Application of Homotopy Perturbation Method to Nonlinear System of PDEs

Jamshad Ahmad¹ and GhulamMohiuddin²
¹Department of Mathematics, Faculty of Sciences, University of Gujrat, Pakistan
²Department of Mathematics, NCBA&E, Gujrat (Campus), Pakistan

Corresponding Author Email: jamshadahmadm@gmail.com

Abstract: In this study, Homotopy Perturbation Method (HPM) is used to obtain the analytically exact solution of linear and nonlinear systems of partial differential equations (PDEs). The efficiency and accuracy of HPM are demonstrated through several test examples. HPM yields solutions in convergent series forms with easily computable terms. Generally, the closed form of the exact solution is obtained without any noise terms. Keywords: Homotopy Perturbation Method, Linear and Nonlinear system of PDEs.

1. Introduction

The nonlinear phenomena played a very significant role in the field of applied Mathematics and mathematical Physics. It is known that phenomena coming from applied physics and engineering for example propagation of waves and shallow water waves can be modeled by systems of linear and nonlinear PDEs. To find accurate, a reliable and efficient method to solve nonlinear system of PDEs is the interest of most of the researchers now a day. Adomian Decomposition Method (ADM) is used to tackle the systems of PDEs which become more difficult when finding the Adomian Polynomials [1-3,8]. Variational Iteration Method (VIM) is also helpful for the approximate solutions of the systems of PDEs [4-5,11]. In recent years, a lot of attention has been given to study the homotopy perturbation method (HPM) by different researchers such as to solve the nonlinear system of PDEs [7-12]. HPM deforms a difficult problem into a set of problems which are easy for solving without any need to transform nonlinear terms. In this article, HPM is applied to some linear and nonlinear system of PDEs to find the exact solutions of the systems. Here HPM is also applied to a two dimensional system of PDEs proving the accuracy, efficiency and reliability of the proposed method.

2. Basic Idea of Homotopy Perturbation Method (HPM)

Consider system of nonlinear differential equations,

\[ L_j(u,v) + N_j(u,v) = f_j(q) \quad q \in \Omega , \]  

with the boundary conditions of,

\[ B_1(u,\frac{\partial u}{\partial n}) = 0, \quad B_2(v,\frac{\partial v}{\partial n}) = 0, \]

where \( L_j \) are linear operators and \( N_j \) are nonlinear operators. The He's homotopy perturbation technique defines the homotopy,

\[ U(q, p) : \Omega \times [0,1] \to R, \]

and

\[ V(q, p) : \Omega \times [0,1] \to R, \]

which satisfies

\[ H_j(U,V,p) = (1-p)[L_j(U,V) - L_j(v_0,u_0)] + p[L_j(U,V) + N_j(U,V) - f_j(q)] = 0 \]  

(2)

Where \( p \in [0,1] \) is an imbedding parameter, \( u_0, v_0 \) are initial approximations which satisfy the boundary conditions. The basic assumption is that the solution of Eq. (2) can be expressed as a power series in \( p \),

\[ U = \sum_{n=0}^{\infty} p^n U_n = U_0 + pU_1 + p^2U_2 + ... \]

\[ V = \sum_{n=0}^{\infty} p^n V_n = V_0 + pV_1 + p^2V_2 + ... \]

The approximate solution of Eq. (1) can be obtained as,
\[ u = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n U_n = U_0 + U_1 + U_2 + \ldots \]
\[ v = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n V_n = V_0 + V_1 + V_2 + \ldots \]

3. Numerical Applications

To demonstrate the effectiveness of the method, we consider the following problems with given initial condition.

**Example 3.1** Consider the following linear System of PDEs
\[ u_x - v_y = 0, \]
\[ v_x + u_y = 0, \]
with the initial conditions of,
\[ u(0, y) = \cos y, \]
\[ v(0, y) = \sin y. \]

Taking inverse operator, we have
\[ u(x, y) = \cos y + L^{-1}(v_y), \]
\[ v(x, y) = \sin y + L^{-1}(u). \]

Let solution of the Eq. (3) be,
\[ u(x, y) = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots \]
\[ v(x, y) = \sum_{n=0}^{\infty} p^n v_n = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \]

Putting in Eq. (4),
\[ u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots = \cos y + pL^{-1}\left[ \frac{\partial}{\partial y} (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) \right] \]
\[ v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots = \sin y - pL^{-1}\left[ \frac{\partial}{\partial y} (u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots) \right]. \]

