Coupled Fixed Point Results In G-Metric Spaces

For $W^*$-Compatible Mappings

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Abstract: In this paper, we consider a new class of pairs of generalized contractive type mappings defined in $G$ – metric spaces. Some coincidence and common fixed point results for these mapping are presented.

Keywords: Coincidence Point, Coupled Fixed Point, Common Coupled Fixed Point, Common Fixed Point, Generalized Metric Space, $W^*$-Compatible Mappings.

1. Introduction and Preliminaries

Mustafa and Sims [5] introduced the notion of complete $G$ – metric spaces as a generalization of complete metric spaces. For details on $G$ – metric spaces, we refer to [5, 6, 7, 8]. The notion of a coupled fixed point in partially ordered metric spaces has been introduced by Bhaskar and Lakshmikantham in (2006)[9]. In this paper ,we prove a common coupled fixed point theorem for two mappings in $G$ – metric spaces.

Definition 1.1 [5] Let $X$ be a nonempty set and $G: X \times X \times X \to \mathbb{R}^+$ a function satisfying the following properties:

$(G_1)$ $G(x, y, z) = 0$ if $x = y = z = 0$,

$(G_2)$ $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$,

$(G_3)$ $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$,

$(G_4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$, symmetry in all three variables,

$(G_5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$

Then the function $G$ is called a generalized metric, or, more specifically, a $G$ – metric on $X$, and the pair $(X, G)$ is called a $G$ – metric space.

Definition 1.2 [5] Let $(X, G)$ be a $G$ – metric space and $(x_n)$ a sequence of points of $X$. A point $x \in X$ is said to be the limit of the sequence $(x_n)$, if $\lim_{m,n \to \infty} G(x_n, x_m) = 0$, and we say that the sequence $(x_n)$ is $G$ – convergent to $x$ or that $(x_n)$ $G$ – converges to $x$.

Thus, $x_n \to x$ in a $G$ – metric space $(X, G)$ if for any $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that...
$G(x,x_n,x_m) < \varepsilon$ for all $m,n \geq k$.

**Proposition 1.1.** [5] Let $(X,G)$ be a $G$–metric space. Then the following are equivalent:

1. $(x_n)$ is $G$–convergent to $x$.
2. $G(x_n,x_n,x) \to 0$ as $n \to \infty$.
3. $G(x_n,x,x) \to 0$ as $n \to \infty$.
4. $G(x_n,x_m,x) \to 0$ as $n,m \to \infty$.

**Definition 1.3** [5] Let $(X,G)$ be a $G$–metric space, a sequence $(x_n)$ is called $G$–Cauchy if for every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n,x_m,x_l) < \varepsilon$, for all $n,m,l \geq k$, that is $G(x_n,x_m,x_l) \to 0$ as $n,m,l \to \infty$.

**Proposition 1.2.** [5] Let $(X,G)$ be a $G$–metric space, then the following statements are equivalent:

1. The sequence $(x_n)$ is $G$–Cauchy.
2. For every $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that $G(x_n,x_m,x_m) < \varepsilon$, for all $n,m \geq k$.

**Definition 1.4** [5] A $G$–metric space $(X,G)$ is called $G$–complete if every $G$–Cauchy sequence in $(X,G)$ is $G$–convergent in $(X,G)$.

**Proposition 1.3.** [5] Let $(X,G)$ be a $G$–metric space. Then, the function $G(x,y,z)$ is jointly continuous in all three of its variables.

**Example 1.1.** [5] Let $(\mathbb{R},d)$ be the usual metric space. Define $G_s$ by

$$G_s(x,y,z) = d(x,y) + d(y,z) + d(x,z)$$

for all $x,y,z \in \mathbb{R}$. Then it is clear that $(\mathbb{R},G_s)$ is a $G$–metric space.

**Proposition 1.4.** [5] Let $(X,G)$ be a $G$–metric space. Then $T:X \to X$ is $G$–continuous at $x \in X$ if and only if it is $G$–sequentially continuous at $x$, that is, whenever $(x_n)$ is $G$–convergent to $x$, $(f(x_n))$ is $G$–convergent to $f(x)$.

