Non-Bayes, Bayes and Empirical Bayes Estimators for the Shape Parameter of Lomax Distribution

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Abstract

Point estimation is one of the core topics in mathematical statistics. In this paper we consider the most common methods of point estimation: non-Bayes, Bayes and empirical Bayes methods to estimate the shape parameter of Lomax distribution based on complete data. The maximum likelihood, moment and uniformly minimum variance unbiased estimators are obtained as non-Bayes estimators. Bayes and empirical Bayes estimators are obtained corresponding to three informative priors "gamma, chi-square and inverted Levy" based on symmetric "squared error" and asymmetric "LINEX and general entropy" loss functions. The estimates of the shape parameter were compared empirically via Monte Carlo simulation study based upon the mean squared error. Among the set of conclusions that have been reached, it is observed that, for all sample sizes and different cases, the performance of uniformly minimum variance unbiased estimator is better than other non-Bayes estimators. Further that, Monte Carlo simulation results indicate that the performance of Bayes and empirical Bayes estimator in some cases are better than non-Bayes for some appropriate of prior distribution, loss function, values of parameters and sample size.

Keywords: Lomax distribution; maximum likelihood estimator; moment estimator; uniformly minimum variance unbiased estimator; Bayes estimator; empirical Bayes estimator; informative prior; squared error loss function; LINEX loss function; general entropy loss function; mean squared error.

1. Introduction

Lomax distribution "also called Pareto type-II distribution or Pearson type-VI distribution [2]" introduced and studied by Lomax (1954) [19]. He used this distribution to analyze business failure data. The probability density function and cumulative distribution function of two parameters Lomax distribution, Lomax (λ, β), are given by [7]:

$$f(t;\lambda,\beta)=\frac{\lambda}{\beta}\left(1+\frac{t}{\beta}\right)^{-(\lambda+1)} \quad ; t > 0 ; \lambda,\beta>0 \qquad (1)$$

$$F(t;\lambda,\beta)=1-\left(1+\frac{t}{\beta}\right)^{-\lambda} \quad ; t > 0 ; \lambda,\beta>0 \qquad (2)$$

The r^{th} moment about the origin of the Lomax distribution is:

$$E(t^r)=\lambda\,\beta^r\,\frac{\Gamma(r+1)\,\Gamma(\lambda-r)}{\Gamma(\lambda+1)} \; ; \lambda>r \; ; r=1,2,\ldots \qquad (3)$$

Where, λ and β are shape and scale parameters, respectively.

Lomax distribution has been received greatest attention from theoretical and statisticians primarily due to its use in reliability and lifetime testing studies [13][22]. Dubey (1970) [12] showed that Lomax distribution can be derived as a special case of a particular compound gamma distribution. Bryson (1974) [10] argued that Lomax distribution provide a very good alternative to common lifetime distributions like exponential, Weibull, or gamma distributions where the experimenter presumes that the population distribution may be heavy-tailed. Tadikamalla (1980)[28] related the Lomax distribution to the Burr family of distributions. Ahsanullah (1991) [4] studied the record statistics of the Lomax distribution with some distributional properties. Balakrishnan and Ahsanullah (1994) [9] obtained some recurrence relations between the moments of record values from Lomax distribution. Saran and Pushkarna (1999) [24] established some recurrence relations for both single and product moments of order statistics from a doubly truncated Lomax distribution. Habibullh and Ahsanullah (2000) [15] addressed the problem of estimating the parameters of Lomax distribution based on generalization order statistics. Al-Awadhi and Ghitany (2001) [5] used the Lomax distribution as a mixing distribution for the Poisson parameter and derived the discrete Poisson-Lomax distribution. Petropoulos and Kourouklis (2004) [23] considered the estimation of a quantile of the common marginal distribution in a multivariate Lomax distribution with unknown location and scale parameters. Nadarajah (2005) [20] derived several properties of the logarithm of the Lomax random variable. Abd Ellah (2006) [3] addressed the problem of Bayesian estimation along with maximum likelihood estimation of the parameters, reliability and hazard functions in the context of record statistics values. Abd-Elfattah et al. (2007) [1] considered the Lomax distribution as an important model of lifetime models and derived the non-Bayesian "maximum likelihood estimator" and Bayesian estimators of sample size in the case of type I censored samples. Kozubowski et al. (2009) [17] considered the problem of maximum likelihood estimation of the parameters. Abd-Elfattah and Alharbey (2010) [2] applied the generalized probability weighted moments method for estimating the parameters. Ashour and Abd-Elfattah (2011) [7] provided the maximum likelihood estimators for the unknown parameters of Lomax distribution and their variance covariance matrix under hybrid censored sample. Giles et al. (2011) [14] evaluated some of the small-sample properties of the maximum likelihood estimators. Nasiri and Hosseini (2012) [21] obtained maximum likelihood, moment and Bayes estimators of one parameter Lomax distribution based on record values. Bayes estimations are calculated for both informative and non-informative priors based on records for quadratic and squared error loss functions. El-Din et al. (2013) [13] discussed non- Bayes, Bayes and Empirical Bayes estimates for the parameters of Lomax model based on progressively type-II censored samples. Okasha (2014) [22] used Bayes and empirical Bayes approaches for obtaining the estimates of the unknown shape parameter and some other life time characteristics such as the reliability and hazard functions of Lomax distribution based on type-II censored data.

2. Different Estimation Methods

2.1 Non-Bayes Estimators of λ

In this subsection, the non-Bayes estimators for the shape parameter, λ, have been obtained in based on maximum likelihood estimation, moment estimation, and uniformly minimum variance unbiased estimation.

