Topological Manifolds With Smooth Fibre Bundles

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ABSTRACT
The purpose of this paper is to develop the basic properties of topological manifolds and smooth fibre bundles. If \( O \) is an open covering of a topological manifold \( M \), then we prove that there exists a refinement \( \{V_{ij}\} \), where \( j \in \mathbb{N} \) and \( i \) runs through a finite set, such that \( V_{ij} \cap V_{ik} = \emptyset \), for each \( i \) and \( j \neq k \). Finally, smooth fibre bundle \((E, \pi, B, F)\) is defined; the projection \( \pi \) is a smooth map from the total space \( E \) to the base space \( B \) and it is shown that every smooth fibre bundle has a finite coordinate representation.

Keywords: Topological manifold, smooth manifold, smooth path, chart, atlas, smooth fibre bundle.

1. INTRODUCTION
The idea of topological manifold with smooth fibre bundle was introduced by H. Whitney [5, 6] and then it was generalized by A. Dold [1] and P. Olum [2]. A number of significant properties of smooth fibre bundle \((E, \pi, B, F)\) were obtained by E. H. Spanier [4], G. Wu [7], M. M. Postnikov [3] and others. We begin with the following definition:

An \( n \)-dimensional topological manifold \( M \) is a Hausdorff space with a countable basis which satisfies the following condition:

- Every point \( a \in M \) has a neighbourhood \( U_a \) which is homomorphic to an open subset of an \( n \)-dimensional real vector space \( E \).

A chart for a topological \( n \)-manifold \( M \) is a triple \((U, u, V)\) where \( U \) is an open subset of \( M \), \( V \) is an open subset of an \( n \)-dimensional real vector space \( E \) and \( u : U \to V \) is a homeomorphism. Because the chart \((U, u, V)\) is determined by the pair \((U, u)\), we will denote a chart by \((U, u)\). An atlas on an \( n \)-manifold \( M \) is a family of charts \( \{(U_a, u_a) : a \in \mathcal{I}\} \), where \( \mathcal{I} \) is an arbitrary indexing set, such that the sets \( U_a \) form a covering of \( M \):

\[
M = \bigcup_{a \in \mathcal{I}} U_a.
\]

An open covering of a topological space \( X \) is said to have order \( \leq p \) if the intersection of every \( p + 1 \) elements of the cover is empty.

Definition 1.1 A topological space \( X \) is said to have Lebesgue dimension \( \leq p \) if every open cover has a locally finite refinement of order \( \leq p + 1 \).

Lemma 1.1 If \( \dim X \leq m \) (\( m \geq 1 \)), then \( \dim(X \times \mathbb{R}) \leq 7m \).

Proof. Let \( \mathcal{O} \) be any open cover of \( X \times \mathbb{R} \). For each \( n \in \mathbb{Z} \), choose an open cover \( \mathcal{W}_n \) of \( X \) with the following property:

If \( W \in \mathcal{W}_n \) and \( t \in [n, n + 2] \), then for some \( \varepsilon > 0 \) and \( \mathcal{O} \in \mathcal{O}, W \times (t - \varepsilon, t + \varepsilon) \subset O \).

We may assume that each \( \mathcal{W}_n \) is locally finite and of order \( \leq m + 1 \) since \( \dim X \leq m \). By considering open sets of the form \( W \times (t - \varepsilon, t + \varepsilon) \) \((W \in \mathcal{W}_n)\), we obtain a locally finite open covering of \( X \times (n, n + 2) \) of order \( \leq 2(m + 1) \). These open coverings together provide an open covering of \( X \times \mathbb{R} \) of order \( \leq 4(m + 1) \leq 7m + 1 \).

Theorem 1.1 Let \( X \) be a normal space with a countable basis. Suppose \( U \) and \( V \) are open sets such that \( \dim U \leq n, \dim V \leq n \) and \( X = U \cup V \). Then \( \dim X \leq n \).
Proof. We choose disjoint open sets $U', V' \subset X$ such that

$$(X - V) \subset U' \subset U \quad \text{and} \quad (X - U) \subset V' \subset V.$$ 

Let $\mathcal{O}$ be an open covering of $X$. By refining $\mathcal{O}$ if necessary we may assume that $\mathcal{O}$ is of the form $\mathcal{O} = \mathcal{O}_U \cup \mathcal{O}_{U'}$, where $\mathcal{O}_U = \{ O_k : k \in \mathbb{N} \}$ is a locally finite open covering of $X$ of order $\leq n + 1$ and $\mathcal{O}_{U'}$ is an open covering of $V'$.