**Definition 1.5** [4] Let $(X,G)$ be a $G$–metric space. A mapping $F:X \times X \to X$ is said to be
continuous if for any two \( G \)-convergent sequences \((x_n)\) and \((y_n)\) converging to \(x\) and \(y\) respectively, \((F(x_n, y_n))\) is \( G \)-convergent to \(F(x, y)\).

**Definition 1.6** [3] An element \((x, y) \in X \times X\) is called a coupled fixed point of the mapping \(F: X \times X \to X\) if \(F(x, y) = x, F(y, x) = y\).

**Definition 1.7** [9] An element \((x, y) \in X \times X\) is called a coupled coincidence point of a mapping \(F: X \times X \to X\) and a mapping \(g: X \to X\) if \(F(x, y) = gx, F(y, x) = gy\).

Note that if \(g\) is the identity mapping, then Definition 1.7 reduces to Definition 1.6.

**Definition 1.8** [1] An element \(x \in X\) is called a common fixed point of a mapping \(F: X \times X \to X\) and \(g: X \to X\) if \(F(x, x) = gx = x\).

Abbas et al. [1] introduced the concept of \(w\)-compatible and \(w^*\)-compatible mappings and utilized this concept to prove an interesting uniqueness theorem of a coupled fixed point for mappings \(F\) and \(g\) in cone metric spaces.

**Definition 1.9** [1] Mappings \(F: X \times X \to X\) and \(g: X \to X\) are called

\((w_1)\) \(w\)-compatible if \(g(F(x, y)) = F(gx, gy)\) whenever \(gx = F(x, y)\) and \(gy = F(y, x)\).

\((w_2)\) \(w^*\)-compatible if \(g(F(x, x)) = F(gx, gx)\) whenever \(gx = F(x, x)\).

**Example 1.2.** [2] Let \(X = \mathbb{R}^+\), define \(F: X \times X \to X\) and \(g: X \to X\) by

\[
F(x, y) = \begin{cases} 
8, & x = 1, y = 0, \\
10, & x = 0, y = 1, \quad \text{and} \\
4 & \text{other wise,}
\end{cases} \quad g(x) = \begin{cases} 
8, & x = 1, \\
10, & x = 0, \\
5, & x = 4, \\
4, & \text{other wise.}
\end{cases}
\]

Then it is clear that \(F\) and \(g\) are \(w\)-compatible but not \(w^*\)-compatible.

**Definition 1.10** [9] Let \(X\) be a nonempty set and \(F: X \times X \to X\) and \(g: X \to X\). One says \(F\) and \(g\) are commutative if for all \(x, y \in X\), \(F(gx, gy) = g(F(x, y))\).

2. Main results

Our first result is the following.

**Theorem 2.1** Let \((X, G)\) be a \(G\)-metric space. Set \(T: X \times X \to X\) and \(g: X \to X\). Assume
there exist $a_1, a_2, a_3 \geq 0$ with $2a_1 + 3a_2 + 3a_3 < 2$ such that

$$G(T(x, y), T(u, v), T(w, z)) \leq \frac{a_1}{2}[G(gx, gu, gw) + G(gy, gv, gz)]$$

$$+ \frac{a_2}{2}[G(gx, T(x, y), T(x, y)) + G(gu, T(u, v), T(u, v)) + G(gy, gv, gz)]$$

$$+ \frac{a_3}{2}[G(gx, T(u, v), T(u, v)) + G(gu, T(x, y), T(x, y)) + G(gy, gv, gz)],$$

for all $x, y, u, v, w, z \in X$. If $T(X \times X) \subseteq g(X)$, $g(X)$ is a $G-$complete subset of $X$, then $T$ and $g$ have a unique common coupled coincidence point. Moreover, if $T$ is $w^*$-compatible with $g$, then $T$ and $g$ have a unique common coupled fixed point.