***<u>Maximum Likelihood Estimator</u>*:** Let $\underline{\boldsymbol{t}} = (\boldsymbol{t_1}, \boldsymbol{t_2}, \ldots\ldots, \boldsymbol{t_n})$ be the life time of a random sample of size n drawn independently from the Lomax distribution defined by (1). Then the likelihood function for the given sample observations is:

$$L(\lambda,\beta|\underline{t}) = \prod_{i=1}^{n} f(t_i|\lambda,\beta) \Longrightarrow L(\lambda,\beta|\underline{t}) = \frac{\lambda^n}{\beta^n} e^{-(\lambda+1)\sum_{i=1}^{n} ln\left(1+\frac{t_i}{\beta}\right)} \qquad (4)$$

The maximum likelihood estimator of λ, denoted by $\hat{\lambda}_{ML}$, yields by taking the derivative of the natural log-likelihood function with respect to λ and setting it equal to zero as:

$$\hat{\lambda}_{ML} = \frac{n}{\sum_{i=1}^{n} ln\left(1+\frac{t_i}{\beta}\right)} = \frac{n}{W} \quad ; \ w = \sum_{i=1}^{n} ln\left(1+\frac{t_i}{\beta}\right) \qquad (5)$$

***<u>Moment Estimator</u>*:** The moment estimator of the shape parameter of Lomax distribution, $\boldsymbol{\lambda}$, denoted by $\hat{\boldsymbol{\lambda}}_{\boldsymbol{MO}}$, yields by equating the first population moment to the first sample moment as:

$$\hat{\lambda}_{MO} = 1 + \frac{\beta}{\bar{t}} \qquad (6)$$

***<u>Uniformly Minimum Variance Unbiased Estimator</u>*:** The probability density function of Lomax distribution is belongs to exponential family. Therefore, $\boldsymbol{W} = \sum_{\mathbf{i=1}}^{\mathbf{n}} \boldsymbol{ln}\left(\mathbf{1}+\frac{\boldsymbol{t_i}}{\boldsymbol{\beta}}\right)$ is a complete sufficient statistic for $\boldsymbol{\lambda}$. Then, depending on the theorem of Lehmann-Scheffe [16], the uniformly minimum variance unbiased estimator of $\boldsymbol{\lambda}$, denoted by $\hat{\boldsymbol{\lambda}}_{\boldsymbol{UMVU}}$ is:

$$\hat{\lambda}_{UMVU} = \frac{n-1}{\sum_{i=1}^{n} ln\left(1+\frac{t_i}{\beta}\right)} = \frac{n-1}{W} \quad ; \ w = \sum_{i=1}^{n} ln\left(1+\frac{t_i}{\beta}\right) \qquad (7)$$

2.2 Bayes Estimators of λ

Previously, we have obtained the non-Bayes point estimators for parameter of interest. In non-Bayes approach, the parameter of interest λ had a fixed but unknown value. In this subsection, Bayesian context, the parameter λ is a random variable with posterior density function.

***<u>Prior and Posterior Density Functions</u>*:** From Bayes' rule the posterior probability density function of the parameter λ given $\underline{\boldsymbol{t}}$, $\boldsymbol{\pi}(\lambda|\underline{\boldsymbol{t}})$, can be expressed as:

$$\boldsymbol{\pi}(\lambda|\underline{\boldsymbol{t}}) = \frac{\boldsymbol{L}(\lambda|\underline{\boldsymbol{t}})\, \boldsymbol{g}(\lambda)}{\int_{\lambda} \boldsymbol{L}(\lambda|\underline{\boldsymbol{t}})\, \boldsymbol{g}(\lambda)\, \boldsymbol{d}\lambda} \qquad (8)$$

The posterior distributions of the parameter λ have been obtained under the assumption of three informative priors:

❖ ***Gamma Prior* [26]:**

$$\boldsymbol{g_1}(\lambda) = \frac{\boldsymbol{b^a}}{\Gamma(\boldsymbol{a})} \lambda^{\boldsymbol{a-1}} \boldsymbol{e}^{-\boldsymbol{b}\lambda} \quad ; \lambda > \boldsymbol{0}\, ;\ \boldsymbol{a}, \boldsymbol{b} > \boldsymbol{0} \qquad (9)$$

❖ ***Chi-Square Prior*, see [8]:**

$$\boldsymbol{g_2}(\lambda) = \frac{\boldsymbol{b}^{\frac{\boldsymbol{a}}{2}}}{\boldsymbol{2}^{\frac{\boldsymbol{a}}{2}}\, \Gamma\left(\frac{\boldsymbol{a}}{\boldsymbol{2}}\right)} \lambda^{\frac{\boldsymbol{a}}{2}-\boldsymbol{1}}\, \boldsymbol{e}^{\frac{-\boldsymbol{b}\lambda}{\boldsymbol{2}}} \quad ;\ \lambda > \boldsymbol{0}\ ;\ \boldsymbol{a}, \boldsymbol{b} > \boldsymbol{0} \qquad (\boldsymbol{10})$$

❖ ***Inverted Levy Prior* [27]:**

$$\boldsymbol{g_3}(\lambda) = \sqrt{\frac{\boldsymbol{b}}{\boldsymbol{2\pi}}}\, \lambda^{-\frac{\boldsymbol{1}}{\boldsymbol{2}}}\, \boldsymbol{e}^{\frac{-\boldsymbol{b}\lambda}{\boldsymbol{2}}} \quad ;\ \lambda > \boldsymbol{0}\, ;\ \boldsymbol{b} > \boldsymbol{0} \qquad (\boldsymbol{11})$$

