = \left(\frac{\partial g_2}{\partial y}\right) / \left(\frac{\partial g_2}{\partial x}\right)$. So, $\frac{dx}{dy} > 0$ if one set of the following sets of conditions holds.

$$\frac{\partial g_2}{\partial y} > 0, \frac{\partial g_2}{\partial x} > 0 \quad \text{OR} \quad \frac{\partial g_2}{\partial y} < 0, \frac{\partial g_2}{\partial x} < 0 \quad \textbf{(3.16)}$$

Then the two isoclines **(3.11)** & **(3.12)** intersect at a unique positive point $(x^*, y^*)$ in the Int.$R_+^4$ of xy – plane .Substituting the value of $x^*$ and $y^*$ in Eq.**(3.9)** & **(3.10)** yield that $z(x^*,y^*) = z^*$ and $w(x^*,y^*) = w^*$ which are positive if and only if the following condition hold.

$$u_{10} < \min \{ u_8 x + u_9 y , u_8 x + u_9 (1-m)y - u_7 \} \qquad \textbf{(3.17)}$$

Accordingly, the positive equilibrium point $E_2$ exists uniquely in the Int. $R_+^4$ if in addition to the conditions **(3.14 - 3.17)** the isoclinic $g_1(x, y) = 0$ intersect the x-axis at the positive value namely $x_1$.

**4. Local stability analysis of system (2):**

In this section the local stability analysis of all feasible equilibrium points of system(**2**) is studied analytically by linearization method as bellow. Note that, from now onward the symbols $\lambda_{ix}$, $\lambda_{iy}$, $\lambda_{iz}$ and $\lambda_{iw}$ represent the eigenvalues of the Jacobian matrix $J(E_i)$ ; i= 0,1,2 that describe the dynamics in the x- direction , y- direction , z- direction and w- direction respectively, where the Jacobian matrix $J(x, y, z, w)$ of the system (**2**) at each of them can be written:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} & \frac{\partial f_3}{\partial w} \\ \frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial z} & \frac{\partial f_4}{\partial w} \end{bmatrix} \qquad \textbf{(4.1)}$$

Where $f_i$ ; i = 1,2,3,4 are given in system (**2**) and

$$\frac{\partial f_1}{\partial x} = 1 - (u_2 + 2x + (1+u_1)y + z);\ \frac{\partial f_1}{\partial y} = u_3 - (1+u_1)x;\ \frac{\partial f_1}{\partial z} = -x;\ \frac{\partial f_1}{\partial w} = 0;$$
$$\frac{\partial f_2}{\partial x} = u_2 + u_1 y;\ \frac{\partial f_2}{\partial y} = u_1 x - (u_4 z + u_3 + u_5);\ \frac{\partial f_2}{\partial z} = -u_4 y;\ \frac{\partial f_2}{\partial w} = 0;$$
$$\frac{\partial f_3}{\partial x} = u_8 z;\ \frac{\partial f_3}{\partial y} = u_9(1-m)z;\ \frac{\partial f_3}{\partial z} = u_8 x + u_9(1-m)y - (u_6 w + u_7 + u_{10});\ \frac{\partial f_3}{\partial w} = u_{11} - u_6 z;$$

$$\frac{\partial f_4}{\partial x} = 0;\ \frac{\partial f_4}{\partial y} = u_9 m z;\ \frac{\partial f_4}{\partial z} = u_7 + u_9 m y + u_6 w;\ \frac{\partial f_4}{\partial w} = u_6 z - (u_{10} + u_{11} + u_{12}).$$

Now, the Jacobian matrix of system (**2**) at $E_0$ can be written as:

$$\mathbf{J_0} = \begin{bmatrix} 1-u_2 & u_3 & 0 & 0 \\ u_2 & -(u_3+u_5) & 0 & 0 \\ 0 & 0 & -(u_7+u_{10}) & u_{11} \\ 0 & 0 & u_7 & -(u_{10}+u_{11}+u_{12}) \end{bmatrix} \qquad \textbf{(4.2)}$$

Accordingly, the characteristic equation of this Jacobian matrix is given by:

$$[\lambda^2 + A_1\lambda + A_2][\lambda^2 + A_3\lambda + A_4] = 0$$

Where:

$A_1 = (u_3 + u_5) - (1 - u_2)$
$A_2 = -(u_3 + u_5 - (u_2 u_5))$
$A_3 = (u_7 + 2u_{10} + u_{11} + u_{12})$
$A_4 = u_7 (u_{10}+u_{12}) + u_{10}(u_{10}+u_{11}+u_{12})$

From which, we obtain that:

$$\left.\begin{aligned} \lambda_{0\,x,y} &= -\frac{A_1}{2} \pm \frac{1}{2}\sqrt{A_1^2 - 4A_2} \\ \lambda_{0\,z,w} &= -\frac{A_3}{2} \pm \frac{1}{2}\sqrt{A_3^2 - 4A_4} \end{aligned}\right\} \quad \textbf{(4.3)}$$

Therefore all the eigenvalues have negative real part provided that the following condition is satisfied:

$$u_2 > 1 + \frac{u_3}{u_5} \quad \textbf{(4.4)}$$

So, the equilibrium point $E_0$ is locally asymptotically stable in the Int.$R_+^4$ **.** However, it is (a saddle point) unstable otherwise.