Set $\mathcal{O}_V = \{ O_k \cap V : k \in \mathbb{N} \}$. Then $\mathcal{O}_V \cup \mathcal{O}_{U'}$ is an open covering of $V$. Let $\mathcal{W}$ be a locally finite refinement of this covering of order $\leq n + 1$. Then $\mathcal{W}$ is the disjoint union of $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$, where $\mathcal{W}^{(1)}$ consists of those open sets contained in $V'$ and $\mathcal{W}^{(2)}$ consists of the others.

We denote the elements of $\mathcal{W}^{(1)}$ and $\mathcal{W}^{(2)}$ by $W_\alpha$ and $W_\beta$ respectively. Thus each $W_\beta$ is contained in some $O_k$. Hence $\mathcal{W}^{(2)}$ is the disjoint union of the subcollections $\mathcal{W}_k^{(2)}$ given by

$$\mathcal{W}_k^{(2)} = \{ W_\beta : W_\beta \subset O_k, \quad W_\beta \notin O_i, \quad i < k \}.$$ 

Now we define open sets $Y_k$ by

$$Y_k = (O_k \cap U') \cup \bigcup_{\beta_j} W_{\beta_j},$$ 

where the union is taken over those $\beta$, such that $W_{\beta_j} \in \mathcal{W}_k^{(2)}$.

Set $\mathcal{Y} = \{ Y_k : k \in \mathbb{N} \}$. We show that $\mathcal{Y}^{(1)} = \mathcal{Y} \cup \mathcal{W}^{(1)}$ is a locally finite refinement and has order $\leq n + 1$.

First note that since the sets $O_k (k \in \mathbb{N})$ cover $U$, the sets $Y_k (k \in \mathbb{N})$ cover $U'$. On the other hand, the $Y_k$ contain all the $W_\beta$ and so the $W_\alpha$ together with the $Y_k$ cover $V$ since $\mathcal{W}$ covers $V$. Since $X = U' \cup V$, it follows that $\mathcal{Y}^{(1)}$ is a cover of $X$.

Next observe that $Y_k \subset O_k$ and so $\mathcal{Y}$ refines $\mathcal{O}$. But $\mathcal{W}$ refines $\mathcal{O}$ and hence $\mathcal{W}^{(1)}$ also refines $\mathcal{O}$. Thus $\mathcal{Y}^{(1)}$ refines $\mathcal{O}$. To show that $\mathcal{Y}^{(1)}$ has order $\leq n + 1$, let

$$x \in Y_{k_1} \cap \cdots \cap Y_{k_p} \cap W_{\alpha_1} \cap \cdots \cap W_{\alpha_q}.$$ 

We distinguish two cases.

Case 1: When $x \in U'$. Since $x \in U'$, so $q = 0$ and $x \in Y_{k_1} \cap \cdots \cap Y_{k_p} \subset O_{k_1} \cap \cdots \cap O_{k_p}$. Hence $p \leq n + 1$ and so $p + q \leq n + 1$.

Case 2: When $x \notin U'$. For each $k_i$ there is an element $W_{\beta_i} \subset \mathcal{W}_k^{(2)}$ such that $x \in W_{\beta_i}$. Moreover, the $W_{\beta_i}$ are necessarily distinct. Thus

$$x \in W_{\beta_1} \cap \cdots \cap W_{\beta_p} \cap W_{\alpha_1} \cap \cdots \cap W_{\alpha_q}.$$ 

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i.e., \( x \) is in \( p + q \) distinct elements of \( \mathcal{W} \). It follows that \( p + q \leq n + 1 \). Distinguishing between the same two cases and using the fact that \( \mathcal{O}_\alpha \) and \( \mathcal{W} \) are locally finite, we see that \( \mathcal{Y}^{(1)} \) is locally finite.

**Lemma 1.2** If a manifold \( M \) has a basis \( \mathcal{O}_\alpha \) such that for each \( \alpha \), \( \dim \mathcal{O}_\alpha \leq p \), then \( \dim M \leq p \) for every open subset \( \mathcal{O} \) of \( M \).

**Proof.** Clearly, if a space \( X \) is the disjoint union of open subsets with \( \text{dimension} \leq p \) then \( \dim X \leq p \). On the other hand, Theorem 1.1 implies that if a manifold \( Q \) is a finite union with \( \text{dimension} \leq p \), then \( \dim Q \leq p \). Hence the lemma is proved.

**Corollary 1.1** If \( \mathcal{O} \) is an open subset of \( \mathbb{R}^n \), then \( \dim \mathcal{O} \leq 7^n \).

**Proposition 1.1** Every topological \( n \)-manifold \( M \) satisfies \( \dim M \leq 7^n \).

**Proof.** Observe that \( M \) admits a basis consisting of open subsets \( \mathcal{O}_\alpha \) homeomorphic to open subsets of \( \mathbb{R}^n \). Hence, by Corollary 1.1, \( \dim \mathcal{O}_\alpha \leq 7^n \). If we replace \( \mathcal{O} \) by \( M \), then Lemma 1.2 yields the proposition.