Via Bayes theorem, combining the likelihood function (4) with the density function of gamma prior (9), chi-square prior (10) and inverted Levy prior (11), results the first, second and third posterior density functions of λ respectively as:

$$\boldsymbol{\pi_1}(\lambda|\underline{\boldsymbol{t}}) = \frac{(\boldsymbol{W}+\boldsymbol{b})^{\boldsymbol{n}+\boldsymbol{a}}}{\Gamma(\boldsymbol{n}+\boldsymbol{a})} \lambda^{\boldsymbol{n}+\boldsymbol{a}-\boldsymbol{1}}\, \boldsymbol{e}^{-\lambda\,(\boldsymbol{W}+\boldsymbol{b})} \qquad (\boldsymbol{12})$$

which implies that: $(\lambda|\underline{t}) \sim \text{Gamma}\left(n+a\,,\ \frac{1}{W+b}\right)$

$$\boldsymbol{\pi_2}(\lambda|\underline{\boldsymbol{t}}) = \frac{\left(\boldsymbol{W}+\frac{\boldsymbol{b}}{\boldsymbol{2}}\right)^{\boldsymbol{n}+\frac{\boldsymbol{a}}{\boldsymbol{2}}}}{\Gamma\left(\boldsymbol{n}+\frac{\boldsymbol{a}}{\boldsymbol{2}}\right)} \lambda^{\boldsymbol{n}+\frac{\boldsymbol{a}}{\boldsymbol{2}}-\boldsymbol{1}}\, \boldsymbol{e}^{-\lambda\left(\boldsymbol{W}+\frac{\boldsymbol{b}}{\boldsymbol{2}}\right)} \qquad (\boldsymbol{13})$$

which implies that: $(\lambda|\underline{t}) \sim \text{Gamma}\left(n+\frac{a}{2}\,,\ \frac{1}{W+\frac{b}{2}}\right)$

$$\pi_3(\lambda|\underline{t}) = \frac{\left(W+\frac{b}{2}\right)^{n+\frac{1}{2}}}{\Gamma\left(n+\frac{1}{2}\right)} \lambda^{n-\frac{1}{2}}\, e^{-\lambda\left(W+\frac{b}{2}\right)} \qquad (14)$$

which implies that: $(\lambda|\underline{t}) \sim \text{Gamma}\left(n+\frac{1}{2}\,,\ \frac{1}{W+\frac{b}{2}}\right)$

***Loss Functions*:** The Bayes estimation of a parameter λ is based in minimization of a Bayesian loss (risk) function, $L(\hat{\lambda}, \lambda)$, defined as an average cost-of-error function [29]:

$$Risk(\hat{\lambda}) = E[L(\hat{\lambda}, \lambda)] = \int_{\lambda} L(\hat{\lambda}, \lambda)\, \pi(\lambda|\underline{t})\, d\lambda$$

There are two types of loss function: symmetric and asymmetric. The symmetric loss function associates equal importance to the losses due to overestimation and underestimation of equal magnitude. We have been adopted the two types of loss function: squared error loss function as a symmetric loss function as well as LINEX and general entropy loss functions as asymmetric loss functions.

❖ ***Squared Error Loss Function*:** The squared error loss function can be expressed as [6]:

$$L(\hat{\lambda}, \lambda) = (\hat{\lambda} - \lambda)^2 \qquad (15)$$

The Bayes estimator of λ based on this loss function, denoted by $\hat{\lambda}_{BS}$, can be obtained as:

$$\hat{\lambda}_{BS} = E_{\pi}(\lambda|\underline{t}) \qquad (16)$$

❖ ***LINEX Loss Function*:** The LINEX loss function can be expressed as [18]:

$$L(\hat{\lambda}, \lambda) = d\left[e^{c(\hat{\lambda}-\lambda)} - c(\hat{\lambda}-\lambda) - 1\right] \ ;\ c \neq 0\ , d > 0 \qquad (17)$$

The Bayes estimator of λ based on this loss function, denoted by $\hat{\lambda}_{BL}$, can be obtained as:

$$\hat{\lambda}_{BL} = -\frac{1}{c} ln\left[E_{\pi}\left(e^{-c\lambda} \mid \underline{t}\right)\right] \qquad (18)$$

Provided that the posterior expectation with respect to the posterior density of parameter λ, $E_{\pi}\left(e^{-c\lambda} \mid \underline{t}\right)$, exists and is finite.

❖ ***General Entropy Loss Function***: This loss function can be expressed as [11]:

$$L(\hat{\lambda},\lambda) = d\left[\left(\frac{\hat{\lambda}}{\lambda}\right)^{p} - p\ ln\left(\frac{\hat{\lambda}}{\lambda}\right) - 1\right] \quad ; p \neq 0\ , d > 0 \qquad (19)$$

The Bayes estimator of λ based on this loss function, denoted by $\hat{\lambda}_{BG}$, can be obtained as:

$$\hat{\lambda}_{BG} = \left[E_{\pi}(\lambda^{-p}|\underline{t})\right]^{-\frac{1}{p}} \qquad (20)$$

Provided that the posterior expectation with respect to the posterior density of λ, $E_{\pi}(\lambda^{-P}|\underline{t})$ exists and is finite. Without any loss of generality it can be assumed that $\boldsymbol{d} = 1$. Now, the Bayes estimators of the shape parameter for Lomax distribution based on squared error, LINEX and general entropy loss functions corresponding to different prior distributions are shown in table (1):

Table (1): The Bayes Estimators of the Shape Parameter (λ) for Lomax Distribution