The Jacobian matrix of system (**2**) at $E_1$ can be written as:

$$J_1 = J(E_1) = [a_{ij}]_{4x4} \quad \textbf{(4.5)}$$

Here:

$a_{11} = 1 - (u_2 + 2\hat{x} + (1+u_1)\hat{y})$ ; $a_{12} = u_3 - (1+u_1)\hat{x}$ ; $a_{13} = -\hat{x}$ ; $a_{14} = 0$ ;
$a_{21} = u_2 + u_1\hat{y}$ ; $a_{22} = u_1\hat{x} - (u_3 + u_5)$ ; $a_{23} = -u_4\hat{y}$ ; $a_{24} = 0$ ; $a_{31} = a_{32} = 0$ ;
$a_{33} = u_8\hat{x} + u_9(1-m)\hat{y} - (u_7 + u_{10})$; $a_{34} = u_{11}$ ; $a_{41} = a_{42} = 0$ ; $a_{43} = u_7 + u_9 m\hat{y}$ ; $a_{44} = -(u_{10}+u_{11}+u_{12})$.

hence, the characteristic equation of this Jacobian matrix is given by:

$[\lambda^2 + R_1\lambda + R_2][\lambda^2 + R_3\lambda + R_4] = 0$

Where:

$R_1 = (u_2 + u_3 + u_5 + 2\hat{x} + (1+u_1)\hat{y}) - (1 + u_1\hat{x})$
$R_2 = (u_1 + u_2 + 2(u_3 + u_5))\hat{x} + (u_3 + u_5(1+u_1))\hat{y} + u_2u_5 - (u_3 + u_5 + 2u_1\hat{x}^2)$
$R3 = (u_7 + 2u_{10} + u_{11} + u_{12}) - (u_8\hat{x} + u_9(1-m)\hat{y})$
$R_4 = u_7(u_{10} + u_{12}) + u_{10}(u_{10} + u_{11} + u_{12}) - (u_8(u_{10}+u_{11}+u_{12})\hat{x} + u_9\hat{y}((1-m)(u_{10} + u_{12}) + u_{11}))$.

From which, we obtain that :

$$\left.\begin{aligned} \lambda_{1\,x,y} &= -\frac{R_1}{2} \pm \frac{1}{2}\sqrt{R_1^2 - 4R_2} \\ \lambda_{1\,z,w} &= -\frac{R_3}{2} \pm \frac{1}{2}\sqrt{R_3^2 - 4R_4} \end{aligned}\right\} \quad \textbf{(4.6)}$$

Therefore all the eigenvalues have negative real part provided that the following conditions are satisfied:

$$\left.\begin{aligned} \hat{x}^2 &< \frac{(u_1+u_2+2(u_3+u_5))\hat{x} + (u_5(1+u_1) + u_3)\hat{y} + u_5(u_2-1) - u_3}{2u_1}, \\ \hat{x} &< \min\{a, b\} \end{aligned}\right\} \quad \textbf{(4.7)}$$

Where:

$$a = \frac{(u_7 + 2u_{10} + u_{11} + u_{12}) - u9(1-m)\hat{y}}{u_8}$$

$$b = \frac{u_7(u10+u12)+u_{10}(u10+u11+u12)-u_9\hat{y}\left((1-m)(u10+u12)+u_{11}\right)}{u_8(u10+u11+u12)}$$

So, the equilibrium point $E_1$ is locally asymptotically stable in the Int.$R_+^4$. However, it is (a saddle point) unstable otherwise**.**

In the following theorem, the local stability conditions of the positive equilibrium point $E_2$ are established**.**

**Theorem (3):** Assume that the positive (coexistence) equilibrium point $E_2=(x^*, y^*, z^*, w^*)$ exists in the Int.$R_+^4$, and the following conditions are satisfied:

$$p_{12}^2 < \frac{2}{3} p_{11} p_{22} \quad \textbf{(4.8)}$$

$$p_{13}^2 < \frac{2}{3} p_{11} p_{33} \quad \textbf{(4.9)}$$

$$p_{23}^2 < \frac{2}{9} p_{22} p_{33} \quad \textbf{(4.10)}$$

$$p_{24}^2 < \frac{2}{3} p_{22} p_{44} \quad \textbf{(4.11)}$$

$$p_{34}^2 < \frac{2}{3} p_{33} p_{44} \quad \textbf{(4.12)}$$

Then $E_2$ is locally asymptotically stable in the $R_+^4$.

