Boundedness and Probabilistic Radius In Probabilistic Norme Spaces

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Abstract

In this paper, we study the boundedness property in probabilistic normed spaces and also we consider the probabilistic radius R_A in Probabilistic normed spaces.

Introduction

Probabilistic normed spaces (PN spaces henceforth) were introduced by Šerstnev in [5] by means of a definition that was closely modeled onthe theory of normed spaces. Here we consistently adopt the now, andin our opinion convincing, definition of PN space given in the paperby Alsina, Schweizer, and Sklar [1] .We recall it. The notation and the concepts used are those of [4,1 and 2].

Definition 1.1. A Probabilistic normed space (briefly a PN space) is a quadruple (V, ν, τ, τ^*), Where V is a real vector space, τ and τ^* are continuous triangle functions with $\tau \le \tau^*$ and ν is a mapping (the probabilistic norm) $\nu : V \to \Delta^+$ such that for every choice of p and q in V the following conditions hold:

(N_1) $\nu_p = \varepsilon_0$ if, and only if, $p = \theta$ (θ is the null vector in V);

(N_2) $\nu_{-p} = \nu_p$;

(N_3) $\nu_{p+q} \ge \tau(\nu_p, \nu_q)$;

(N_4) $\nu_p \le \tau^*(\nu_{\lambda p}, \nu_{(1-\lambda)p})$ for every $\lambda \in [0, 1]$.

A Menger PN space under T is a PN space (V, ν, τ, τ^*) in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, for some continuous t-norm T and its t-conorm T^*; it is denoted by (V, ν, T).

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A PN space is called a Šerstnev space if it satisfies (N_1) and (N_3) and the following condition, $\nu_{\alpha p}(x) = \nu_p(x/|\alpha|)$, for every $\alpha \in R - \{0\}$ and for every $x > 0$, (Š) which clearly implies (N_2) and also (see [1]) (N_4) in the strengthened form, $\nu_p = \tau_M(\nu_{\lambda p}, \nu_{(1-\lambda)p})$, for every λ in [0, 1].

There is a natural topology in a PN space (V, ν, τ, τ^*), called the strong topology; it is defined by the neighbourhoods,

$$N_p(t) := \{q \in V : \nu_{q-p}(t) > 1 - t\} = \{q \in V : d_s(\nu_{q-p}, \varepsilon_0) < t\},$$

Where $t > 0$. Here d_s is the modified Sibly metric [6].

Given a nonempty set A in a PN space (V, ν, τ, τ^*) its Probabilistic radius R_A is defined by

$$R_A(x) := \begin{cases} L^- \Phi_A(x), & x \in [0, +\infty[, \\ 1, & x = +\infty, \end{cases}$$

Where $L^-f(x)$ denotes the left limit of the function f at the point x and
$\Phi_A(x) := \inf\{\nu_p(x) : p \in A\}$.

The following definition sharpens that of [4, section 4.4] as we detail in section 4.

Definition 1.2. A nonempty set A in a PN space (V, ν, τ, τ^*) is said to be:

(a) certainly bounded, if $R_A(x_0) = 1$ for some $x_0 \in]0, +\infty[$;

(b) perhaps bounded, if one has $R_A(x) < 1$, for every $x \in]0, +\infty[$, and $L^- R_A(+\infty) = 1$;

(c) perhaps unbounded, if $R_A(x_0) > 0$ for some $x_0 \in]0, +\infty[$ and $L^- R_A(+\infty) \in]0, 1[$;

(d) certainly unbounded, if $L^- R_A(+\infty) = 0$, i.e., if $R_A = \varepsilon_\infty$.

Moreover, A will be said to be D-bounded if either (a) or (b) holds, i.e., if $R_A \in D^+$; otherwise, i.e., if $R_A \in \Delta^+ \setminus D^+$, A will said to be D-unbounded.

Theorem 1.3. In a PN space (V, ν, τ, τ^*), the probabilistic radius has the following properties:

(a) for every nonempty set A, $R_A = R_{A^-}$ where A^- denotes the closure of A in the strong topology;

(b) $R_{A \cup B} \geq \tau(R_A, R_B)$, if A and B are nonempty.