Prior	Squared Error	LINEX	General entropy
Gamma	$\hat{\lambda}_{BS_1} = \frac{n+a}{W+b}$	$\hat{\lambda}_{BL_1} = \frac{n+a}{c} ln\left(1+\frac{c}{W+b}\right)$	$\hat{\lambda}_{BG_1} = \left(\frac{\Gamma(n+a)}{\Gamma(n+a-p)}\right)^{\frac{1}{p}} \frac{1}{W+b}$
Chi-Square	$\hat{\lambda}_{BS_2} = \frac{n+\frac{a}{2}}{W+\frac{b}{2}}$	$\hat{\lambda}_{BL_2} = \frac{n+\frac{a}{2}}{c} ln\left(1+\frac{c}{W+\frac{b}{2}}\right)$	$\hat{\lambda}_{BG_2} = \left(\frac{\Gamma\left(n+\frac{a}{2}\right)}{\Gamma\left(n+\frac{a}{2}-p\right)}\right)^{\frac{1}{p}} \frac{1}{W+\frac{b}{2}}$
Inverted Levy	$\hat{\lambda}_{BS_3} = \frac{n+\frac{1}{2}}{W+\frac{b}{2}}$	$\hat{\lambda}_{BL_3} = = \frac{n+\frac{1}{2}}{c} ln\left(1+\frac{c}{W+\frac{b}{2}}\right)$	$\hat{\lambda}_{BG_3} = \left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}-p\right)}\right)^{\frac{1}{p}} \frac{1}{W+\frac{b}{2}}$

2.3 Empirical Bayes Estimators of λ

The Bayes estimators in previous subsection are seen to depend on the hyper-parameter ‘*b*’. When ‘*b*’ is unknown, we may use the empirical Bayes approach to get its estimate from likelihood function and probability density function of prior distribution [25]. Now, from likelihood function (4) and gamma, chi-square and inverted Levy prior distributions (9, 10, 11), we calculate the marginal pdf of *T*, with densities:

$$f_1(t|b) = \frac{\Gamma(n+a)\ b^{a}\ e^{-W}}{\Gamma(a)\beta^{n}(W+b)^{n+a}} \qquad (21)$$

$$f_2(t|b) = \frac{b^{\frac{a}{2}}\,\Gamma\left(n+\frac{a}{2}\right)\ e^{-W}}{2^{\frac{a}{2}}\,\Gamma\left(\frac{a}{2}\right)\ \beta^{n}\left(W+\frac{b}{2}\right)^{n+\frac{a}{2}}} \qquad (22)$$

$$f_3(t|b) = \sqrt{\frac{b}{2\pi}}\frac{\Gamma\left(n+\frac{1}{2}\right)\ e^{-W}}{\beta^{n}\left(W+\frac{b}{2}\right)^{n+\frac{1}{2}}} \qquad (23)$$

Assume that ‘*a*’ is known, then based on $f(t|b)$ we obtain an estimate, $\hat{b}$ of *b*. The maximum likelihood estimators of ‘*b*’ under the assumption of gamma, chi-square and inverted Levy priors, denoted by $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$ respectively, are calculating by taking the derivative of the natural log for (21), (22) and (23) and setting it equal to zero. The empirical Bayes estimators of the shape parameter for Lomax distribution based on squared error, LINEX and general entropy loss functions corresponding to different prior distributions are shown in table (2):

Table (2): The Empirical Bayes Estimators of the Shape Parameter (λ) for Lomax Distribution

Prior	Squared Error	LINEX	General entropy
Gamma	$\hat{\lambda}_{EBS_1}=\frac{n+a}{W+\hat{b}_1}$	$\hat{\lambda}_{EBL_1}=\frac{n+a}{c}ln\left(1+\frac{c}{W+\hat{b}_1}\right)$	$\hat{\lambda}_{EBG_1}=\left(\frac{\Gamma(n+a)}{\Gamma(n+a-p)}\right)^{\frac{1}{p}}\frac{1}{W+\hat{b}_1}$
Chi-Square	$\hat{\lambda}_{EBS_2}=\frac{n+\frac{a}{2}}{W+\frac{\hat{b}_2}{2}}$	$\hat{\lambda}_{EBL_2}=\frac{n+\frac{a}{2}}{c}ln\left(1+\frac{c}{W+\frac{\hat{b}_2}{2}}\right)$	$\hat{\lambda}_{EBG_2}=\left(\frac{\Gamma\left(n+\frac{a}{2}\right)}{\Gamma\left(n+\frac{a}{2}-p\right)}\right)^{\frac{1}{p}}\frac{1}{W+\frac{\hat{b}_2}{2}}$
Inverted Levy	$\hat{\lambda}_{EBS_3}=\frac{n+\frac{1}{2}}{W+\frac{\hat{b}_3}{2}}$	$\hat{\lambda}_{EBL_3}=\frac{n+\frac{1}{2}}{c}ln\left(1+\frac{c}{W+\frac{\hat{b}_3}{2}}\right)$	$\hat{\lambda}_{EBG_3}=\left(\frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}-p\right)}\right)^{\frac{1}{p}}\frac{1}{W+\frac{\hat{b}_3}{2}}$
Where: $\hat{b}_1=\hat{b}_2=\frac{aW}{n}, \hat{b}_3=\frac{W}{n}, W=\sum_{i=1}^{n}ln\left(1+\frac{t_i}{\beta}\right)$			

3. Monte Carlo Simulation Study and Results

In this section, Monte Carlo simulation study has been conducted to assess the behavior of different estimators for the unknown shape parameter of Lomax distribution. The simulation design consists of four basic steps which are:

***Step* (*1*):** Set the default values (true values) for the parameters of Lomax distribution which are varied into six cases to observe their effect on the estimates when $\lambda>\beta, \lambda=\beta, \lambda<\beta$.

Parameter	Cases					
	I	II	III	IV	V	VI
λ	2.1	2.1	2.1	3	3	3
β	0.24	1	2.1	1	3	4

The numbers of sample size used are (n = 10, 30 and 50) to represent small, moderate, and large dataset. The default values of the hyper-parameters of prior distributions (a,b) chosen to be $(9,3)$. The values of LINEX loss function constant (c) and general entropy loss function constant (p) used are $(c=-0.8,0.8)$ and $(p=-0.5,0.5)$. The number of sample replicated (L) chosen to be (3000).