**proof:** The Jacobian matrix $J(E_2)$ of system (**2**) is computed as follows:

$$J_2 = J(E_2) = [b_{ij}]_{4x4} \quad \textbf{(4.13)}$$

Where:

$b_{11} = 1-(u_2+2x^*+(1+u_1)y^*+z^*)$ ; $b_{12} = u_3-(1+u_1)x^*$ ; $b_{13} = -x^*$ ; $b_{14} = 0$ ;

$b_{21} = u_2+u_1y^*$ ; $b_{22} = u_1x^*-(u_3+u_5+u_4z^*)$ ; $b_{23} = -u_4y^*$ ; $b_{24} = 0$ ; $b_{31} = u_8z^*$ ;

$b_{32} = u_9(1-m)z^*$; $b_{33} = u_8x^*+u_9(1-m)y^*-(u_7+u_{10}+u_6w^*)$ ; $b_{34} = u_{11}-u_6z^*$ ;

$b_{41} = 0$ ; $b_{42} = u_9mz^*$ ; $b_{43} = u_7+u_9my^*+u_6w^*$ ; $b_{44} = u_6z^*-(u_{10}+u_{11}+u_{12})$ .

It is easy to verify that, the linearized system of system (**2**) can be written as:

$\frac{dw}{dt} = J(E_2)\,w$ , where $w = (w_1,w_2,w_3,w_4)^t$ and $w_i = x_i - x_i^*$ .

Now, consider the following function

$$V_2(w_1,w_2,w_3,w_4) = \frac{w_1^2}{2} + \frac{w_2^2}{2} + \frac{w_3^2}{2} + \frac{w_4^2}{2}$$

Clearly, $V_2: R_+^4 \to R$ is $C^1$ positive definite function. Now, by differentiating $V_2$ with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dV_2}{dt} = -p_{11}w_1^2 + p_{12}w_1w_2 - p_{22}w_2^2 - p_{33}w_3^2 + p_{13}w_1w_3 + p_{23}w_2w_3 - p_{44}w_4^2 + p_{24}w_2w_4 + p_{34}w_3w_4$$

Here:

$p_{11} = u_2+2x^*+(1+u_1)y^*+z^*-1$; $p_{12} = u_2+u_3+u_1y^*-(1+u_1)x^*$; $p_{13} = u_8z^*-x^*$ ;

$p_{22} = u_3+u_5 + u_4z^* - u_1x^*$; $p_{23} = u_9(1\text{-}m)\, z^* - u_4y^*$; $p_{33} = u_7+u_{10}+u_6w^* - (u_8x^* + u_9(1\text{-}m)\, y^*)$;
$p_{34} = u_7 + u_{11} + u_9my^* + u_6w^* - u_6z^*$; $p_{44} = (u_{10}+ u_{11}+ u_{12}) - u_6z^*$; $p_{24} = u_9mz^*$ .

According, to conditions (4.8 – 4.12) we get:

$$\frac{dv_2}{dt} \leq - \left[\frac{\sqrt{p_{11}}}{\sqrt{2}}\, w_1 - \frac{\sqrt{p_{22}}}{\sqrt{3}}\, w_2\right]^2 - \left[\frac{\sqrt{p_{11}}}{\sqrt{2}}\, w_1 - \frac{\sqrt{p_{33}}}{\sqrt{3}}\, w_3\right]^2 - \left[\frac{\sqrt{p_{22}}}{\sqrt{3}}\, w_2 - \frac{\sqrt{p_{33}}}{\sqrt{3}}\, w_3\right]^2$$

$$- \left[\frac{\sqrt{p_{22}}}{\sqrt{3}}\, w_2 - \frac{\sqrt{p_{44}}}{\sqrt{2}}\, w_4\right]^2 - \left[\frac{\sqrt{p_{33}}}{\sqrt{3}}\, w_3 - \frac{\sqrt{p_{44}}}{\sqrt{2}}\, w_4\right]^2$$

Then $\frac{dv_2}{dt} < 0$ under the given conditions and hence $V_2$ is strictly Lyapunov function. Thus $E_2$ is a locally asymptotically stable in the $R_+^4$ . And hence the proof is complete. ■

**5. Global stability analysis of system (2):**

In this section the global stability analysis of the equilibrium points, which are locally asymptotically stable of system (**2**) is studied analytically by use the suitable of Lyapunov function as shown in the following theorems.

**Theorem (4):** Assume that the vanishing equilibrium point $E_0$ of system (**2**) is locally asymptotically stable in the $R_+^4$, and the following condition is satisfied:

$$u_4 > \frac{u_9}{u_8} \qquad \textbf{(5.1)}$$

Then $E_0$ is globally asymptotically stable on the region $\Omega \subset R_+^4$ , where $\Omega = \{ (x,y,z,w) : x > 1 \}$.