**Theorem 1.2** Let \( \mathcal{O} \) be an open covering of a topological manifold \( M \). Then there exists a refinement \( \{V_{ij}\} \), where \( j \in \mathbb{N} \) and \( i \) runs through a finite set, such that for each \( i \)

\[ V_{ij} \cap V_{ik} = \emptyset, \quad j \neq k. \]

**Proof.** Let \( \mathcal{O} \) be any open covering of \( M \). According to Proposition 1.1 there exists a locally finite refinement of finite order. Thus we may assume that \( \mathcal{O} \) is locally finite and of finite order \( p \). Moreover, we may assume that \( \mathcal{O} \) is indexed by \( \mathbb{N} \), \( \mathcal{O} = \{\mathcal{O}_j : j \in \mathbb{N}\} \).

Now we proceed by induction on \( p \). If \( p = 1 \), there is nothing to prove. Assume that the theorem holds for coverings of order \( p - 1 \) and that \( \mathcal{O} \) has order \( p \). For each distinct set \( \nu_1 < \cdots < \nu_{p+1} \) of \( (p+1) \) indices let

\[ \mathcal{O}_{\nu_1 \cdots \nu_{p+1}} = \bigcap_{k=1}^{p+1} \mathcal{O}_{\nu_k}. \]

Since \( \mathcal{O} \) has order \( p \) these sets are disjoint. Denote them by \( V_{ij} \) \((i = 1, 2, \cdots)\) and set

\[ V_1 = \bigcup V_{1i}. \]

Next choose open sets \( U_j \) so that

\[ \bar{U}_j \subset \mathcal{O}_j \text{ and } \bigcup_j U_j = M. \]

Let \( A \) denote the union of all sets of the form

\[ \bar{U}_{\nu_1} \cap \cdots \cap \bar{U}_{\nu_{p+1}} \left( \nu_1 < \cdots < \nu_{p+1} \right). \]
Then $A$ is closed because the $O_j$ are locally finite. Now the $U_j$ provide a locally finite covering of $M - A$ of order $p - 1$. Since $M = (M - A) \cup V$, the theorem follows by induction.

2. SMOOTH MANIFOLD

Let $M$ be a topological manifold and let $\{(U_\alpha, u_\alpha) : \alpha \in I \}$ be an atlas for $M$. Consider two neighbourhoods $U_\alpha, U_\beta$ such that $U_{\alpha\beta} = U_\alpha \cup U_\beta \neq \emptyset$. Then a homeomorphism

$$u_{\alpha\beta} : u_\alpha(U_{\alpha\beta}) \to u_\alpha(U_{\alpha\beta})$$

is defined by $u_{\alpha\beta} = u_\alpha \circ u^{-1}_\beta$. This map is called the identification map for $U_\alpha$ and $U_\beta$. By definition

$$u_{\beta\gamma} \circ u_{\alpha\beta} = u_{\alpha\gamma}$$

in $u_\alpha(U_{\alpha\beta})$ and $u_{\alpha\gamma}(x) = x, x \in u_\alpha(U_\alpha)$. The atlas $\{(U_\alpha, u_\alpha)\}$ is called smooth if all its identification maps are smooth.

Two smooth atlases are equivalent if their union is again a smooth atlas; i.e., $\{(U_\alpha, u_\alpha)\}$ and $\{(V_\alpha, v_\alpha)\}$ are equivalent if all the maps

$$v_\gamma \circ u^{-1}_\alpha : u_\alpha(U_\alpha \cap V_\gamma) \to v_\gamma(U_\alpha \cap V_\gamma)$$

and their inverses are smooth. A smooth structure on $M$ is an equivalence class of smooth atlases on $M$. A topological manifold endowed with a smooth structure is called a smooth manifold.

Hence we shall use the word "manifold" in the sense of a smooth manifold. An atlas for a manifold will mean a member of its smooth structure and the term chart will refer to a member of an atlas.