Equating powers of “p”,
\[ p^0, \quad u_0(x, y) = \cos y \]
\[ v_0(x, y) = \sin y \]
\[ p^1, \quad u_1(x, y) = L^{-1}\left( \frac{\partial v_0}{\partial y} \right) \]
\[ v_1(x, y) = -L^{-1}\left( \frac{\partial u_0}{\partial y} \right) \]
\[ u_2(x, y) = x \cos y \]
\[ v_2(x, y) = x \sin y \]
\[ p^2, \quad u_2(x, y) = L^{-1} \left( \frac{\partial v_1}{\partial y} \right) \]
\[ v_2(x, y) = -L^{-1} \left( \frac{\partial u_1}{\partial y} \right) \]
\[ u_2(x, y) = \frac{x^2}{2!} (\cos y) \]
\[ v_2(x, y) = \frac{x^2}{2!} (\sin y) \]
\[ p^3, \quad u_3(x, y) = L^{-1} \left( \frac{\partial v_2}{\partial y} \right) \]
\[ v_3(x, y) = -L^{-1} \left( \frac{\partial u_2}{\partial y} \right) \]
\[ u_3(x, y) = \frac{x^3}{3!} (\cos y) \]
\[ v_3(x, y) = \frac{x^3}{3!} (\sin y) . \]

So, the solution of the system of PDEs (3) is given by,
\[ u(x, y) = \lim_{n \to \infty} \sum_{n=0}^{\infty} p^n u_n = u_0 + u_1 + u_2 + u_3 + ... \]
\[ v(x, y) = \lim_{n \to \infty} \sum_{n=0}^{\infty} p^n v_n = v_0 + v_1 + v_2 + v_3 + ... \]

Consequently, we have
\[ u(x, y) = \cos y + x \cos y + \frac{x^2}{2!} \cos y + \frac{x^3}{3!} \cos y + ... \]
\[ u(x, y) = \cos y [1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...] \]
\[ u(x, y) = e^x \cos y \]
\[ v(x, y) = \sin y + x \sin y + \frac{x^2}{2!} \sin y + \frac{x^3}{3!} \sin y + ... \]
\[ v(x, y) = \sin y [1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...] \]
\[ v(x, y) = e^x \sin y . \]

Example 3.2 Consider another example of linear system of PDEs
\[ u_t - v_x + u + v = 0 \]
\[ v_t - u_x + u + v = 0 , \]
with the initial conditions of,
\[ u(x,0) = \sinh x \]
\[ v(x,0) = \cosh x . \]

Taking inverse operator, we have
\[ u(x,t) = \sinh x + L^{-1} (v_x - u - v) \]
\[ v(x,t) = \cosh x + L^{-1} (u_x - u - v) \]
\[ u(x, t) = \sum_{n=0}^{\infty} p^nu_n = u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots \]
\[ v(x, t) = \sum_{n=0}^{\infty} p^nv_n = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots \]

Putting these values in Eq. (6),
\[ u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots = \sinh x + pL^{-1} \left[ \frac{\partial}{\partial x} (v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots) - (u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots) \right] \]
\[ v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots = \cosh x + pL^{-1} \left[ \frac{\partial}{\partial x} (u_0 + pu_1 + p^2u_2 + p^3u_3 + \ldots) - (v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots) \right] \]

Equating powers of “p”,
\[ p^0, \quad u_0(x, t) = \sinh x \]
\[ v_0(x, t) = \cosh x \]
\[ p^1, \quad u_1(x, t) = L^{-1} \left( \frac{\partial v_0}{\partial x} - u_0 - v_0 \right) \]
\[ v_1(x, t) = L^{-1} \left( \frac{\partial u_0}{\partial x} - u_0 - v_0 \right) \]
\[ u_1(x, t) = -t(\cosh x) \]
\[ v_1(x, t) = -t(\sinh x) \]
\[ p^2, \quad u_2(x, t) = L^{-1} \left( \frac{\partial v_1}{\partial x} - u_1 - v_1 \right) \]
\[ v_2(x, t) = L^{-1} \left( \frac{\partial u_1}{\partial x} - u_1 - v_1 \right) \]
\[ u_2(x, t) = \frac{t^2}{2!} \sinh x \]
\[ v_2(x, t) = \frac{t^2}{2!} \cosh x \]
\[ p^3, \quad u_3(x, t) = L^{-1} \left( \frac{\partial v_2}{\partial x} - u_2 - v_2 \right) \]
\[ v_3(x, t) = L^{-1} \left( \frac{\partial u_2}{\partial x} - u_2 - v_2 \right) \]
\[ u_3(x, t) = -\frac{t^3}{3!} \cosh x \]
\[ v_3(x, t) = -\frac{t^3}{3!} \sinh x \]
\[ p^4, \quad u_4(x, t) = L^{-1} \left( \frac{\partial v_3}{\partial x} - u_3 - v_3 \right) \]
\[ v_4(x, t) = L^{-1} \left( \frac{\partial u_3}{\partial x} - u_3 - v_3 \right) \]
\[ u_4(x,t) = \frac{t^4}{4!} \sinh x \]