Proof. (a) Because $A \subset A^-$, and as a consequence, $R_A \geq R_{A^-}$, one has Only to show the converse inequality $R_A \leq R_{A^-}$. When (V, ν, τ, τ^*) is endowed with the strong topology and Δ^+ is endowed with the topology of weak convergence, i.e., the topology of the modified Sibly metric d_s ,the probabilistic norm $\nu : V \to \Delta^+$ is uniformly continuous (see [2]); in other words, for every $\eta \in]0, 1[$ there exists $\delta = \delta(\eta) > 0$ such that $d_s(\nu_p, \nu_q) < \eta$ whenever $d_s(\nu_{p-q}, \varepsilon_0) < \delta$.

Now, for every $p \in A^-$, there exists $q(p) \in A$ such that

$$d_s(\nu_{p-q(p)}, \varepsilon_0) < \delta;$$

therefore $d_s(\nu_p, \nu_{q(p)}) < \eta$. In particular, for every $t \in]0,1/\eta[$, we have

$$\nu_p(t) \geq \nu_{q(p)}(t - \eta) - \eta.$$

Then, for $t \in]0,1/\eta[$,

$$\Phi_{A^-}(t) = \inf_{p\in A^-} \nu_p(t) \geq \inf_{p\in A^-} \nu_{q(p)}(t-\eta) - \eta$$

$$= \inf_{p\in A} \nu_{q(p)}(t-\eta) - \eta$$

$$\geq \inf_{p\in A} \nu_{q(p)}(t-\eta) - \eta = \Phi_A(t-\eta) - \eta.$$

Therefore, if $t \in]0,1/\eta[$, then

$$R_{A^-}(t) \geq R_A(t-\eta) - \eta.$$

This latter inequality holds for every $\eta \in]0, 1[$ and for every $t \in]0, 1/\eta[$.Thus, letting $\eta \to 0$ and using the left-continuity of R_A yields that, for every $t > 0$,

$$R_{A^-}(t) \geq R_A(t).$$

(b) For every $p \in A \cup B$ and for every $q \in B$ we have that

$$\nu_p = \tau(\nu_p, \varepsilon_0) \geq \tau(\nu_p, \nu_q) \geq \tau(\nu_p, R_B),$$

because $R_B \leq \nu_q$ for all $q \in B$. Therefore, if $p \in A$ we have

$$\nu_p \geq \tau(R_A, R_B).$$

Repeating the same argument for $p \in A \cup B$ and $q \in A$ leads tothe inequality $\nu_p \geq \tau(R_A, R_B)$ for every $p \in B$. Now the last two inequalities yield the assertion. □

Definition 1.4. The PN space (V, ν, τ, τ^*) is said to satisfy the DI-condition if the probabilistic norm ν is such that, for all $\alpha \in R\backslash\{0\}$, $x \in R$ and $p \in V$,

$$\nu_{\alpha p}(x) = \nu_p(\Phi(\alpha, x)),$$

where $\Phi : R \times [0, +\infty[\to [0, +\infty[$ satisfies

$$\lim_{x\to+\infty} \Phi(\alpha, x) = +\infty \text{ and } \lim_{\alpha\to 0} \Phi(\alpha, x) = +\infty.$$

Theorem 1.5. Let (V, ν, τ, τ^*) be a PN space that satisfies the DI-condition. Then for a subset $A \subset V$ the following statements are equivalent:

(a) A is D-bounded.

(b) A is bounded, namely, for every $n \in N$ and for every $p \in A$, there is $k \in N$ such that $\nu_{p/k}(1/n) > 1-1/n$.

(c) A is topologically bounded.

Proof. (a) ⇒ (b). Let A any D-bounded subset of V. By definition of D-boundedness, the probabilistic radius R_A of A (see[3]) is a distance d.f. such that $\lim_{x\to+\infty} R_A(x) = 1$, therefore, for every n ∈ N there exists $x_n > 0$ such that $R_A(x_n) > 1 - 1/n$.