**Definition 2.1** Let $M, N$ be manifolds and assume that $\varphi: M \to N$ is a continuous map. Let $\{(U_\alpha, u_\alpha)\}$ and $\{(V_\alpha, v_\alpha)\}$ be atlases for $M$ and $N$ respectively. Then $\varphi$ determines continuous maps

$$\varphi_{i\alpha} : u_\alpha(U_{\alpha} \cap \varphi^{-1}(V_\gamma)) \to v_\gamma(V_\gamma)$$

by

$$\varphi_{i\alpha} = v_\gamma \circ \varphi \circ u^{-1}_\alpha.$$  

We say that $\varphi: M \to N$ is smooth if the maps $\varphi_{i\alpha}$ are smooth. This definition is independent of the choice of atlases for $M$ and $N$. Moreover, if $\varphi: M \to N$ and $\mu: N \to P$ are smooth maps, then $\mu \circ \varphi: M \to P$ is smooth. The set of smooth maps from $M$ to $N$ is denoted by $\mathcal{C}(M; N)$.

**Definition 2.2** A smooth map $\varphi: M \to N$ is called a diffeomorphism if it has a smooth inverse $\varphi^{-1}: N \to M$. Every diffeomorphism is a homeomorphism. Two manifolds $M$ and $N$ are diffeomorphic if there exists a diffeomorphism $\varphi: M \to N$.

**Definition 2.3** A smooth function on a manifold $M$ is a smooth map $f: M \to \mathbb{R}$. If $f$ and $g$ are two such functions, then smooth functions $\lambda f + \mu g$ and $fg$ are defined by

$$(\lambda f + \mu g)(x) = \lambda f(x) + \mu g(x), \quad \lambda, \mu \in \mathbb{R}$$  

$$(fg)(x) = f(x)g(x), \quad x \in M.$$  

These operations make the set of smooth functions on $M$ into an algebra over $\mathbb{R}$, which we denote by $\mathcal{C}(M)$. The unit element of $\mathcal{C}(M)$ is the constant function $M \mapsto 1$. If $M$ and $N$ are $\mathcal{C}(M)$-modules, we
denote their tensor product over $\mathcal{S}(M)$ by $M \otimes_M N$. The module of $\mathcal{S}(M)$-linear maps of $M$ into $N$ will be denoted by $\text{Hom}_M(M; N)$. Now suppose that $\varphi: M \to N$ is a smooth map. $\varphi$ determines an algebra homomorphism $\varphi^*: \mathcal{S}(N) \to \mathcal{S}(M)$ given by

$$\varphi^* f = f \circ \varphi, \quad f \in \mathcal{S}(N).$$

If $\varphi$ is surjective, $\varphi^*$ is injective. If $\psi: N \to Q$ is a second smooth map, then $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

**Definition 2.4** The carrier (or support) of a smooth function $f$ on $M$ is the closure of the set $\{x \in M : f(x) \neq 0\}$. We denote this set by $\text{carr} f$.

If $Q$ is an open subset of $M$ and $f$ is a smooth function on $Q$ whose carrier is closed in $M$, then $f$ extends to the smooth function $g$ on $M$, given by

$$g(x) = \begin{cases} f(x), & x \in Q \\ 0, & x \in M - \text{carr} f. \end{cases}$$

In particular, if $f \in \mathcal{S}(M)$ has carrier in $Q$ and $h \in \mathcal{S}(Q)$, a smooth function $f \cdot h \in \mathcal{S}(M)$ is given by

$$(f \cdot h) = f(h(x)), \quad x \in Q \quad \text{and} \quad (f \cdot h)(x) = 0, \quad x \notin \text{carr} f.$$  

Next, suppose that $\{U_a\}$ is a locally finite family of open sets of $M$, and let $f_a \in \mathcal{S}(M)$ satisfy $\text{carr} f_a \subseteq U_a$. Then for each $a \in M$ there is a neighbourhood $V(a)$ which meets only finitely many of the $U_a$. Thus in this neighbourhood $\sum_a f_a$ is a finite sum. It follows that a smooth function $f$ on $M$ is defined by

$$f(x) = \sum_a f_a(x), \quad x \in M.$$  

**Lemma 2.1** Let $E$ be a Euclidean space and $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha < \beta$. There exists a smooth function $h: E \to [0, 1] \subseteq \mathbb{R}$ such that $h(x) = 1$, for $|x|^2 \leq \alpha$, $h(x) = 0$, for $|x|^2 \geq \beta$.

**Proof.** Given real numbers $\alpha, \beta$ with $\alpha < \beta$, there exists a smooth non-decreasing function $g: \mathbb{R} \to [0, 1]$ such that

$$g(t) = \begin{cases} 0, & t \leq \alpha \\ 1, & t \geq \beta. \end{cases}$$

If we define $h$ by $h(x) = 1 - g(|x|^2)$, then $h(x) = 1$, for $|x|^2 \leq \alpha$ and $h(x) = 0$, for $|x|^2 \geq \beta$.