***Step* (2):** Generate data distributed as Lomax distribution with parameters (λ,β), through the adoption of inverse transformation method, by using the formula:

$$t_i=F^{-1}(U_i)=\beta\left[(1-U_i)^{-\frac{1}{\lambda}}-1\right] \quad ; i=1,2,\dots,n \qquad (24)$$

Where U is a random variable distributed as uniform distribution for the period (0,1).

***Step* (3):** Calculate the non-Bayes, Bayes and empirical Bayes estimators of the unknown shape parameter of Lomax distribution according to the formulas that have been obtained.

***Step* (4):** After the shape parameter is estimated, mean squared error (MSE) is calculated to compare the estimation methods, where:

$$MSE(\hat{\lambda})=\frac{\sum_{j=1}^{L}(\hat{\lambda}_j-\lambda)^2}{L} \qquad (25)$$

$\hat{\lambda}_j$: is the estimate of λ at the j^{th} replicate (run).

The simulation program is written by using MATLAB (R2011b) program. The simulation results of MSE are tabulated in tables (3)…(8)

4. Conclusions and Recommendations

The most important conclusions of Monte-Carlo simulation results for estimating the shape parameter of Lomax distribution, with assumption the scale parameter is known, are:

- Among non-Bayes estimators, table (3), the performance of the uniformly minimum variance unbiased estimator (UMVU) is higher than that of other estimators "maximum likelihood (ML) and moment (MO) estimators". As well as the performance of ML estimator is better than that of MO estimator for all different cases and all sample sizes, i.e. MSE values for (UMVU<ML<MO)

- When $\lambda = 2.1$ and $n > 10$, table (4), Bayes estimator corresponding to inverted Levy prior based on squared error loss function represent the best Bayes estimator comparing to other Bayes estimators for all different values of β. When $\lambda = 3$, Bayes estimator corresponding to gamma prior based on LINEX $(c = 0.8)$ loss function is represent the best Bayes estimator comparing to other Bayes estimators for all different values of β and all sample sizes as well as when $\lambda = 2.1$ for $n = 10$.
- Among the empirical Bayes estimators, table (5), the performance of empirical Bayes estimator corresponding to inverted Levy prior based on LINEX $(c = 0.8)$ loss function is better than that of other estimators for all different cases and all sample sizes.
- For all cases and all sample sizes, LINEX $(c = 0.8)$ record full appearance as best loss function associated with Bayes estimates corresponding to gamma and chi-square priors as well as with empirical Bayes estimates corresponding to all different priors. It is important to mention that LINEX $(c = -0.8)$ and squared error record appearance as best loss functions associated with Bayes estimates corresponding to inverted Levy prior.
- According to results of $\beta = 1$, MSE values of non-Bayes and empirical Bayes "corresponding to different priors" estimators of shape parameter are increasing as the shape parameter value increase from $\lambda = 2.1$ up to $\lambda = 3$ for all sample sizes. As well as that is true for Bayes estimator corresponding to chi-square and inverted Levy priors for all sample sizes and corresponding to gamma prior for $n \geq 30$.
- For all sample sizes, the MSE values associated with non-Bayes and empirical Bayes estimators are increased with increasing the shape and scale parameters (λ, β) together from (2.1) up to (3). As well as that is true for the MSE values associated with Bayes estimators corresponding to chi-square and inverted Levy priors for all sample sizes and corresponding to gamma prior for $n \geq 30$.
- For all different cases and all sample sizes, the performance of Bayes estimators based on LINEX and general entropy loss functions with positive value of c and p respectively is better than that with negative value corresponding to gamma and chi-square priors while that is true for empirical Bayes corresponding to all different priors.
- With some cases, loss functions and sample sizes, the performance of chi-square prior is better than that of gamma and inverted Levy priors. In spite of that chi-square prior doesn't record any appearance as best prior.
- The MSE values of the empirical Bayes estimators corresponding to all different priors based on squared error loss function are identical.
- The MSE values associated with each non-Bayes, Bayes and empirical Bayes estimates corresponding to each prior and every loss function reduces with the increase in the sample size. Also, the results show a convergence between most of the estimators to increase the sample sizes.
- For all sample sizes, tables (6,7,8), the MSE values for the best Bayes estimators with different cases are less than MSE values for the best empirical Bayes estimators which in turn are less than non-Bayes estimators, i.e., MSE values for the best (Bayes $<$ empirical Bayes $<$ non-Bayes) estimators.

In the light of the conclusions that have been obtained for estimating the shape parameter of Lomax distribution, some recommended have been put forward:

- Using UMVU estimator as a non-Bayes estimator.
- As Bayes estimators, using Bayes estimator corresponding to inverted Levy prior based on squared error loss function when $\lambda = 2.1$ for $n > 10$ and using Bayes estimator corresponding to gamma prior based on LINEX $(c = 0.8)$ loss function when $\lambda = 3$.
- As empirical Bayes estimators, using empirical Bayes estimator corresponding to inverted Levy prior based on LINEX $(c = 0.8)$ loss function.