**Proof.** Let $x_0$ and $y_0$ be in $X$. Since $T(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = T(x_0, y_0)$ and $gy_1 = T(y_0, x_0)$. Analogously, there exist $x_2, y_2 \in X$ such that $gx_2 = T(x_1, y_1)$ and $gy_2 = T(y_1, x_1)$. Continuing this process, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$gx_{n+1} = T(x_n, y_n) \text{ and } gy_{n+1} = T(y_n, x_n) \text{ for all } n \geq 0$$

From by (2.1), we have

$$G(gx_n, gx_{n+1}, gx_{n+1}) = G(T(x_{n-1}, y_{n-1}), T(x_n, y_n), T(x_n, y_n))$$

$$\leq \frac{a_1}{2}[G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)]$$

$$+ \frac{a_2}{2}[G(gx_{n-1}, T(x_{n-1}, y_{n-1}), T(x_{n-1}, y_{n-1})) + G(gx_n, T(x_n, y_n), T(x_n, y_n)) + G(gy_{n-1}, gy_n, gy_n)]$$

$$+ \frac{a_3}{2}[G(gx_{n-1}, T(u, v), T(u, v)) + G(gu, T(x, y), T(x, y)) + G(gy, gv, gz)]$$

$$= \frac{a_1}{2}[G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)]$$

$$+ \frac{a_2}{2}[G(gx_{n-1}, gx_n, gx_n) + G(gx_n, gx_{n+1}, gx_{n+1}) + G(gy_{n-1}, gy_n, gy_n)]$$

$$+ \frac{a_3}{2}[G(gx_{n-1}, gx_{n+1}, gx_{n+1}) + G(gx_n, gx_n, gx_n) + G(gy_{n-1}, gy_n, gy_n)].$$

Thus, we obtain

$$G(gx_n, gx_{n+1}, gx_{n+1}) \leq \frac{a_1 + a_2 + a_3}{2 - a_2 - a_3}[G(gx_{n-1}, gx_n, gx_n) + G(gy_{n-2}, gy_{n-1}, gy_{n-1})],$$

(2.3)
and
\[ G(g_{n+1}, g_{n+1}, g_{n+1}) = G(T(y_{n-1}, x_{n-1}), T(y_n, x_n), T(y_n, x_n)) \]
\[ \leq \frac{a_1}{2} [G(g_{n-1}, g_{n-1}, g_{n}) + G(g_{n-1}, g_{n}, g_{n})] \]
\[ + \frac{a_2}{2} [G(g_{n+1}, T(y_{n-1}, x_{n-1}), T(y_{n-1}, x_{n-1})) + G(g_{n}, T(y_{n}, x_0), T(y_{n}, x_0))] \]
\[ + G(g_{n-1}, g_{n}, g_{n})] + \frac{a_3}{2} [G(g_{n+1}, T(y_{n-1}, x_{n-1}), T(y_{n-1}, x_{n-1})) + G(g_{n-1}, g_{n}, g_{n})] \]
\[ = \frac{a_1}{2} [G(g_{n-1}, g_{n-1}, g_{n}) + G(g_{n-1}, g_{n}, g_{n})] \]
\[ + \frac{a_2}{2} [G(g_{n-1}, g_{n-1}, g_{n}) + G(g_{n}, g_{n+1}, g_{n+1}) + G(g_{n-1}, g_{n}, g_{n})] \]
\[ + \frac{a_3}{2} [G(g_{n-1}, g_{n+1}, g_{n+1}) + G(g_{n}, g_{n}, g_{n}) + G(g_{n-1}, g_{n}, g_{n})] \]

Thus, we obtain
\[ G(g_{n+1}, g_{n+1}, g_{n+1}) \leq \frac{a_1 + a_2 + a_3}{2 - a_2 - a_3} [G(g_{n-1}, g_{n}, g_{n}) + G(g_{n-1}, g_{n}, g_{n})]. \] (2.4)

From (2.3) and (2.4), we have
\[ G(g_{n}, g_{n+1}, g_{n+1}) + G(g_{n}, g_{n+1}, g_{n+1}) \leq \frac{2(a_1 + a_2 + a_3)}{2 - a_2 - a_3} [G(g_{n-1}, g_{n}, g_{n}) + G(g_{n-1}, g_{n}, g_{n})]. \]