**Proposition 2.1** Let $K, O$ be subsets of $M$ such that $K$ is closed, $O$ is open and $K \subseteq O$. There exists a smooth function $f$ such that

1. $\text{carr} f$ is contained in $O$
2. $0 \leq f(x) \leq 1, x \in M$
3. $f(x) = 1, \ x \in K$. 

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Proof. We choose open sets \( U_a \subseteq M \) and compact sets \( K_a \subseteq U_a \), subject to the following conditions

(a) \( \{U_a\}, M - K \) is a locally finite open cover of \( M \).

(b) Each \( U_a \) is diffeomorphic to \( \mathbb{R}^n \) and \( \bar{U}_a \subseteq O \).

(c) \( \cup K_a = K \).

It follows from Lemma 2.1 that there are smooth functions \( h_a \) in \( U_a \) such that \( \text{carr} h_a \) is compact and \( h_a(x) = 1 \), \( x \in K_a \). In particular \( \text{carr} h_a \) is closed in \( M \). We extend the \( h_a \) to smooth functions \( f_a \) in \( M \) with \( \text{carr} f_a = \text{carr} h_a \subseteq U_a \). Then we can form \( \sum_a f_a \) in \( S(M) \). Evidently,

\[ \text{carr} \sum_a f_a \subseteq \cup \bar{U}_a \subseteq O \quad \text{and} \quad (\sum_a f_a)(x) \geq 1, \quad x \in K. \]

Finally, we choose a smooth map \( g: \mathbb{R} \to [0, 1] \) so that \( g(0) = 0 \) and \( g(t) = 1, \ t \geq 1 \). Then the function \( f = g \circ (\sum_a f_a) \) satisfies the desired conditions. Hence the proposition is proved.

Definition 2.5 A partition of unity, subordinate to a locally finite open covering \( \{U_a\} \) of \( M \) is a family \( \{f_a\} \) of smooth functions on \( M \) satisfying

(a) \( 0 \leq f_a(x) \leq 1, \ x \in M \)

(b) \( \text{carr} f_a \subseteq U_a \)

(c) \( \sum f_a = 1 \).

Theorem 2.1 Every locally finite open covering of a manifold admits a subordinate partition of unity \( \{f_a\} \).

Proof. Let \( \{U_a\} \) be such a covering of \( M \) and let \( \{V_a\} \) be a second covering such that \( \bar{V}_a \subseteq U_a \). In view of Proposition 2.1, there are non-negative smooth functions \( g_a \) on \( M \) which have carriers in \( U_a \) and take the value 1 at points of \( \bar{V}_a \). Thus \( g = \sum g_a \) is smooth and positive. Set \( f_a = g_a/g \) Every locally finite open covering \( \{U_a\} \) of a manifold \( M \) admits a subordinate partition of unity \( \{f_a\} \).

Definition 2.6 Let \( a \) be a fixed point of \( M \). Two members \( f, g \) of \( \mathcal{S}(M) \) will be called \( a \)-equivalent if and only if there is a neighbourhood \( U \) of \( a \) such that \( (x) = g(x), \ x \in U \). The equivalence classes so obtained are called function germs at \( a \). We write \( [f]_a \) for the germ represented by \( f \in \mathcal{S}(M) \).

Let \( M, N \) be smooth manifolds and \( \varphi, \psi \) be smooth maps of \( M \) into \( N \). We say that \( \varphi \) is homotopic to \( \psi \), and write \( \varphi \sim \psi \), if there exists a smooth map \( H: \mathbb{R} \times M \to N \) such that \( H(0,x) = \varphi(x) \) and \( H(1,x) = \psi(x), \ x \in M \). \( H \) is called a homotopy. Homotopy is an equivalence relation in the set of smooth maps \( M \to N \).

Lemma 2.2 \( \varphi, \psi: M \to N \) are homotopic if and only if there is a smooth map \( K: \mathbb{R} \times M \to N \) such that \( K(t,x) = \varphi(x), \ t \leq 0 \) and \( K(t,x) = \psi(x), \ t \geq 1 \).

Proof. If \( K \) exists, then obviously \( \varphi \sim \psi \). If \( \varphi \sim \psi \), let \( H \) be a homotopy. Choose a smooth function \( g: \mathbb{R} \to \mathbb{R} \) such that

\[ g(t) = 0, \ t \leq 0 \quad \text{and} \quad g(t) = 1, \ t \geq 1. \]

Then set \( K(t,x) = H(g(t), x) \). Therefore, \( K(t,x) = \varphi(x), \ t \leq 0 \) and \( K(t,x) = \psi(x), \ t \geq 1 \). Hence \( \varphi, \psi: M \to N \) are homotopic if and only if there is a smooth map \( K: \mathbb{R} \times M \to N \) such that \( K(t,x) = \varphi(x), \ t \leq 0 \) and \( K(t,x) = \psi(x), \ t \geq 1 \).
3. SMOOTH PATHS
A smooth path on $M$ is a smooth map $\varphi: \mathbb{R} \to M$. A manifold is called smoothly path-connected if, for every two points $a, b \in M$, there exists a smooth path $\varphi$ such that $\varphi(0) = a$ and $\varphi(1) = b$.