\[ v_4(x,t) = \frac{t^4}{4!} \cosh x . \]

So the solution of the system of Eq. (5) is,

\[ u(x,t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n = u_0 + u_1 + u_2 + u_3 + \ldots \]

\[ v(x,t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n v_n = v_0 + v_1 + v_2 + v_3 + \ldots \]

\[ u(x,t) = \sinh x - t(\cosh x) + \frac{t^2}{2!}(\sinh x) - \frac{t^3}{3!}(\cosh x) + \frac{t^4}{4!}(\sinh x) + \ldots \]

\[ v(x,t) = \sinh(x - t) \]

and,

\[ v(x,t) = \cosh x - t(\sinh x) + \frac{t^2}{2!}(\cosh x) - \frac{t^3}{3!}(\sinh x) + \frac{t^4}{4!}(\cosh x) + \ldots \]

\[ v(x,t) = \cosh(x + t) . \]

**Example 3.3** Now consider the nonlinear system of PDEs

\[ v_t = vv_x + wv_y \]

\[ w_t = ww_x + vw_y , \]

with the initial conditions of,

\[ v(x, y, 0) = w(x, y, 0) = x + y . \]

Taking inverse operator, we have

\[ v(x, x, t) = x + y + L^{-1}(vv_x + wv_y) \]

\[ w(x, x, t) = x + y + L^{-1}(ww_x + vw_y) . \]

Let solution of the Eq. (8) be,

\[ v(x, y, t) = \sum_{n=0}^{\infty} p^n v_n = v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots \]

\[ w(x, y, t) = \sum_{n=0}^{\infty} p^n w_n = w_0 + pw_1 + p^2w_2 + p^3w_3 + \ldots \]

Putting these values in Eq. (8),

\[ v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots = x + y + pL^{-1}[(v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots) + (w_0 + pw_1 + p^2w_2 + p^3w_3 + \ldots)] \frac{\partial}{\partial x} \]

\[ \frac{\partial}{\partial x} (v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots) \]

\[ w_0 + pw_1 + p^2w_2 + p^3w_3 + \ldots = x + y + pL^{-1}[(w_0 + pw_1 + p^2w_2 + p^3w_3 + \ldots) + (v_0 + pv_1 + p^2v_2 + p^3v_3 + \ldots)] \frac{\partial}{\partial x} \]

Equating powers of “p”,

139
$p^0$, \quad v_0(x, y, t) = x + y \\
\quad w_0(x, y, t) = x + y \\

$P^1$, \quad v_1(x, y, t) = L^{-1}[v_0 \frac{\partial v_0}{\partial x} + \frac{\partial v_0}{\partial y}] \\
\quad w_1(x, y, t) = L^{-1}[w_0 \frac{\partial w_0}{\partial x} + \frac{\partial w_0}{\partial y}] \\
\quad v_1(x, y, t) = 2(x + y)t \\
\quad w_1(x, y, t) = 2(x + y)t \\

$P^2$, \quad v_2(x, y, t) = L^{-1}[(v_0 \frac{\partial v_0}{\partial x} + v_1 \frac{\partial v_0}{\partial y}) + (w_0 \frac{\partial w_0}{\partial x} + w_1 \frac{\partial w_0}{\partial y})] \\
\quad w_2(x, y, t) = L^{-1}[(w_0 \frac{\partial w_0}{\partial x} + w_1 \frac{\partial w_0}{\partial y}) + (v_0 \frac{\partial v_0}{\partial y} + v_1 \frac{\partial v_0}{\partial y})] \\
\quad v_2(x, y, t) = 4(x + y)t^2 \\
\quad w_2(x, y, t) = 4(x + y)t^2 . \\