Set \( a_n = G(g_{n}, g_{n+1}, g_{n+1}) + G(g_{n}, g_{n+1}, g_{n+1}) \) and \( \lambda = \frac{2(a_1 + a_2 + a_3)}{2 - a_2 - a_3} \), then the sequence \( \{a_n\} \) is decreasing as
\[ 0 \leq a_n \leq \lambda a_{n-1} \leq \lambda^2 a_{n-2} \leq \ldots \leq \lambda^n a_0 \]

which implies
\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} [G(g_{n}, g_{n+1}, g_{n+1}) + G(g_{n}, g_{n+1}, g_{n+1})] = 0. \]

Thus,
\[ \lim_{n \to \infty} G(g_{n}, g_{n+1}, g_{n+1}) = 0, \quad \text{and} \quad \lim_{n \to \infty} G(g_{n}, g_{n+1}, g_{n+1}) = 0. \] (2.5)

Next, let us prove that \( \{g_{n}\} \) and \( \{g_{n}\} \) are \( G - \)Cauchy sequences. In fact, for \( m > n \), we have
\[ G(g_{n}, g_{m}, g_{m}) + G(g_{n}, g_{m}, g_{m}) \leq G(g_{n}, g_{m}, g_{m}) + G(g_{n}, g_{m}, g_{m}) + G(g_{n}, g_{m+1}, g_{m+1}) \]
\[ + G(g_{n+1}, g_{m+1}, g_{m+1}) + G(g_{n+1}, g_{m+2}, g_{m+2}) \]
\[ + \ldots + G(gx_{m-1}, gx_m, gx_m) + G(gy_{m-1}, gy_m, gy_m) \]
\[ = a_n + a_{n+1} + \ldots + a_{m-1} \]
\[ \leq \lambda^n a_0 + \lambda^{n+1} a_0 + \ldots + \lambda^{m-1} a_0 = (\lambda^n + \lambda^{n+1} + \ldots + \lambda^{m-1})a_0 \]
\[ \leq \frac{\lambda^n}{1 - \lambda} a_0. \]

Letting \( n, m \to \infty \), we have
\[ \lim_{n, m \to \infty} G(gx_n, gx_m, gx_m) + G(gy_n, gy_m, gy_m) = 0. \]

This imply that \( \{gx_n\} \) and \( \{gy_n\} \) are \( G \)-Cauchy sequences in \( g(X) \). By \( G \)-completeness of \( g(X) \), there exists \( gx, gy \in g(X) \) such that \( \{gx_n\} \) and \( \{gy_n\} \) converge to \( gx \) and \( gy \), respectively. We claim that \( g(x) = T(x, y) \) and \( g(y) = T(y, x) \). Indeed, from (2.1), we have
\[ G(gx_{n+1}, T(x, y), T(x, y)) = G(T(x, y_n), T(x, y), T(x, y)) \]
\[ \leq \frac{a_1}{2}[G(gx_n, g(x), g(x)) + G(gy_n, g(y), g(y))] \]
\[ + \frac{a_2}{2}[G(gx_n, T(x, y_n), T(x, y_n)) + G(g(x), T(x, y), T(x, y)) \]
\[ + G(gy_n, g(y), g(y))] + \frac{a_3}{2}[G(gx_n, T(x, y), T(x, y)) \]
\[ + G(g(x), T(x, y_n), T(x, y_n) + G(gy_n, g(y), g(y))]. \]

Letting \( n \to \infty \), and using the fact that \( G \) is continuous on its variables, we get that
\[ G(g(x), T(x, y), T(x, y)) \leq \frac{a_2 + a_3}{2} G(g(x), T(x, y), T(x, y)). \]