**proof:** Consider the following function:

$$V_0\,(x,y,z,w) = c_1\, x + c_2 y + c_3\, z + c_4\, w$$

Clearly, $V_0$: $R_+^4 \to R$ is $C^1$ positive definite function, where $c_i$; $i = 1,2,3,4$ are positive constants to be determined. Now, by differentiating $V_0$ with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dv_0}{dt} = - c_1\, x^2 +\big(c_1(1\text{-}u_2) + c_2u_2\big)\, x -\big(c_1(1+u_1) - c_2u_1\big)\, xy -\big(c_1 - c_3u_8\big)\, xz +\big(c_1u_3 - c_2(u_3+ u_5\,)\big)\, y\, \big(c_2u_4 - c_3u_9(1\text{-}m) - c_4u_9m\big)yz - \big(c_3 - c_4\big)\, u_6zw -\big(c_3(\, u_7+u_{10}) - c_4u_7\big)z+\big(\, c_3u_{11} - c_1(\, u_{10}+u_{11}+u_{12}\,)\big)\, w$$

So, by choosing the positive constants as:

$c_1 = c_2 = 1$ and $c_3 = c_4 = \dfrac{1}{u_8}$ , it is obtain that:

$$\frac{dv_0}{dt} = - x^2 + x - xy - u_5y - \Big(\, u_4 - \frac{u_9}{u_8}\,\Big)\, yz - \Big(\,\frac{u_{10}}{u_8}\,\Big)\, z - \Big(\,\frac{u_{10}+u_{12}}{u_8}\,\Big)\, w$$

Therefore, according to condition **(5.1)** we obtain that: $\frac{dv_0}{dt} \leq x\,(1 - x)$, then $\frac{dv_0}{dt} < 0$, when $x > 1$ and hence $V_0$ is strictly Lyapunov function.

Therefore, $E_0$ is a globally asymptotically stable on the region $\Omega \subset R_+^4$ , and the proof is complete. ■

According to the above theorem its easy to concludes that, the basin of attraction of the vanishing equilibrium point is $B(E_0) = \Omega = \{(x, y, z, w) \in R_+^4 : x > 1\}$ .

**Theorem (5):** Assume that the predator free equilibrium point $E_1$ of system (**2**) is locally asymptotically stable in the $R_+^4$, in addition to condition (**5.1**) the following conditions are satisfied:

$$\Big(-1 + \frac{u_3}{x} + \frac{u_2}{y}\,\Big)^2 < 4\,\Big(\,1 + \frac{u_3\,\hat{y}}{x\,\hat{x}}\,\Big)\Big(\,\frac{u_2\,\hat{x}}{y\,\hat{y}}\,\Big) \qquad \textbf{(5. 2)}$$

$$\hat{x} + u_4\,\hat{y} < \frac{u_{10}}{u_8} \qquad \textbf{(5. 3)}$$

Then $E_1$ is globally asymptotically stable in the $R_+^4$ .

**proof:** Consider the following function:

$$V_1(x,y,z,w) = c_1\left(x - \hat{x} - \hat{x}\ln\frac{x}{\hat{x}}\right) + c_2\left(y - \hat{y} - \ln\frac{y}{\hat{y}}\right) + c_3 z + c_4 w$$

Clearly, $V_1: R_+^4 \to R$ is $C^1$ positive definite function, where $c_i$ ; $i = 1,2,3,4$ are positive constants to be determined. Now, by differentiating $V_1$ with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dv_1}{dt} = -c_1\left(1 + \frac{u_3\hat{y}}{x\hat{x}}\right)(x-\hat{x})^2 - c_2\left(\frac{u_2\hat{x}}{y\hat{y}}\right)(y-\hat{y})^2 + \left(-c_1\left(1 + u_1 - \frac{u_3}{x}\right) + c_2\left(u_1 + \frac{u_2}{y}\right)\right)(x-\hat{x})(y-\hat{y}) + (c_1\hat{x} + c_2u_4\hat{y} - c_3(u_7+u_{10}) + c_4u_7)z - (c_1 - c_3u_8)xz - (c_2u_4 - c_3u_9(1-m) - c_4u_9m)yz + (c_3u_{11} - c_4(u_{10}+u_{11}+u_{12}))w - (c_3 - c_4)u_6zw.$$

So, by choosing the positive constants as:

$c_1 = c_2 = 1$ and $c_3 = c_4 = \frac{1}{u_8}$ , it is obtain that:

$$\frac{dv_1}{dt} = -q_{11}(x-\hat{x})^2 + q_{12}(x-\hat{x})(y-\hat{y}) - q_{22}(y-\hat{y})^2 - q_{23}yz + B_1 z - B_2 w$$

Here:

$$q_{11} = \left(1 + \frac{u_3\hat{y}}{x\hat{x}}\right);\ q_{12} = \left(-1 + \frac{u_3}{x} + \frac{u_2}{y}\right);\ q_{22} = \left(\frac{u_2\hat{x}}{y\hat{y}}\right);\ q_{23} = \left(u_4 - \frac{u_9}{u_8}\right);\ B_1 = \left(\hat{x} + u_4\hat{y} - \frac{u_{10}}{u_8}\right);$$
$$B_2 = \left(\frac{u_{10}+u_{12}}{u_8}\right).$$