So solution of the Eq. (7) is given by, \\
\quad v(x, y, t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n v_n = v_0 + v_1 + v_2 + v_3 + ... \\
\quad w(x, y, t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n w_n = w_0 + w_1 + w_2 + w_3 + ... \\

Putting values, we have, \\
\quad v(x, y, t) = (x + y) + 2(x + y)t + 4(x + y)t^2 + ... \\
\quad v(x, y, t) = (x + y)[1 + 2t + 4t^2 + ...] \\
\quad w(x, y, t) = (x + y) + 2(x + y)t + 4(x + y)t^2 + ... \\
\quad w(x, y, t) = (x + y)[1 + 2t + 4t^2 + ...] . \\

**Example 3.4** Consider another system of nonlinear PDEs \\
\quad u_x + v_x w_y - v_y w_x = -u \\
\quad v_x + w_x u_y + u_y w_x = v \\
\quad w_x + u_x v_y + u_y v_x = w , \\

with the initial condition of, \\
\quad u(x, y, 0) = e^{x+y}, \quad v(x, y, 0) = e^{x-y}, \quad w(x, y, 0) = e^{y-x} . \\

Taking inverse operator, we have \\
\quad u(x, y, 0) = e^{x+y} - L^{-1}(u - v_x w_y + v_y w_x) \\
\quad v(x, y, t) = e^{x-y} + L^{-1}(v - w_x u_y - u_y w_x) \\
\quad w(x, y, t) = e^{y-x} + L^{-1}(w - u_x v_y - u_y v_x) . \\

Let solution of the Eq. (10) be,
\[ u(x, y, t) = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots \]

\[ v(x, y, t) = \sum_{n=0}^{\infty} p^n v_n = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \]

\[ w(x, y, t) = \sum_{n=0}^{\infty} p^n w_n = w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \ldots \]

Putting these values in Eq. (10),

\[ u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots = e^{x+y} - L^{-1}[(u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots) - \frac{\partial}{\partial x} (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) \frac{\partial}{\partial y} (w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \ldots)] \]

\[ v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots = e^{x+y} - L^{-1}[(v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) - \frac{\partial}{\partial x} (w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \ldots) \frac{\partial}{\partial y} (u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots)] \]

\[ w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \ldots = e^{x+y} - L^{-1}[(w_0 + pw_1 + p^2 w_2 + p^3 w_3 + \ldots) - \frac{\partial}{\partial x} (u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots) \frac{\partial}{\partial y} (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots)] \]

Equating powers of “\( p \)”:

For \( p^0 \),

\[ u_0(x, y, t) = e^{x+y} \]

\[ v_0(x, y, t) = e^{x+y} \]

\[ w_0(x, y, t) = e^{x+y} \]

For \( p^1 \),

\[ u_1(x, y, t) = -u_0 \begin{bmatrix} \frac{\partial v_0}{\partial x} \\ \frac{\partial w_0}{\partial y} \end{bmatrix} + \begin{bmatrix} \frac{\partial v_0}{\partial y} \\ \frac{\partial w_0}{\partial x} \end{bmatrix} \]

\[ v_1(x, y, t) = v_0 - \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial w_0}{\partial y} \end{bmatrix} - \begin{bmatrix} \frac{\partial u_0}{\partial y} \\ \frac{\partial w_0}{\partial x} \end{bmatrix} \]

\[ w_1(x, y, t) = w_0 - \begin{bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \end{bmatrix} - \begin{bmatrix} \frac{\partial u_0}{\partial y} \\ \frac{\partial v_0}{\partial x} \end{bmatrix} \]

\[ u_1(x, y, t) = -te^{x+y} \]

\[ v_1(x, y, t) = te^{x+y} \]

\[ w_1(x, y, t) = te^{y-x} \]
\[ p^2, \quad u_2(x, y, t) = -u_1 - \left[ \frac{\partial v_0}{\partial x} \frac{\partial w_1}{\partial y} + \frac{\partial v_1}{\partial x} \frac{\partial w_0}{\partial y} \right] + \left[ \frac{\partial v_1}{\partial y} \frac{\partial w_0}{\partial x} + \frac{\partial v_0}{\partial y} \frac{\partial w_1}{\partial x} \right] \]