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Table (3): MSE Values for Non-Bayes Estimators of λ with Different Cases

Case	*n*	ML	MO	UMVU	Best Estimator
I	10	0.6924156	0.7928562	0.5195667	UMVU
	30	0.1837935	0.2205734	0.1659697	UMVU
	50	0.0945257	0.1177104	0.0902156	UMVU
II	10	0.7567706	0.8645757	0.5691602	UMVU
	30	0.1746109	0.2097958	0.1588529	UMVU
	50	0.0998795	0.1258586	0.0939855	UMVU
III	10	0.7413184	0.8449798	0.5538372	UMVU
	30	0.1749264	0.2119030	0.1595775	UMVU
	50	0.0958524	0.1212523	0.0905668	UMVU
IV	10	1.5079181	1.6319121	1.1242461	UMVU
	30	0.3488927	0.3867222	0.3165483	UMVU
	50	0.1906341	0.2164611	0.1800680	UMVU
V	10	1.4953797	1.6238960	1.1045519	UMVU
	30	0.3648184	0.3960929	0.3315562	UMVU
	50	0.2092135	0.2350960	0.1964496	UMVU
VI	10	1.5753702	1.7032210	1.1697876	UMVU
	30	0.3505929	0.3870746	0.3164375	UMVU
	50	0.1915445	0.2149299	0.1815260	UMVU

Table (4): MSE Values for Bayes Estimators of λ with Different Cases

Case	*n*	Prior	Loss Function					Best Loss
			Squared Error	LINEX		General Entropy		
				c= - 0.8	c= 0.8	p= - 0.5	p= 0.5	
I	10	Gamma	0.4209589	0.6382777	0.2805309	0.3871345	0.3259441	LIN(0.8)
		Chi- Square	0.4688394	0.7742312	0.3007371	0.4296691	0.3619228	LIN(0.8)
		Inverted Levy	0.2892625	0.2857887	0.3346616	0.3098095	0.3626562	LIN(-0.8)
	Best Prior		I Levy	I Levy	Gamma	I Levy	Gamma	
	30	Gamma	0.1758869	0.2180729	0.1436139	0.1681386	0.1539424	LIN(0.8)
		Chi- Square	0.1742364	0.2155819	0.1444809	0.1673784	0.1552564	LIN(0.8)
		Inverted Levy	0.1299505	0.1351867	0.1318421	0.1318364	0.1372669	Squared
	Best Prior		I Levy	I Levy	I Levy	I Levy	I Levy	
	50	Gamma	0.0941950	0.1090917	0.0824312	0.0913138	0.0860793	LIN(0.8)
		Chi- Square	0.0921272	0.1051265	0.0824704	0.0898839	0.0859963	LIN(0.8)
		Inverted Levy	0.0805481	0.0823537	0.0821643	0.0815494	0.0841664	Squared
	Best Prior		I Levy	I Levy	I Levy	I Levy	I Levy	
II	10	Gamma	0.4307917	0.6518213	0.2880308	0.3966973	0.3349749	LIN(0.8)
		Chi- Square	0.4886429	0.8060028	0.3142996	0.4486857	0.3793995	LIN(0.8)
		Inverted Levy	0.2982702	0.3002198	0.3408746	0.3182839	0.3700984	Squared
	Best Prior		I Levy	I Levy	Gamma	I Levy	Gamma	
	30	Gamma	0.1676134	0.2078642	0.1369771	0.1602110	0.1466971	LIN(0.8)
		Chi- Square	0.1655487	0.2045121	0.1377642	0.1591115	0.1478182	LIN(0.8)
		Inverted Levy	0.1273896	0.1310296	0.1305867	0.1296124	0.1357040	Squared
	Best Prior		I Levy	I Levy	I Levy	I Levy	I Levy	
	50	Gamma	0.1012966	0.1176753	0.0881955	0.0980856	0.0921977	LIN(0.8)
		Chi- Square	0.0984488	0.1130575	0.0873523	0.0958429	0.0912377	LIN(0.8)
		Inverted Levy	0.0814241	0.0834525	0.0819526	0.0821121	0.0841100	Squared
	Best Prior		I Levy	I Levy	I Levy	I Levy	I Levy	
III	10	Gamma	0.4339072	0.6568331	0.2897009	0.3994029	0.3368846	LIN(0.8)
		Chi- Square	0.4891338	0.8075124	0.3136329	0.4486687	0.3784107	LIN(0.8)
		Inverted Levy	0.2926855	0.2944066	0.3353042	0.3124724	0.3638807	Squared
	Best Prior		I Levy	I Levy	Gamma	I Levy	Gamma	
	30	Gamma	0.1656986	0.2056243	0.1353610	0.1583946	0.1450751	LIN(0.8)
		Chi- Square	0.1644508	0.2031860	0.1368946	0.1581192	0.1470336	LIN(0.8)
		Inverted Levy	0.1278156	0.1313484	0.1311261	0.1301212	0.1363746	Squared

	Best Prior		I Levy	I Levy	I Levy	I Levy	I Levy	
	50	Gamma	0.0973734	0.1131279	0.0848316	0.0942820	0.0886312	LIN(0.8)
		Chi- Square	0.0944763	0.1083779	0.0840074	0.0920053	0.0876673	LIN(0.8)
		Inverted Levy	0.0794859	0.0809631	0.0805016	0.0802916	0.0825223	Squared
	Best Prior		I Levy	I Levy	I Levy	I Levy	I Levy	
IV	10	Gamma	0.2512371	0.4242360	0.2032488	0.2409582	0.2282416	LIN(0.8)
		Chi- Square	0.4890435	0.9262249	0.3361726	0.4509820	0.4108411	LIN(0.8)
		Inverted Levy	0.7532214	0.5919364	0.9835630	0.8219570	0.9795409	LIN(-0.8)
	Best Prior		Gamma	Gamma	Gamma	Gamma	Gamma	
	30	Gamma	0.1888777	0.2377626	0.1649938	0.1846381	0.1784801	LIN(0.8)
		Chi- Square	0.2498248	0.3225491	0.2126592	0.2434845	0.2338094	LIN(0.8)
		Inverted Levy	0.2702601	0.2542718	0.3070943	0.2801516	0.3030672	LIN(-0.8)
	Best Prior		Gamma	Gamma	Gamma	Gamma	Gamma	
	50	Gamma	0.1324744	0.1536801	0.1210544	0.1305168	0.1276020	LIN(0.8)
		Chi- Square	0.1574465	0.1845595	0.1423563	0.1549164	0.1510355	LIN(0.8)
		Inverted Levy	0.1645456	0.1586546	0.1790145	0.1683556	0.1771851	LIN(-0.8)
	Best Prior		Gamma	Gamma	Gamma	Gamma	Gamma	