Note that, $B_1$ is negative provided that condition (**5.3**) is satisfied, while $q_{23}$ is positive provided that condition (**5.1**) hold. Therefore, by using the given conditions. We obtain that:

$$\frac{dv_1}{dt} \le -\left(\sqrt{q_{11}}(x-\hat{x}) - \sqrt{q_{22}}(y-\hat{y})\right)^2$$

Then $\frac{dv_1}{dt} < 0$ under the given conditions and hence $V_1$ is strictly Lyapunov function. Thus $E_1$ is globally asymptotically stable in the $R_+^4$ . And hence the proof is complete. ■

In the following theorem, the conditions of the globally asymptotically stable of the positive equilibrium point $E_2$ are established.

**Theorem (6):** Assume that the positive (coexistence) equilibrium point $E_2$ of system (**2**) is locally asymptotically stable in the $R_+^4$, and the following conditions are satisfied:

$$p_{12}^2 < p_{11}\,p_{22} \quad \textbf{(5.4)}$$

$$p_{13}^2 < \frac{2}{3}\,p_{11}\,p_{33} \quad \textbf{(5.5)}$$

$$p_{23}^2 < \frac{2}{3}\,p_{22}\,p_{33} \quad \textbf{(5.6)}$$

$$p_{34}^2 < \frac{4}{3}\,p_{33}\,p_{44} \quad \textbf{(5.7)}$$

$$m < \frac{u_4}{u_9} \quad \textbf{(5.8)}$$

$$R_1 z + R_2 < \mu_1 + \mu_2 + \mu_3 + \mu_4 \quad \textbf{(5.9)}$$

Where:

$p_{11} = \left(1 + \frac{u_3 y^*}{x x^*}\right)$; $p_{12} = \left(-1 + \frac{u_3}{x} + \frac{u_2}{y}\right)$; $p_{22} = \left(\frac{u_2 x^*}{y y^*}\right)$; $p_{33} = \left(\frac{u_{11} w^*}{z z^*}\right)$; $p_{13} = u_8$ ; $p_{23} = u_9(1-m)$; $p_{44} = \left(\frac{z^*(u_7 + u_9 m y^*)}{w w^*}\right)$; $p_{34} = \left(\frac{u_{11}}{z} + \frac{u_7}{w}\right)$; $R_1 = (x^* + u_4 y^*)$; $R_2 = \left(\frac{u_9\, m}{w}\right.$

$y^*z^*w^*$ ) and $\mu_1=\left[\frac{\sqrt{p_{11}}}{\sqrt{2}}(x-x*)-\frac{\sqrt{p_{22}}}{\sqrt{2}}(y-y*)\right]^2$; $\mu_2=\left[\frac{\sqrt{p_{11}}}{\sqrt{2}}(x-x*)-\frac{\sqrt{p_{33}}}{\sqrt{3}}(z-z*)\right]^2$;

$\mu_3=\left[\frac{\sqrt{p_{22}}}{\sqrt{2}}(y-y*)-\frac{\sqrt{p_{33}}}{\sqrt{3}}(z-z*)\right]^2$; $\mu_4=\left[\frac{\sqrt{p_{33}}}{\sqrt{3}}(z-z*)-\sqrt{p_{44}}\,(w-w*)\right]^2$;

Then, $E_2$ is globally asymptotically stable in the $R_+^4$ .

**proof:** Consider the following function:

$$V_2(x,y,z,w) = k_1\left(x - x^* - x^*\ln\frac{x}{x^*}\right) + k_2\left(y - y^* - y^*\ln\frac{y}{y^*}\right) + k_3\left(z - z^* - z^*\ln\frac{z}{z^*}\right) + k_4\left(w - w - w^*\ln\frac{w}{w^*}\right)$$

Clearly, $V_2\colon R_+^4 \to R$ is $C^1$ positive definite function, where $k_i$ ; i = 1,2,3,4 are positive constants to be determined. Now, by differentiating $V_2$ with respect to time t and doing some algebraic manipulation, gives that:

$$\frac{dv_2}{dt} = -k_1\left(1+\frac{u_3\,y^*}{x\,x^*}\right)(x-x^*)^2 + \left(-k_1\left(1+u_1-\frac{u_3}{x}\right)+k_2\left(u_1+\frac{u_2}{y}\right)\right)(x-x^*)(y-y^*) - k_2\left(\frac{u_2\,x^*}{y\,y^*}\right)(y-y^*)^2 - k_3\left(\frac{u_{11}\,w^*}{z\,z^*}\right)(z-z^*)^2 + k_3u_8(x-x^*)(z-z^*) + k_3u_9(1\text{-}m)(y-y^*)(z-z^*) - k_4\left(\frac{z^*(u_7+u_9my^*)}{w\,w^*}\right)(w-w^*)^2 + \left(-k_3\left(u_6-\frac{u_{11}}{z}\right)+k_4\left(u_6-\frac{u_7}{w}\right)\right)(z-z^*)(w-w^*) + (k_1x^*+k_2u_4y^*)z + k_4\left(\frac{u_9\,m}{w}y^*z^*w^*\right) - (k_2u_4-k_4u_9m)\,yz - k_1xz - k_4\frac{u_9\,m}{w}w^*yz - k_4u_9m\,y^*z^*$$