Hence \( g(x) = T(x, y) \). Similarly, we may show that \( g(y) = T(y, x) \). Then, \( (gx, gy) \) is a coupled point of coincidence of mappings \( T \) and \( g \). Now we prove that \( gx = gy \). By (2.1), we have
\[ G(g(x), g(y), g(y)) = G(T(x, y), T(y, x), T(y, x)) \]
\[ \leq \frac{a_1}{2}[G(g(x), g(y), g(y)) + G(g(y), g(x), g(x))] \]
\[ + \frac{a_2}{2}[G(g(x), T(x, y), T(x, y)) + G(g(y), T(y, x), T(y, x)) + G(g(y), g(x), g(x))] \]
\[ + \frac{a_3}{2}[G(g(x), T(y, x), T(y, x)) + G(g(y), T(x, y), T(x, y)) + G(g(y), g(x), g(x))]. \]
\[
= \frac{a_1 + a_2}{2} G(g(x), g(y), g(y)) + \frac{a_1 + a_2 + 2a_3}{2} G(g(y), g(x), g(x)).
\]

Similarly, we may show that
\[
G(g(y), g(x), g(x)) \leq \frac{a_1 + a_2 + 2a_3}{2} G(g(x), g(y), g(y)) + \frac{a_1 + a_2}{2} G(g(y), g(x), g(x)).
\]

Therefore
\[
G(g(x), g(y), g(y)) + G(g(y), g(x), g(x)) \leq \frac{2a_1 + a_2 + 3a_3}{2} [G(g(x), g(y), g(y)) + G(g(y), g(x), g(x))]
\]

\[
< G(g(x), g(y), g(y)) + G(g(y), g(x), g(x)).
\]

which is a contradiction. So \( g(x) = g(y) \). We conclude that \( T(x, y) = g(x) = g(y) = T(y, x) \).

Thus, \( (g(x), g(x)) \) is a coupled point of coincidence of mappings \( T \) and \( g \). Now, if there is another \( x_i \in X \) such that \( (g(x_i), g(x_i)) \) is a coupled point of coincidence of mappings \( T \) and \( g \), then

\[
G(g(x), g(x_i), g(x_i)) = G(T(x, x), T(x_i, x_1), T(x_i, x_i))
\]

\[
\leq \frac{a_1}{2} [G(g(x), g(x_i), g(x_i)) + G(g(x), g(x_i), g(x_i))]
\]

\[
+ \frac{a_2}{2} [G(g(x), T(x, x), T(x, x)) + G(g(x), T(x_i, x_1), T(x_i, x_i)) + G(g(x), g(x_i), g(x_i))]
\]

\[
+ \frac{a_3}{2} [G(g(x), T(x_i, x_1), T(x_i, x_i)) + G(g(x_i), T(x, x), T(x, x)) + G(g(x), g(x_i), g(x_i))]
\]

\[
= (a_1 + a_2 + a_3)G(g(x), g(x_i), g(x_i)) + \frac{a_3}{2} G(g(x_i), g(x), g(x)).
\]

Similarly, we may show that
\[
G(g(x_i), g(x), g(x)) \leq (a_1 + a_2 + a_3)G(g(x_i), g(x), g(x)) + \frac{a_3}{2} G(g(x), g(x_i), g(x_i)).
\]

Therefore
\[
G(g(x), g(x_i), g(x_i)) + G(g(x_i), g(x), g(x)) \leq \frac{2a_1 + 2a_2 + 3a_3}{2} [G(g(x), g(x_i), g(x_i)) + G(g(x_i), g(x), g(x))].
\]

It implies that \( G(g(x), g(x_i), g(x_i)) = G(g(x_i), g(x), g(x)) = 0 \) and so \( g(x) = g(x_i) \). Hence, \( (g(x), g(x)) \) is a unique coupled point of coincidence of mappings \( T \) and \( g \). Now, we show that \( T \) and \( g \) have common coupled fixed point. For this, let \( u = g(x) \). Then, we have \( u = g(x) = T(x, x) \).