V	10	Gamma	0.2678059	0.4485666	0.2110825	0.2550297	0.2395036	LIN(0.8)
		Chi- Square	0.5066402	0.9687069	0.3528798	0.4757152	0.4319642	LIN(0.8)
		Inverted Levy	0.7557265	0.5992942	0.9831671	0.8233931	0.9789622	LIN(-0.8)
	Best Prior		Gamma	Gamma	Gamma	Gamma	Gamma	
	30	Gamma	0.1965522	0.2463540	0.1719607	0.1922661	0.1860145	LIN(0.8)
		Chi- Square	0.2603529	0.3348389	0.2218890	0.2539096	0.2440291	LIN(0.8)
		Inverted Levy	0.2791828	0.2645349	0.3150218	0.2889777	0.3117005	LIN(-0.8)
	Best Prior		Gamma	Gamma	Gamma	Gamma	Gamma	
	50	Gamma	0.1437432	0.1674556	0.1301677	0.1414429	0.1378488	LIN(0.8)
		Chi- Square	0.1716942	0.2020976	0.1538185	0.1687346	0.1640024	LIN(0.8)
		Inverted Levy	0.1723071	0.1689878	0.1846116	0.1757413	0.1838274	LIN(-0.8)
	Best Prior		Gamma	Gamma	Gamma	Gamma	Gamma	
VI	10	Gamma	0.2600914	0.4390406	0.2078941	0.2475763	0.2331639	LIN(0.8)
		Chi- Square	0.5001097	0.9641520	0.3470110	0.4695904	0.4266267	LIN(0.8)
		Inverted Levy	0.7544944	0.5997126	0.9817738	0.8224008	0.9784242	LIN(-0.8)
	Best Prior		Gamma	Gamma	Gamma	Gamma	Gamma	
	30	Gamma	0.1895114	0.2398847	0.1644222	0.1850092	0.1783343	LIN(0.8)
		Chi- Square	0.2507902	0.3255288	0.2120461	0.2441096	0.2337662	LIN(0.8)
		Inverted Levy	0.2663479	0.2515768	0.3022578	0.2759772	0.2983807	LIN(-0.8)
	Best Prior		Gamma	Gamma	Gamma	Gamma	Gamma	
	50	Gamma	0.1326237	0.1533469	0.1216317	0.1307717	0.1280657	LIN(0.8)
		Chi- Square	0.1578726	0.1844656	0.1432432	0.1554636	0.1518219	LIN(0.8)
		Inverted Levy	0.1667764	0.1605234	0.1815634	0.1706893	0.1797218	LIN(-0.8)
	Best Prior		Gamma	Gamma	Gamma	Gamma	Gamma	

Table (5): MSE Values for Empirical Bayes Estimators of λ with Different Cases

Case	*n*	Prior	Loss Function					Best Loss
			Squared Error	LINEX		General Entropy		
				c= - 0.8	c= 0.8	p= - 0.5	p= 0.5	
I	10	Gamma	0.6924156	0.9697528	0.5235522	0.6629485	0.6098955	LIN(0.8)
		Chi- Square	0.6924156	1.0902191	0.4857770	0.6543010	0.5881686	LIN(0.8)
		Inverted Levy	0.6924156	1.3365110	0.4359868	0.6408775	0.5570511	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
	30	Gamma	0.1837935	0.2132448	0.1625423	0.1795290	0.1721855	LIN(0.8)
		Chi- Square	0.1837935	0.2177787	0.1602977	0.1790044	0.1709410	LIN(0.8)
		Inverted Levy	0.1837935	0.2231430	0.1578829	0.1784168	0.1696017	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
	50	Gamma	0.0945257	0.1032931	0.0883198	0.0933061	0.0913596	LIN(0.8)
		Chi- Square	0.0945257	0.1041414	0.0879127	0.0932130	0.0911650	LIN(0.8)
		Inverted Levy	0.0945257	0.1050434	0.0875061	0.0931176	0.0909737	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
II	10	Gamma	0.7567706	1.0710899	0.5682896	0.7252691	0.6682160	LIN(0.8)
		Chi- Square	0.7567706	1.2102818	0.5262475	0.7160022	0.6446804	LIN(0.8)
		Inverted Levy	0.7567706	1.5015511	0.4707725	0.7015942	0.6107172	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
	15	Gamma	0.4349196	0.5608423	0.3521966	0.4203219	0.3945088	LIN(0.8)
		Chi- Square	0.4349196	0.5982315	0.3376841	0.4171399	0.3867041	LIN(0.8)
		Inverted Levy	0.4349196	0.6561800	0.3200401	0.4128736	0.3768900	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
	50	Gamma	0.0998795	0.1101208	0.0923442	0.0983201	0.0957004	LIN(0.8)
		Chi- Square	0.0998795	0.1110978	0.0918332	0.0981991	0.0954234	LIN(0.8)
		Inverted Levy	0.0998795	0.1121347	0.0913197	0.0980747	0.0951466	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
III	10	Gamma	0.7413184	1.0449496	0.5572145	0.7098159	0.6527772	LIN(0.8)
		Chi- Square	0.7413184	1.1778083	0.5159708	0.7005498	0.6292555	LIN(0.8)
		Inverted Levy	0.7413184	1.4519956	0.4614607	0.6861441	0.5953245	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
	30	Gamma	0.1749264	0.2021367	0.1556163	0.1711480	0.1647631	LIN(0.8)
		Chi- Square	0.1749264	0.2063537	0.1536052	0.1706861	0.1637030	LIN(0.8)
		Inverted Levy	0.1749264	0.2113503	0.1514506	0.1701697	0.1625723	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
	50	Gamma	0.0958524	0.1054133	0.0889262	0.0944234	0.0920623	LIN(0.8)
		Chi- Square	0.0958524	0.1063307	0.0884626	0.0943131	0.0918169	LIN(0.8)
		Inverted Levy	0.0958524	0.1073050	0.0879979	0.0941999	0.0915729	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
IV	10	Gamma	1.5079181	2.6431437	1.0067869	1.4434695	1.3267640	LIN(0.8)
		Chi- Square	1.5079181	3.5957561	0.9114887	1.4245117	1.2786300	LIN(0.8)
		Inverted Levy	1.5079181	3.6696966	0.7975467	1.3950378	1.2091839	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
	30	Gamma	0.3488927	0.4331272	0.2972029	0.3409792	0.3275524	LIN(0.8)
		Chi- Square	0.3488927	0.4470195	0.2924384	0.3400104	0.3253132	LIN(0.8)
		Inverted Levy	0.3488927	0.4637261	0.2875179	0.3389270	0.3229202	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
	50	Gamma	0.1906341	0.2197145	0.1724446	0.1877649	0.1830400	LIN(0.8)
		Chi- Square	0.1906341	0.2226631	0.1713747	0.1875437	0.1825507	LIN(0.8)
		Inverted Levy	0.1906341	0.2258199	0.1703290	0.1873166	0.1820649	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
V	10	Gamma	1.4953797	2.4604090	1.0149214	1.4298897	1.3111844	LIN(0.8)
		Chi- Square	1.4953797	2.9479643	0.9211262	1.4106180	1.2621682	LIN(0.8)
		Inverted Levy	1.4953797	4.1238572	0.8078500	1.3806484	1.1913642	LIN(0.8)