So, by choosing the positive constants as:

$k_1 = k_2 = k_3 = k_4 = 1$ , it is obtain that:

$$\frac{dv_2}{dt} = -p_{11}(x-x^*)^2 + p_{12}(x-x^*)(y-y^*) - p_{22}(y-y^*)^2 - p_{33}(z-z^*)^2 + p_{13}(x-x^*)(z-z^*) + p_{23}(y-y^*)(z-z^*) - p_{44}(w-w^*)^2 + p_{34}(z-z^*)(w-w^*) + R_1z + R_2$$

Therefore, according to the conditions (**5.5 - 5.8**) we obtain that:

$$\frac{dv_2}{dt} \le -(\mu_1+\mu_2+\mu_3+\mu_4) + R_1z + R_2$$

Then $\frac{dv_2}{dt} < 0$ under the condition (**5.9**) and hence $V_2$ is strictly Lyapunov function. Thus $E_2$ is globally asymptotically stable in the $R_+^4$ . And hence the proof is complete. ■

## 6. Numerical analysis of system (2):

In this section the dynamical behavior of system (**2**) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (**2**) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters that satisfies stability conditions of the positive equilibrium point, system (**2**) has a globally asymptotically stable positive equilibrium point as shown in Fig. (**1**).

Note that, from now onward the red , blue , sky blue and green colors are used to describing the trajectories of the susceptible prey x , infected prey y , susceptible predator z and infected predator w respectively .

$$\left.\begin{array}{l} u_1 = 0.5,\ u_2 = 0.1,\ u_3 = 0.1,\ u_4 = 0.5,\ u_5 = 0.5,\ u_6 = 0.3,\ u_7 = 0.2,\ u_8 = 0.5, \\ u_9 = 0.5,\ u_{10} = 0.1,\ u_{11} = 0.3,\ u_{12} = 0.2,\ m = 0.6\ . \end{array}\right\} \quad \textbf{(6.1)}$$