Thus, for every p ∈ A,

$$\nu_p(x_n) \geq R_A(x_n) > 1 - 1/n .$$

Since, in view of the DI-condition,

$$\lim_{\lambda\to 0} \Phi(\lambda, 1/n) = +\infty,$$

for every n ∈ N, there exists λ′ ∈ R such that $\Phi(\lambda', 1/n) \geq x_n$.

Therefore,

$$\nu_{\lambda' p}(1/n) = \nu_p(\Phi(\lambda', 1/n)) \geq \nu_p(x_n) \geq R_A(x_n) > 1 - 1/n.$$

As a consequence letting k = [1/ λ′], where [t] denotes the integral partof t, one has

$$\nu_{p/k}(1/n) > 1 - 1/n ,$$

namely, A is bounded.

(b) ⇒ (a). Let A be a bounded subset of V and consider the neighbourhood of $N_\theta(1/n)$.

Then there exists λ_0 ∈R such that for every p ∈ V, $p = \lambda_0 q$ for some $q \in N_\theta(1/n)$.

Because of the DI-condition,

$$\lim_{x\to+\infty} \Phi(\lambda_0, x) = +\infty,$$

for every n ∈ N ; then, there exists $x_0 > 0$ such that, $\Phi(\lambda_0, x_0) \geq 1$. Then, for $x \geq x_0$,

$$\nu_p(x) = \nu_{\lambda_0 q}(x) \geq \nu_{\lambda_0 q}(x_0) = \nu_q(\Phi(\lambda_0, x_0)) \geq \nu_q(1) \geq \nu_q(1/n) > 1 - 1/n ,$$

So that

$$R_A(x) \geq 1 - 1/n ,$$

i.e., R_A is in D^+ .

(a) ⇒ (c). Let A any D-bounded subset of V. One has, as above,

$$\nu\alpha_n p_n (x_n) = \nu p_n (\Phi(\alpha_n , x_n)) \geq R_A (x_n) > 1 - 1/n,$$

which implies

$$\alpha_n p_n \longrightarrow \theta_{n\to+\infty} .$$

(c) ⇒ (a). Let A be a subset of V which is not D-bounded. Then

$$\lim_{x\to+\infty} R_A (x) = \gamma < 1.$$

By definition of R_A, for every $n \in N$ there is $p_n \in A$ such that, for every $x > 0$,

$$\nu_{pn} (x) < 1+\gamma/2< 1.$$

Then for every $x > 0$,

$$\nu\alpha_n p_n (x) = \nu_{pn} (\Phi(\alpha_n , x)) <1+\gamma/2<1,$$

which shows that $\nu\alpha_n p_n$ does not tend to ε_0, even if it has a weak limit, viz., $(\alpha_n p_n)$ does not tend to θ in the strong topology; in other words, A is not topologically bounded. □

Example 1.6. Let (V, ‖ 0 ‖) be a normed space and, for $\alpha \in]0, 1[$, Let

$\nu : V \longrightarrow \Delta^+$ be given by

$$\nu_p (x) = \begin{cases} 0, & x\leq 0 \\ \ln(1 + x)/ \ln(1 + x)+ \| p \| , & x \in]0, +\infty[, \| p \|< 1 \\ a\ln(1 + x)/ \ln(1 + x)+ \| p \| , & x \in]0, +\infty[, \| p \|\geq 1 \\ 1, & x=+\infty \end{cases}$$

Then

(1) $(V, \nu, \tau_\pi , \tau_M)$ is a PN space satisfying the DI-condition with

$$\Phi(\lambda, x) = (1 + x)^{1/\|\lambda\|} - 1;$$

(2) $(V, \nu, \tau_\pi , \tau_M)$ is a TV PN space;

(3) the subset $A = \{p : \| p \|< 1\}$ is both D-bounded and bounded. Only property (2) needs to be checked. For every sequence (λ_n) Of real numbers that converges to 0 as n tends to +∞, and for every $p \in V$, one has $\lambda_n p \to \theta$ in the strong topology of V; in fact, For every $x \in R^+$, $\lim_{n\to\infty} \nu_{\lambda n}\, p (x) = 1$, namely,

$$\lim \nu_{\lambda n}\, p = \varepsilon_0 .$$

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