\[ v_2(x, y, t) = v_1 = - \left[ \frac{\partial w_0}{\partial x} \frac{\partial u_0}{\partial y} + \frac{\partial w_1}{\partial x} \frac{\partial u_1}{\partial y} \right] - \left[ \frac{\partial u_1}{\partial y} \frac{\partial w_0}{\partial x} + \frac{\partial u_0}{\partial y} \frac{\partial w_1}{\partial x} \right] \]

\[ w_2(x, y, t) = w_1 = - \left[ \frac{\partial u_0}{\partial x} \frac{\partial v_1}{\partial y} + \frac{\partial u_1}{\partial x} \frac{\partial v_0}{\partial y} \right] - \left[ \frac{\partial v_0}{\partial y} \frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} \frac{\partial u_0}{\partial x} \right] \]

\[ u_2(x, y, t) = \frac{t^2}{2!} e^{x+y} \]

\[ v_2(x, y, t) = \frac{t^2}{2!} e^{x-y} \]

\[ w_2(x, y, t) = \frac{t^2}{2!} e^{y-x} \]

\[ p^3, \quad u_3(x, y, t) = -u_2 - \left[ \frac{\partial v_0}{\partial x} \frac{\partial w_2}{\partial y} + \frac{\partial v_1}{\partial x} \frac{\partial w_1}{\partial y} + \frac{\partial v_2}{\partial x} \frac{\partial w_0}{\partial y} \right] + \left[ \frac{\partial v_2}{\partial y} \frac{\partial w_0}{\partial x} + \frac{\partial v_1}{\partial y} \frac{\partial w_1}{\partial x} + \frac{\partial v_0}{\partial y} \frac{\partial w_2}{\partial x} \right] \]

\[ v_3(x, y, t) = v_2 = - \left[ \frac{\partial w_0}{\partial x} \frac{\partial u_2}{\partial y} + \frac{\partial w_1}{\partial x} \frac{\partial u_1}{\partial y} + \frac{\partial w_2}{\partial x} \frac{\partial u_0}{\partial y} \right] - \left[ \frac{\partial u_0}{\partial y} \frac{\partial w_2}{\partial x} + \frac{\partial u_1}{\partial y} \frac{\partial w_1}{\partial x} + \frac{\partial u_2}{\partial y} \frac{\partial w_0}{\partial x} \right] \]

\[ w_3(x, y, t) = w_2 = - \left[ \frac{\partial u_0}{\partial x} \frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x} \frac{\partial v_0}{\partial y} \right] - \left[ \frac{\partial v_0}{\partial y} \frac{\partial u_2}{\partial x} + \frac{\partial v_1}{\partial y} \frac{\partial u_1}{\partial x} + \frac{\partial v_2}{\partial y} \frac{\partial u_0}{\partial x} \right] \]

\[ u_3(x, y, t) = -\frac{t^3}{3!} e^{x+y} \]

\[ v_3(x, y, t) = \frac{t^3}{3!} e^{x-y} \]

\[ w_3(x, y, t) = \frac{t^3}{3!} e^{y-x} \]

So the solution of the system of PDEs in (9) is given as,
\[ u(x, y, t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n = u_0 + u_1 + u_2 + u_3 + \ldots \]

\[ u(x, y, t) = e^{x+y} - te^{x+y} + \frac{t^2}{2!} e^{x+y} - \frac{t^3}{3!} e^{x+y} + \ldots \]

\[ u(x, y, t) = e^{x+y} \left[ 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ldots \right] \]

\[ u(x, y, t) = e^{x+y} e^{-t} = e^{x+y-t} \]

\[ v(x, y, t) = \lim_{q \to 1} \sum_{n=0}^{\infty} q^n v_n = v_0 + v_1 + v_2 + v_3 + \ldots \]

\[ v(x, y, t) = e^{-x-y} + te^{-x-y} + \frac{t^2}{2!} e^{-x-y} + \frac{t^3}{3!} e^{-x-y} + \ldots \]

\[ v(x, y, t) = e^{-x-y} \left[ 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots \right] \]

\[ v(x, y, t) = e^{-x-y} e^t = e^{-x+y} \]

\[ w(x, y, t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n w_n = w_0 + w_1 + w_2 + w_3 + \ldots \]

\[ w(x, y, t) = e^{y-x} + te^{y-x} + \frac{t^2}{2!} e^{y-x} + \frac{t^3}{3!} e^{y-x} + \ldots \]

\[ w(x, y, t) = e^{y-x} \left[ 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots \right] \]

\[ w(x, y, t) = e^{y-x} e^t = e^{y-x+t} \]

Hence the solution of the system of PDEs given in (9) is,

\[ u(x, y, t) = e^{x+y-t} \]

\[ v(x, y, t) = e^{-x+y+t} \]

\[ w(x, y, t) = e^{y-x+t} \].