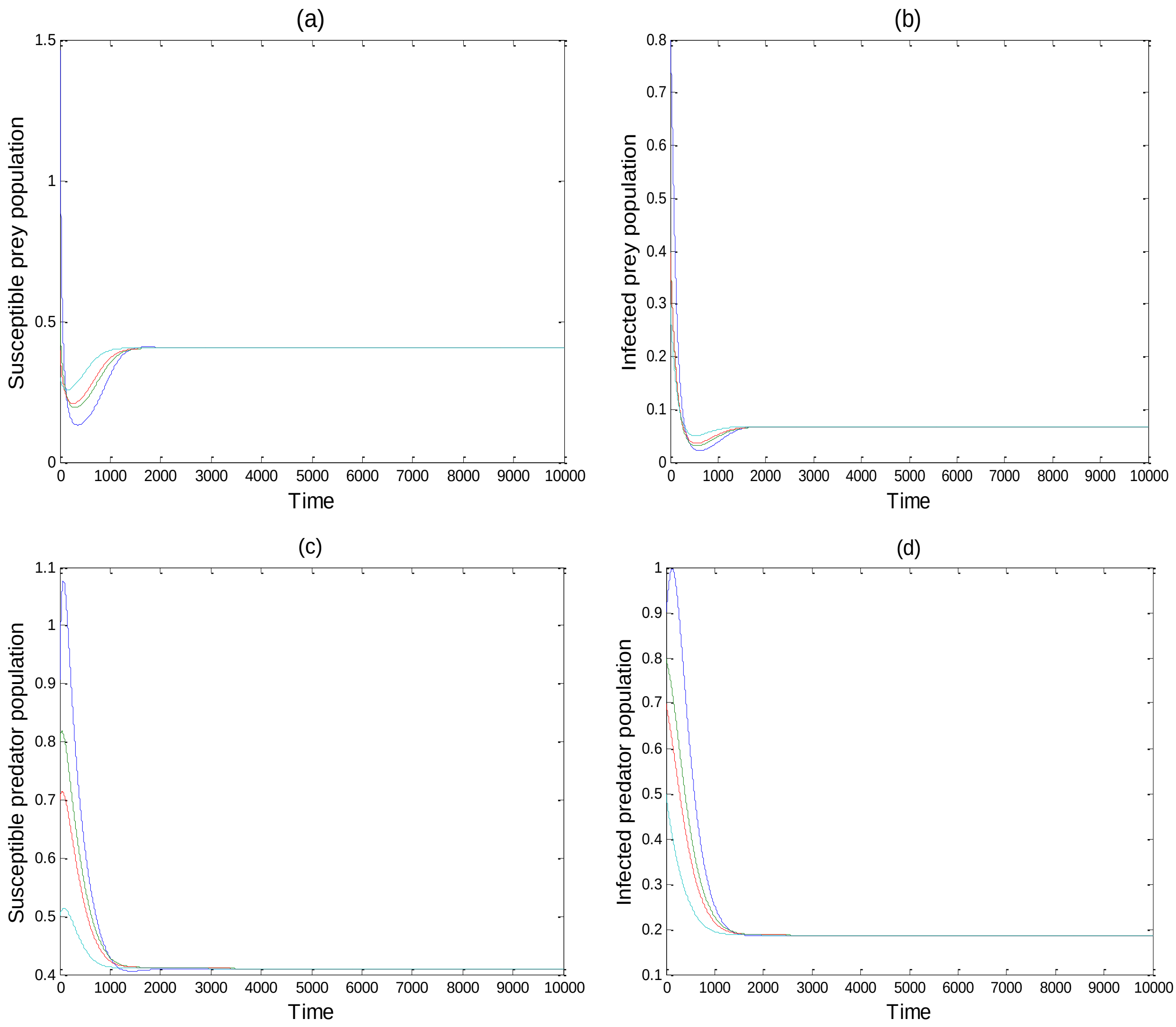


**Fig (1): Time series of the solution of system (2) that started from four different initial points (1.5, 0.8, 0.9, 0.9) , (0.5, 0.4, 0.8, 0.9) , (0.4, 0.4, 0.7, 0.7) and (0.3, 0.3, 0.5, 0.5) for the data given by Eq. (6.1). (a) trajectories of *x* as a function of time, (b) trajectories of *y* as a function of time, (c) trajectories of *z* as a function of time, (d) trajectories of *w* as a function of time.**

Clearly, figure (**1**) shows that system (**2**) has a globally asymptotically stable as the solution of system (**2**) approaches asymptotically to the positive equilibrium point $E_2$= (0.41, 0.4, 0.18, 0.06) starting from four different initial points and this is confirming our obtained analytical results**.**

Now in order to discuss the effect of the parameters values of system (**2**) on the dynamical behavior of the system, the system is solved numerically for the data given in Eq. (**6.1**) with varying one parameter each time. It is observed that varying the parameters values $\mathbf{u_i}$ ; i = 1,3,4,5,6,9,11,12 and **m**, do not have any effect on the dynamical behavior of system (**2**) and the solution of the system still approaches to positive equilibrium point $E_2= (x^*,y^*,z^*,w^*)$. However, varying the infection rates of susceptible prey and predator $\mathbf{u_2}$ and $\mathbf{u_7}$, respectively keeping other parameters fixed as given in equation (**6.1**), leads to extinction in predator species as shown in Fig. (**2**), for the parameters given by Eq. (**6.1**) with $\mathbf{u_2} = \mathbf{u_7} = 0.001$ **.**

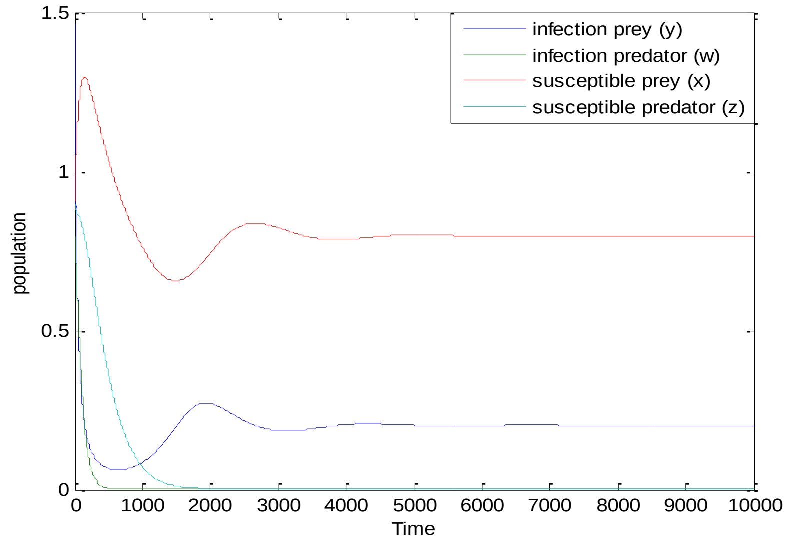


**Fig. (2): Time series of the solution of system (2) for the data given by Eq. (6.1) with $u_2 = u_7 = 0.001$.**
From the above figure it is clear that as the predation process of predator decrease, the trajectory of the system (**2**) approaches asymptotically to the predator free equilibrium point $E_1$= ( 0.79 , 0.2 , 0 , 0 ). Also when $0.001 < \boldsymbol{u_2} \leq 0.0099$ and $0.001 < \boldsymbol{u_7} < 0.0036$, then the solution of the system still approaches to the predator free equilibrium point $E_1 = (x^*, y^*, 0, 0)$.
Finally, the dynamical behavior at the vanishing equilibrium point $E_0$ is investigated by choosing the infection rate of susceptible prey $1.2 \leq \mathbf{u_2} < 34.94$ and keeping other parameters fixed as given in Eq.(**6.1**), as shown in Fig.(**3**).