**Proposition 3.1** If $a, b$ are points of a connected manifold $M$, there is a smooth path $\varphi$ on $M$ such that

$$
\varphi(t) = \begin{cases} 
  a, & t \leq 0 \\
  b, & t \geq 1 
\end{cases}
$$

In particular, $M$ is smoothly path-connected.

**Proof.** By Lemma 2.2, the smooth path $\varphi$ exists if and only if the inclusion maps

$$
j_a: \{\text{point}\} \to a \in M \quad \text{and} \quad j_b: \{\text{point}\} \to b \in M
$$

are homotopic. Since homotopy is an equivalence relation, so an equivalence relation is induced on the points of $M$:

$$
a \sim b \quad \text{if and only if} \quad a \text{ can be joined to } b \text{ by some } \varphi.
$$

If $M = \mathbb{R}^n$, the proposition is obviously true. Thus, in general, if $(U, u, \mathbb{R}^n)$ is a chart in $M$, then all the points of $U$ are equivalent. Hence the equivalence classes are all open and $M$ is their disjoint union. Since $M$ is connected, there is only one class; i.e., every $a, b \in M$ are equivalent. Hence the proposition is proved.

**Lemma 3.2** There is a smooth function $f$ on $\mathbb{R}$ such that

1. $\text{carr } f \subset [-3, 3]
2. 0 \leq f(t) \leq 1, \ t \in \mathbb{R} \quad \text{and} \quad f(0) = 1
3. |f'(t)| < 1, \ t \in \mathbb{R}.

**Proof.** If we define $f$ by

$$
f(t) = \begin{cases} 
  \exp\left(-\frac{t^2}{9-t^2}\right), & t \in (-3, 3) \\
  0, & \text{otherwise}
\end{cases}
$$

then

$$
\text{carr } f \subset [-3, 3]
0 \leq f(t) \leq 1, \ t \in \mathbb{R} \quad \text{and} \quad f(0) = 1.
$$

Differentiating $f(t)$ we get

$$
f'(t) = \begin{cases} 
  -\frac{18t}{(9-t^2)^2} \exp\left(-\frac{t^2}{9-t^2}\right), & t \in (-3, 3) \\
  0, & \text{otherwise}
\end{cases}
$$
Hence, $|f'(t)| < 1$, $t \in \mathbb{R}$.

**Corollary 3.1** There exists a diffeomorphism $\varphi$ of $\mathbb{R}^n$ such that

1. $\varphi(0, \ldots, 0) = (1, 0, \ldots, 0)$
2. $\varphi(x) = x$, for every $x = (\xi^1, \ldots, \xi^n)$ such that $\max |\xi^i| > 3$.

**Proof.** We define $\varphi$ by

$$\varphi(\xi^1, \ldots, \xi^n) = \left( \xi^1 + \prod_{i=1}^{n} f(\xi^i), \xi^2, \ldots, \xi^n \right)$$

where $f: \mathbb{R} \rightarrow \mathbb{R}$ is the function of Lemma 3.2. Then the Jacobian of $\varphi$ is given by

$$\det \varphi'(x) = 1 + f'(\xi^1) \prod_{i=1}^{n} f(\xi^i).$$

As $\det \varphi'(x) > 0$, so $\varphi$ is a local diffeomorphism. To see that $\varphi$ is a global diffeomorphism, it is only necessary to note that it induces a bijection on each of the lines

$$\xi^2 = \xi_0^2, \xi^3 = \xi_0^3, \ldots, \xi^n = \xi_0^n.$$

Thus $\varphi$ satisfies conditions (1) and (2) from the properties of $f$.

**Theorem 3.1** Let $C$ be a closed subset of a manifold $M$ such that $M - C$ is nonvoid and connected. Let $a$, $b$ be arbitrary points of $M - C$. Then there is a diffeomorphism $\varphi: M \rightarrow M$ homotopic to $\iota_M$ and such that $\varphi(a) = b$ and $\varphi(x) = x$, $x \in C$.

**Proof.** Let $\sim$ be the equivalence relation on $M - C$ defined by $x_1 \sim x_2$ if and only if there is a diffeomorphism $\varphi: M \rightarrow M$, homotopic to $\iota_M$, such that $\varphi(x_1) = x_2$ and $\varphi(x) = x$, $x \in C$.