By \( w^* \) – compatibility of \( T \) and \( g \), we have

\[
g(u) = g(g(x)) = g(T(x, x)) = T(g(x), g(x)) = T(u, u).
\]

Then, \( (g(u), g(u)) \) is a coupled point of coincidence of mappings \( T \) and \( g \). By the uniqueness of
coupled point of coincidence, we have \( g(x) = g(u) \). Therefore, \((u, u)\) is the common coupled fixed point of \( T \) and \( g \). To prove the uniqueness, let \( v \in X \) with \( v \neq u \) such that \((v, v)\) is the common coupled fixed point of \( T \) and \( g \). Then, using (2.1),

\[
G(u, v, v) = G(T(u, u), T(v, v), T(v, v)) \leq \frac{a_1}{2}[G(gu, gv, gv) + G(gu, gv, gv)]
\]

\[
+ \frac{a_2}{2}[G(gu, T(u, u), T(u, u)) + G(gv, T(v, v), T(v, v)) + G(gu, gv, gv)]
\]

\[
+ \frac{a_3}{2}[G(gu, T(v, v), T(v, v)) + G(gv, T(u, u), T(u, u)) + G(gu, gv, gv)]
\]

\[
= (a_1 + a_2 + a_3)G(u, v, v) + \frac{a_3}{2}G(v, u, u).
\]

Similarly, we may show that

\[
G(v, u, u) \leq (a_1 + a_2 + a_3)G(v, u, u) + \frac{a_1}{2}G(u, v, v).
\]

Hence,

\[
G(u, v, v) + G(v, u, u) \leq \frac{2a_1 + a_2 + 3a_3}{2}[G(u, v, v) + G(v, u, u)].
\]

Since \( \frac{2a_1 + a_2 + 3a_3}{2} < 1 \), so that \( G(u, v, v) = G(v, u, u) = 0 \) and \( u = v \). Thus \( T \) and \( g \) have a unique common coupled fixed point. In Theorem 2.1, take \( w = u \) and \( z = v \), to obtain the following corollary.

**Corollary 2.2** Let \((X, G)\) be a \( G \)-metric space. Set \( T : X \times X \to X \) and \( g : X \to X \). Assume there exist \( a_1, a_2, a_3 \geq 0 \) with \( 2a_1 + 3a_2 + 3a_3 < 2 \) such that

\[
G(T(x, y), T(u, v), T(u, v)) \leq \frac{a_1}{2}[G(gx, gu, gu) + G(gy, gv, gv)]
\]

\[
+ \frac{a_2}{2}[G(gx, T(x, y), T(x, y)) + G(gu, T(u, v), T(u, v)) + G(gy, gv, gv)]
\]

\[
+ \frac{a_3}{2}[G(gx, T(u, v), T(u, v)) + G(gu, T(x, y), T(x, y)) + G(gy, gv, gv)],
\]

for all \( x, y, u, v, w, z \in X \). If \( T(X \times X) \subseteq g(X) \), \( g(X) \) is a \( G \)-complete subset of \( X \), then \( T \) and \( g \) have a unique common coupled coincidence point. Moreover, if \( T \) is \( w^* \)-compatible with \( g \), then \( T \) and \( g \) have a unique common coupled fixed point.

Now, putting \( g = I_x \) (the identity map of \( X \)) in the Theorem 2.1, we obtain

**Corollary 2.3** Let \((X, G)\) be a complete \( G \)-metric space. Assume \( T : X \times X \to X \) be a function
satisfying (2.1) (with \( g = I_X \) ) for all \( x, y, u, v, w, z \in X \). Then \( T \) has a unique fixed point.

By choosing \( a_1, a_2, \) and \( a_3 \) suitably, one can deduce some corollaries from Theorem 2.1.

For example, if \( a_1 = 2k \) and \( a_2 = a_3 = 0 \) in Theorem 2.1, then the following corollary is obtained which extends and generalizes the comparable results of [10].

**Corollary 2.4** Let \((X, G)\) be a \( G \)-metric space. Set \( T : X \times X \rightarrow X \) and \( g : X \rightarrow X \). Assume there exist \( k \in [0, \frac{1}{2}) \) such that

\[
G(T(x, y), T(u, v), T(w, z)) \leq k[G(gx, gu, gw) + G(gy, gv, gz)],
\]

for all \( x, y, u, v, w, z \in X \). If \( T(X \times X) \subseteq g(X) \), \( g(X) \) is a \( G \)-complete subset of \( X \), then \( T \) and \( g \) have a unique common coupled coincidence point. Moreover, if \( T \) is \( w^* \)-compatible with \( g \), then \( T \) and \( g \) have a unique common coupled fixed point.