	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
	30	Gamma	0.3648184	0.4518895	0.3110018	0.3567268	0.3429459	LIN(0.8)
		Chi- Square	0.3648184	0.4662207	0.3059996	0.3557350	0.3406380	LIN(0.8)
		Inverted Levy	0.3648184	0.4834490	0.3008186	0.3546254	0.3381671	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
	50	Gamma	0.2092135	0.2420508	0.1878152	0.2058735	0.2002148	LIN(0.8)
		Chi- Square	0.2092135	0.2453378	0.1865017	0.2056136	0.1996111	LIN(0.8)
		Inverted Levy	0.2092135	0.2488509	0.1852061	0.2053464	0.1990065	LIN(0.8)
	Best Prior		-----	Gamma	I Levy	I Levy	I Levy	
VI	10	Gamma	1.5753702	2.7187648	1.0517656	1.5078664	1.3851847	LIN(0.8)
		Chi- Square	1.5753702	3.4661943	0.9511604	1.4879805	1.3343625	LIN(0.8)
		Inverted Levy	1.5753702	4.0221500	0.8300146	1.4570336	1.2607063	LIN(0.8)
	Best Prior		------	Gamma	I Levy	I Levy	I Levy	
	30	Gamma	0.3505929	0.4370342	0.2971456	0.3423228	0.3281941	LIN(0.8)
		Chi- Square	0.3505929	0.4512552	0.2921788	0.3413081	0.3258200	LIN(0.8)
		Inverted Levy	0.3505929	0.4683485	0.2870352	0.3401725	0.3232745	LIN(0.8)
	Best Prior		------	Gamma	I Levy	I Levy	I Levy	
	50	Gamma	0.1915445	0.2201811	0.1737591	0.1887930	0.1843009	LIN(0.8)
		Chi- Square	0.1915445	0.2230914	0.1727209	0.1885815	0.1838401	LIN(0.8)
		Inverted Levy	0.1915445	0.2262079	0.1717077	0.1883643	0.1833838	LIN(0.8)
	Best Prior		------	Gamma	I Levy	I Levy	I Levy	

Table (6): MSE Values for the Best Non-Bayes Estimators of λ with Different Cases

n	I	II	III	IV	V	VI
10	UMVU	UMVU	UMVU	UMVU	UMVU	UMVU
	0.5195667	0.5691602	0.5538372	1.1242461	1.1045519	1.1697876
30	UMVU	UMVU	UMVU	UMVU	UMVU	UMVU
	0.1659697	0.1588529	0.1595775	0.3165483	0.3315562	0.3164375
50	UMVU	UMVU	UMVU	UMVU	UMVU	UMVU
	0.0902156	0.0939855	0.0905668	0.1800680	0.1964496	0.1815260

Table (7): MSE Values for the Best Bayes Estimators of λ with Different Cases

n	I	II	III	IV	V	VI
10	GLIN(0.8)	GLIN(0.8)	GLIN(0.8)	GLIN(0.8)	GLIN(0.8)	GLIN(0.8)
	0.2805309	0.2880308	0.2897009	0.2032488	0.2110825	0.2078941
30	ILS	ILS	ILS	GLIN(0.8)	GLIN(0.8)	GLIN(0.8)
	0.1299505	0.1273896	0.1278156	0.1649938	0.1719607	0.1644222
50	ILS	ILS	ILS	GLIN(0.8)	GLIN(0.8)	GLIN(0.8)
	0.0805481	0.0814241	0.0794859	0.1210544	0.1301677	0.1216317

Table (8): MSE Values for the Best Empirical Bayes Estimators of λ with Different Cases

n	I	II	III	IV	V	VI
10	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)
	0.4359868	0.4707725	0.4614607	0.7975467	0.8078500	0.8300146
30	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)
	0.1578829	0.1510920	0.1514506	0.2875179	0.3008186	0.2870352
50	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)	ILLIN(0.8)
	0.0875061	0.0913197	0.0879979	0.1703290	0.1852061	0.1717077