We shall show that the equivalence classes are open. In fact, if $a \in M - C$, let $(U, u, \mathbb{R}^n)$ be a chart of $M$ such that $a \in U \subset M - C$. If $b \in U$ is arbitrary, we can compose $u$ with a transformation of $\mathbb{R}^n$, if necessary, and assume that $u(a) = 0, u(b) = (1, 0, \ldots, 0)$. Applying Corollary 3.1, we obtain a diffeomorphism $\varphi_0: U \rightarrow U$ such that $\varphi_0(a) = b$ and $\varphi_0$ is the identity outside a compact set $K$ such that $b \in K \subset U$. Then $\varphi: M \rightarrow M$ defined by

$$\varphi(x) = \begin{cases} \varphi_0(x), & x \in U \\ x, & x \notin U \end{cases}$$

is a diffeomorphism which establishes the equivalence of $a$ and $b$; hence all points of $U$ are equivalent to $a$. Since the equivalence classes are open and $M - C$ is connected, all points of $M - C$ are equivalent, as required. Hence the theorem is proved.

**Proposition 3.2** Let $M$ be a connected manifold of dimension $n \geq 2$ and $\{a_1, \ldots, a_k\}, \{b_1, \ldots, b_k\}$ be two finite subsets of $M$. Then there is a diffeomorphism $\varphi: M \rightarrow M$, homotopic to $\iota_M$, such that $\varphi(a_1) = b_i$ ($i = 1, \ldots, k$).
Proof. If \( k = 1 \), the result follows from Theorem 3.1 with \( C = \emptyset \). Suppose that the result has been proved for \( k - 1 \); i.e. a diffeomorphism \( \varphi_0 \) of \( M \), homotopic to \( t_B \) has been found such that \( \varphi_0(a_i) = b_i, \ i = 1, \cdot \cdot \cdot , k - 1 \). Since \( M - \{ b_1, \cdot \cdot \cdot , b_{k-1} \} \) is nonvoid and connected, we obtain, from Theorem 3.1, a diffeomorphism \( \varphi_1 \) of \( M \), homotopic to \( t_B \) such that \( \varphi_1(\varphi_0(a_k)) = b_k \) and \( \varphi_1(b_i) = b_i \) for \( i = 1, \cdot \cdot \cdot , k \). If we set \( \varphi = \varphi_1 \circ \varphi_0 \), then \( \varphi(a_i) = b_i (i = 1, \cdot \cdot \cdot , k) \).

4. SMOOTH FIBRE BUNDLES

Let \( \pi: E \to B \) be a smooth map between manifolds. The \( \pi \) map will be said to have the local product property with respect to a manifold \( F \) if there is an open covering \( \{ U_a \} \) of \( B \) and a family \( \{ \psi_a \} \) of diffeomorphisms

\[
\psi_a: U_a \times F \to \pi^{-1}(U_a),
\]

such that

\[
\pi \psi_a(x, y) = x, \ x \in U_a, \ y \in F.
\]

Definition 4.1 A smooth fibre bundle is a four-tuple \( (E, \pi, B, F) \) where \( \pi: E \to B \) is a smooth map which has the local product property with respect to \( F \). A local decomposition for \( \pi \) is called a coordinate representation for the fibre bundle.

We call \( E \) the total or bundle space, \( B \) the base space, and \( F \) the typical fibre. For each \( x \in B \), the set \( F_x = \pi^{-1}(x) \) will be called the fibre over \( x \). Every fibre is a closed subset of \( E \), and \( E \) is the disjoint union of the fibres.

A smooth cross-section of a fibre bundle \( (E, \pi, B, F) \) is a smooth map \( \sigma: B \to E \) such that \( \pi \circ \sigma = t_B \). If \( \{(U_a, \psi_a)\} \) is a coordinate representation for the bundle, we obtain bijections \( \psi_{a,x}: F \to F_x, x \in U_a \), defined by

\[
\psi_{a,x}(y) = \psi_a(x, y), \ y \in F.
\]

In particular, if \( x \in U_{a\beta} \), we obtain maps \( \psi^{-1}_{\beta\alpha} \circ \psi_{a,x}: F \to F \). These are diffeomorphisms. Since \( \psi_a \) and \( \psi_{\beta} \) define diffeomorphisms of \( U_{a\beta} \times F \) onto \( \pi^{-1}(U_{a\beta}) \), they determine a diffeomorphism \( \psi_{\beta a} = \psi^{-1}_{\beta} \circ \psi_a \) of \( U_{a\beta} \) onto itself. But

\[
\psi_{\beta a}(x, y) = \left( x, \psi^{-1}_{\beta x} \psi_{a,x}(y) \right), x \in U_{a\beta}, \ y \in F,
\]

and hence \( \psi^{-1}_{\beta x} \circ \psi_{a,x} \) is a diffeomorphism of \( F \). We now show that \( \varphi_B \) is always smooth. In fact, if \( \{(U_a, \psi_a)\} \) is a local decomposition for \( \pi \) and \( y \in F \) is fixed, then

\[
\varphi_B(x) = (\pi' \circ \varphi \circ \psi_a)(x, y), \ x \in U_a.
\]

Hence \( \varphi_B \) is smooth on each member \( U_a \) of a covering of \( B \).