Now, we introduce an example to support the usability of our results.

**Example 2.1.** Let \( X = [0, 1] \). Define \( T : X \times X \rightarrow X \) by \( T(x, y) = \frac{1}{16} x^2 y^2 \) and define \( g : X \rightarrow X \) by \( g(x) = \frac{1}{2} x^2 \).

Define a \( G \)-metric on \( X \) by \( G(x, y, z) = |x - y| + |x - z| + |y - z| \) for all \( x, y, z \in X \).

By routine calculations, the reader can easily verify that the following assumptions hold:

1. \( T(X \times X) \subseteq g(X) \);
2. \( g(X) \) is a \( G \)-complete subset of \( X \);
3. \( T \) is \( w^* \)-compatible with \( g \).

Here, we show only that \( T \) and \( g \) are condition (2.1) in Theorem 2.1 is satisfied for all real numbers \( a_1, a_2, a_3 \) with \( 0 \leq 2a_1 + 3a_2 + 3a_3 < 2 \). Since \( |xy - uv| \leq |x - u| + |y - v| \) holds for all \( x, y, u, v \in X \), we have

\[
G(T(x, y), T(u, v), T(w, z)) = G\left(\frac{1}{16} x^2 y^2, \frac{1}{16} u^2 v^2, \frac{1}{16} w^2 z^2\right)
= \frac{1}{16} |x^2 y^2 - u^2 v^2| + \frac{1}{16} |x^2 y^2 - w^2 z^2| + \frac{1}{16} |u^2 v^2 - w^2 z^2|
\]
\[
\frac{1}{16} \left[ \left| x^2 - u^2 \right| + \left| y^2 - v^2 \right| + \left| x^2 - w^2 \right| + \left| y^2 - z^2 \right| + \left| u^2 - w^2 \right| + \left| v^2 - z^2 \right| \right]
\]
\[
\leq \frac{1}{8} \left[ \frac{1}{2} x^2 - \frac{1}{2} u^2 \right] + \left| \frac{1}{2} x^2 - \frac{1}{2} w^2 \right| + \left| \frac{1}{2} u^2 - \frac{1}{2} w^2 \right| + \left| \frac{1}{2} y^2 - \frac{1}{2} v^2 \right| + \left| \frac{1}{2} y^2 - \frac{1}{2} z^2 \right| + \left| \frac{1}{2} v^2 - \frac{1}{2} z^2 \right| + \left| \frac{1}{2} x^2 - \frac{1}{2} y^2 \right| + \left| \frac{1}{2} x^2 - \frac{1}{2} z^2 \right| + \left| \frac{1}{2} u^2 - \frac{1}{2} x^2 \right| + \left| \frac{1}{2} u^2 - \frac{1}{2} y^2 \right| + \left| \frac{1}{2} u^2 - \frac{1}{2} z^2 \right| \]
\[
\leq \frac{1}{2} \left[ G(gx, gu, gw) + G(gy, gv, gz) \right]
\]
\[
+ \frac{1}{2} \left[ G(gx, T(x, y), T(x, y)) + G(gu, T(u, v), T(u, v)) + G(gy, gv, gz) \right]
\]
\[
+ \frac{1}{2} \left[ G(gx, T(u, v), T(u, v)) + G(gu, T(x, y), T(x, y)) + G(gy, gv, gz) \right].
\]

Thus, (2.1) is satisfied with \( a_1 = \frac{1}{4} \) and \( a_2 = a_3 = \frac{1}{8} \) where \( 2a_1 + 3a_2 + 3a_3 < 2 \). Hence, all the conditions of Theorem 2.1 are satisfied. Moreover, \((0,0)\) is the unique common coupled fixed point of \( T \) and \( g \).

References