**Example 3.5** Consider two dimensional system of nonlinear PDEs

\[ u_t = u_{xx} + u_{yy} + 2(u(u_x + u_y) - (uv)_x - (uv)_y \]

\[ v_t = v_{xx} + v_{yy} + 2(v(v_x + v_y) - (uv)_x - (uv)_y \]

with the initial conditions of,

\[ u(x, y, 0) = v(x, y, 0) = \cos(x + y) \].

Taking inverse operator, we get

\[ L^{-1} [L(u)] = L^{-1} \left[ u_{xx} + u_{yy} + 2(u(u_x + u_y) - (uv)_x - (uv)_y \right] \]

\[ L^{-1} [L(u)] = L^{-1} \left[ v_{xx} + v_{yy} + 2(v(v_x + v_y) - (uv)_x - (uv)_y \right] \]

\[ u(x, y, t) = \cos(x + y) + L^{-1} \left[ u_{xx} + u_{yy} + 2(u(u_x + u_y) - (uv)_x - (uv)_y \right] \]

\[ v(x, y, t) = \cos(x + y) + L^{-1} \left[ v_{xx} + v_{yy} + 2(v(v_x + v_y) - (uv)_x - (uv)_y \right]. \]
\[ u(x, y, t) = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots \]
\[ v(x, y, t) = \sum_{n=0}^{\infty} p^n v_n = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \]

Putting in Eq. (12)
\[ u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots = \cos(x + y) + pL^{-1} \left[ \frac{\partial^2}{\partial y^2} (u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots) \right] \]
\[ \frac{\partial}{\partial x} \left( u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots \right) + \frac{\partial}{\partial y} \left( u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots \right) \]
\[ \{ \frac{\partial}{\partial x} (u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots) + \frac{\partial}{\partial y} (u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots) \} \]
\[ \frac{\partial}{\partial x} \{ (u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots) (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) \} \]
\[ \frac{\partial}{\partial y} \{ (u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots) (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) \} \]

\[ v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots = \cos(x + y) + pL^{-1} \left[ \frac{\partial^2}{\partial y^2} (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) \right] \]
\[ \frac{\partial}{\partial x} \left( v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \right) + \frac{\partial}{\partial y} \left( v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \right) \]
\[ \{ \frac{\partial}{\partial x} (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) + \frac{\partial}{\partial y} (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) \} \]
\[ \frac{\partial}{\partial x} \{ (u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots) (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) \} \]
\[ \frac{\partial}{\partial y} \{ (u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots) (v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots) \} \]

Equating powers of “p”,
\[ p^0, \quad u_0(x, y, t) = \cos(x + y) \]
\[ v_0(x, y, t) = \cos(x + y) \]
\[ p^1, \quad u_1(x, y, t) = L^{-1} \left[ \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_0}{\partial y^2} + 2u_0 \left( \frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial y} \right) - \frac{\partial}{\partial y} (u_0 v_0) - \frac{\partial}{\partial y} (u_0 v_0) \right] \]
\[ v_1(x, y, t) = L^{-1} \left[ \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} + 2v_0 \left( \frac{\partial v_0}{\partial x} + \frac{\partial v_0}{\partial y} \right) - \frac{\partial}{\partial y} (u_0 v_0) - \frac{\partial}{\partial y} (u_0 v_0) \right] \]
\[ u_1(x, y, t) = -2t \cos(x + y) \]
\[ v_1(x, y, t) = -2t \cos(x + y) \]
\[ p^2, \quad u_2(x, y, t) = L^{-1}\left[ \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2 u_1}{\partial y^2} + 2u_0 \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_0}{\partial y} \right) + 2u_1 \left( \frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial y} \right) - \frac{\partial}{\partial x} (u_0 v_1 + u_1 v_0) \right] \]