Let \( (E'', \pi'', B'', F'') \) be a third fibre bundle and assume that \( \varphi: E \to E', \ \varphi': E' \to E'' \) are fibre preserving. Then \( \varphi' \circ \varphi: E \to E'' \) is fibre preserving and \( (\varphi' \circ \varphi)_B = \varphi'_B \circ \varphi_B \).

Proposition 4.1 Let \( M \) be a set which is the union of a countable collection \( \{ W_i \} \) of subsets such that
(a) For each $i \in \mathbb{N}$, there is a bijection $\varphi_i : W_i \to M_i$ where $M_i$ is an $n$-manifold.

(b) For every pair $i, j$ the subsets $\varphi_i(W_{ij}) \subset M_i$ and $\varphi_j(W_{ij}) \subset M_j$ are open and the map

$$\varphi_{ji} = \varphi_j \circ \varphi_i^{-1} : \varphi_i(W_{ij}) \to \varphi_j(W_{ij})$$

is a diffeomorphism.

(c) For distinct points $a_i \in W_i$ and $a_j \in W_j$, there are disjoint subsets $U_i, U_j$ such that

$$a_i \in U_i \subset W_i, \quad a_j \in U_j \subset W_j$$

and then, by definition, $\varphi_i(U_i), \varphi_j(U_j)$ are open.

Then there is a unique smooth manifold structure on $M$ such that the $W_i$ are open and the $\varphi_i$ are diffeomorphisms.

**Theorem 4.1** Let $B, F$ be manifolds and let $E$ be a set. Assume that a surjective map $\pi : E \to B$ is given with the following properties:

(a) There is an open covering $\{U_a\}$ of $B$ and a family $\{\psi_a\}$ of bijections $\psi_a : U_a \times F \to \pi^{-1}U_a$.

(b) For every $x \in U_a, y \in F$, $\pi \psi_a(x, y) = x$.

(c) The maps $\psi_{a\beta} : U_{a\beta} \times F \to U_{a\beta} \times F$ defined by $\psi_{a\beta}(x, y) = (\psi_{a\beta}^{-1} \circ \psi_a)(x, y)$ are diffeomorphisms.

Then there is exactly one manifold structure on $E$ for which $(E, \pi, B, F)$ is a fibre bundle with coordinate representation $\{(U_a, \psi_a)\}$.

**Proof.** We may assume that $\{a\}$ is countable and thus apply Proposition 4.1, with $W_a = \pi^{-1}U_a$, $\varphi_a = \psi_a^{-1}$, and $M_a = U_a \times F$ to obtain a unique manifold structure on $E$ such that the $\varphi_a$ are diffeomorphisms. Hypothesis (b) then says that the restriction of $\pi$ to $\pi^{-1}U_a$ is $\psi_a \circ \psi_a^{-1}$ where $\pi_a : U_a \times F \to U_a$ denotes the projection onto the first factor. Since $\pi_a$ is smooth, $\pi$ is smooth on $\pi^{-1}U_a$. Hence $\pi$ is smooth on $E$ and then, by definition, $\{(U_a, \psi_a)\}$ is a local decomposition for $\pi$. Hence $(E, \pi, B, F)$ is a fibre bundle with coordinate representation $\{(U_a, \psi_a)\}$.

**Theorem 4.2** Every smooth fibre bundle has a finite coordinate representation.

**Proof.** Let $\{(U_a, \psi_a)\}$ be any coordinate representation for $(E, \pi, B, F)$. We choose a refinement $\{V_{ij} : i = 1, \ldots, p; j \in \mathbb{N}\}$ of $\{U_a\}$ such that $V_{ij} \cap V_{ik} = \emptyset$ for $j \neq k$. Let $V_i = \bigcup_j V_{ij}$ and define $\psi_i : V_i \times F \to \pi^{-1}V_i$ by

$$\psi_i(x, y) = \psi_{ij}(x, y) \quad \text{if} \quad x \in V_{ij}, \quad y \in F$$

where $\psi_{ij}$ is the restriction of some $\psi_a$. Thus every smooth fibre bundle has a finite coordinate representation.
REFERENCES

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