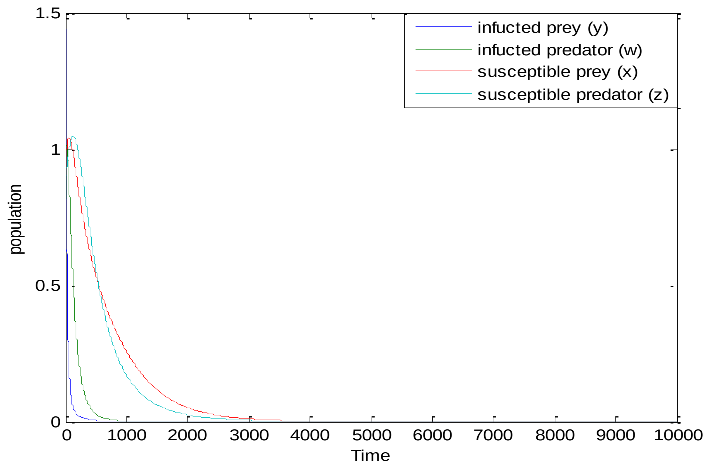


**Fig. (3):Time series of the solution of system (2) for the data given by Eq. (6.1) with $1.2 \leq u_2 < 34.94$.**

Obviously, Fig.(**3**) shows clearly the convergence of the solution of system (**2**) to the vanishing equilibrium point $E_0 = (0,0,0,0)$ when the parameter increase up to a specific values. Clearly the used values in Fig. (**3**) satisfy the stability conditions of the vanishing equilibrium point **.**

**7. Conclusion and Discussion**

In this paper, we proposed and analyzed an eco-epidemiological mathematical model that described the dynamical behavior of prey -predator model with Lotka -Volterra type of functional response and linear incidence rate for the disease in prey and predator respectively. It is assumed that the disease is transmitted from a prey to predator during the predation process, also the disease transmitted within the same species by two ways: from an external source as well as through contact with the infected individuals. The model included four non-linear autonomous differential equations that describe the dynamics of four different population namely susceptible prey ***x***, infected prey ***y***, susceptible predator ***z*** and infected predator ***w*** .The boundedness of the system (**2**) has been discussed. The dynamical behavior of system (**2**) has been investigated locally as well as globally. Further, it is observed that the vanishing equilibrium point (**$E_0$**) always exist, and it is locally asymptotically stable point if and only if condition (4.4) hold, in addition to that it is globally on the region $\Omega \subset R^4_+$ if the condition (5.1) hold. The predator free equilibrium point (**$E_1$**) exists under the conditions (3.3-3.4), it is locally asymptotically stable point if and only if the condition (4.7) hold as well as it is globally if the conditions (5.1-5.3) hold. The positive equilibrium point (**$E_2$**) of system (**2**) exists provided that the conditions (3.14-3.17) are hold and the isoclinic $g_1(x, y) = 0$ intersect the $x$ -axis at the positive value namely $x_1$. It is locally asymptotically stable point if and only if conditions (4.8-4.12) hold, in addition it is globally if the conditions (5.5-5.9) are hold. To understand the effect of varying each parameter on the global dynamics of system (**2**) and to confirm our above analytical results, system (**2**) has been solved numerically and the following results are obtained:

1- The system (**2**) does not have periodic dynamic.
2- For the set of hypothetical parameters values given Eq. (**6.1**), system (**2**) approaches asymptotically to a globally asymptotically stable point $E_2 = (0.41, 0.4, 0.18, 0.06)$. Further, with varying one parameter each time, it is observed that varying the parameters values $\mathbf{u_i}$ ,$i = 1,3,4,5,6,9,11,12$ and $\mathbf{m}$, do not have any effect on the dynamical behavior of system (**2**) and the solution of the system still approaches to positive equilibrium point $E_2 = (x^*, y^*, z^*, w^*)$ .
3- As the infection rates parameters $\mathbf{u_2}$ and $\mathbf{u_7}$ for susceptible prey and predator in system (**2**) decreases keeping other parameters fixed as in Eq.(**6.1**),then the susceptible and infected predator will face extinction and the solution of the system (**2**) still stable and approaches asymptotically to the predator free equilibrium point $E_1 = (x^*, y^*, 0, 0)$ **.**
4- As the infection rate of susceptible prey $\mathbf{u_2}$ increase with keeping other parameters as in Eq.(**6.1**) , then the solution of system (**2**) approaches asymptotically to the vanishing equilibrium point $E_0 = (0,0,0,0)$ **.**

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