\[ v_2(x, y, t) = L^{-1}\left[ \frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + 2v_0 \left( \frac{\partial v_1}{\partial x} + \frac{\partial v_0}{\partial y} \right) + 2v_1 \left( \frac{\partial v_0}{\partial x} + \frac{\partial v_0}{\partial y} \right) - \frac{\partial}{\partial x} (u_0 v_1 + u_1 v_0) \right] \]

\[ u_2(x, y, t) = \frac{(2t)^2}{2!} \cos(x + y) \]

\[ v_2(x, y, t) = \frac{(2t)^2}{2!} \cos(x + y) \]

\[ p^3, \quad u_3(x, y, t) = L^{-1}\left[ \frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} + 2u_0 \left( \frac{\partial u_2}{\partial x} + \frac{\partial u_0}{\partial y} \right) + 2u_2 \left( \frac{\partial u_0}{\partial x} + \frac{\partial u_0}{\partial y} \right) - \frac{\partial}{\partial x} (u_0 v_2 + u_2 v_0) \right] \]

\[ v_3(x, y, t) = L^{-1}\left[ \frac{\partial^2 v_2}{\partial x^2} + \frac{\partial^2 v_2}{\partial y^2} + 2v_0 \left( \frac{\partial v_2}{\partial x} + \frac{\partial v_0}{\partial y} \right) + 2v_2 \left( \frac{\partial v_0}{\partial x} + \frac{\partial v_0}{\partial y} \right) - \frac{\partial}{\partial x} (u_0 v_2 + u_2 v_0) \right] \]

\[ u_3(x, y, t) = -\frac{(2t)^3}{3!} \cos(x + y) \]

\[ v_3(x, y, t) = -\frac{(2t)^3}{3!} \cos(x + y) \]

So the solution of the Eq. (11) is given by,

\[ u(x, y, t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n u_n = u_0 + u_1 + u_2 + u_3 + ... \]

\[ v(x, y, t) = \lim_{p \to 1} \sum_{n=0}^{\infty} p^n v_n = v_0 + v_1 + v_2 + v_3 + ... \]

Consequently, we have

\[ u(x, y, t) = \cos(x + y) - 2t \cos(x + y) + \frac{(2t)^2}{2!} \cos(x + y) - \frac{(2t)^3}{3!} \cos(x + y) + ... \]

\[ u(x, y, t) = \cos(x + y) \left[ 1 - \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + ... \right] \]

\[ = \cos(x + y) e^{-2t} \]

\[ v(x, y, t) = \cos(x + y) - 2t \cos(x + y) + \frac{(2t)^2}{2!} \cos(x + y) - \frac{(2t)^3}{3!} \cos(x + y) + ... \]

\[ = \cos(x + y) \left[ 1 - \frac{(2t)^2}{2!} - \frac{(2t)^3}{3!} + ... \right] \]

\[ = \cos(x + y) e^{-2t} \]
Hence the solution of the system of Eq. (11) is given by,
\[ u(x, y, t) = v(x, y, t) = e^{-2t} \cos(x + y). \]

4. Conclusion
In this paper, the Homotopy Perturbation Method (HPM) was employed successfully for solving linear and nonlinear system of partial differential equations. The HPM is applied without determining the Adomian’s polynomials, unrealistic assumptions, and transformation formulas. The results given here provide further evidence of the usefulness of Homotopy Perturbation Method. The HPM is clearly very efficient and powerful technique to find exact solutions of the nonlinear systems of PDEs and can be extended to other mathematical problems.

References
The IISTE is a pioneer in the Open-Access hosting service and academic event management. The aim of the firm is Accelerating Global Knowledge Sharing.

More information about the firm can be found on the homepage: http://www.iiste.org

CALL FOR JOURNAL PAPERS

There are more than 30 peer-reviewed academic journals hosted under the hosting platform.

Prospective authors of journals can find the submission instruction on the following page: http://www.iiste.org/journals/ All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Paper version of the journals is also available upon request of readers and authors.

MORE RESOURCES

Book publication information: http://www.iiste.org/book/

Academic conference: http://www.iiste.org/conference/upcoming-conferences-call-for-paper/

IISTE Knowledge Sharing Partners

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digital Library, NewJour, Google Scholar