Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.5, No.1, 2015 www.iiste.org

Minimax Estimation of the Scale Parameter of Laplace Distribution under Squared-Log Error Loss Function

Huda, A. Rasheed [1*] Emad,F.AL-Shareefi [2]

1. AL-Mustansiriya University, Collage of science, Dept. of Math.
2. Foundation of Technical Education

* E-mail: iraqalnoor1@gmail.com

Abstract

In this paper, we obtained Minimax estimator of the scale parameter θ for the Laplace distribution under the Squared log error loss function by applying the theorem of Lehmann [1950], and compared it with Minimax estimator under Quadratic loss function in addition of Maximum Likelihood Estimator according to Monte-Carlo simulation study. The performance of these estimators is compared depending on the mean squared errors (MSE's).

Keywords: Minimax estimator, Laplace distribution, Bayes estimator, Squared-log error loss function, Jeffery prior, Mean squared error.

1. Introduction

The classical Laplace distribution with mean zero and variance σ^2 was introduced by Laplace in 1774 . The distribution is symmetrical and Leptokurtic[1]. Also called the double exponential [2].

This distribution has been used for modeling data that have heavier tails than those of the normal distribution and analyze engineering, financial, industrial, environmental, and biological data (Kotz et al., 2001).[3], it used for modeling data that have heavier tails than those of the normal distribution.

The probability density function of a Laplace distributed random variable is given by [3]:

$$f(x|a,\theta) = \frac{1}{2\theta} exp\left[-\frac{|x-a|}{\theta}\right] \qquad -\infty < x < \infty \qquad (1)$$

Where $a \in (-\infty,\infty)$ and $\theta > 0$ are location and scale parameters, respectively.

The cumulative distribution function is given by:

$$F(x|\, a,\theta) = \begin{cases} 1-\frac{1}{2}\exp\left[\frac{a-x}{\theta}\right] & \text{for } x \geq a \\ \frac{1}{2}\exp\left[\frac{x-a}{\theta}\right] & \text{for } x < a \end{cases}$$

With moment –generating function

$$M_x(t) = \frac{e^{t\mu}}{1-b^2t^2}$$

We can obtain the Maximum likelihood estimator for the scale parameter θ , as follows:

Let $X_1, X_2, \ldots, X_n$ be a random sample from density (1) (when α is known). The likelihood function is given by

$$L(x_i;a,\theta) = \left(\frac{1}{2\theta}\right)^n exp\left[-\frac{\sum_{i=1}^{n}|x_i-a|}{\theta}\right]$$

By taking the log and differentiating partially, with respect to θ

$$\frac{\partial \ln L(x_i;a,\theta)}{\partial\,\theta} = \frac{-n}{\theta} + \frac{\sum_{i=1}^{n}|x_i-a|}{\theta^2} = 0 \qquad (2)$$

The MLE of θ is the solution of equation (2) after equating the first derivative to zero:

$$\hat{\theta}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n}|x_i - a| = \frac{T}{n} \text{ , } where\ T = \sum_{i=1}^{n}|x_i - a| \quad (3)$$

2. Bayes Estimator using Jeffery Prior Information [5][6]

Let us assume that, θ has non-information prior density, defined as

$$g \propto \sqrt{I(\theta)} \quad (4)$$

Where $I(\theta)$ represented Fisher information which defined as follows:

$$I(\theta) = -nE\left[\frac{\partial^2 \ln f(x;a,\theta)}{\partial\theta^2}\right] \text{ , hence,}$$

$$g(\theta) = k\sqrt{-nE\left(\frac{\partial^2 \ln f(x;a,\theta)}{\partial\theta^2}\right)} \quad (5)$$

$$\ln f(x;a,\theta) = -\ln(2\theta) - \frac{|x-a|}{\theta}$$

$$\frac{\partial \ln f(x;a,\theta)}{\partial\theta} = -\frac{1}{2\theta} - \frac{|x-a|}{\theta^2}$$

$$\frac{\partial^2 \ln f(x;a,\theta)}{\partial\theta^2} = \frac{1}{2\theta^2} - \frac{2|x-a|}{\theta^3}$$

$$E\left[\frac{\partial^2 \ln f(x;a,\theta)}{\partial\theta^2}\right] = \frac{1}{2\theta^2} - \frac{2}{\theta^3}E[|x-a|] = \frac{-3}{2\theta^2} \quad (6)$$

After Substitution (6) into (4), we get

$$g(\theta) = k\sqrt{-n\left(-\frac{3}{2\theta^2}\right)} = \frac{k}{\theta}\sqrt{\frac{3n}{2}} \quad (7)$$

Now, Combining the prior (7) with the likelihood function, we have the posterior distribution of θ with Jeffrey's prior information given by $h(\theta|\underline{x})$:

$$h(\theta|\underline{x}) = \frac{g(\theta)L(\theta;x_1x_2\ldots\ldots x_n)}{\int_0^\infty g(\theta)L(\theta;x_1x_2\ldots\ldots x_n)d\theta}$$

$$h(\theta|\underline{x}) = \frac{\frac{1}{\theta^{n+1}}exp\left[-\frac{\sum_{i=1}^{n}|x_i-a|}{\theta}\right]}{\int_0^\infty \frac{1}{\theta^{n+1}}exp\left[-\frac{\sum_{i=1}^{n}|x_i-a|}{\theta}\right]d\theta} \quad (8)$$

On simplification, we have

$$h(\theta|\underline{x}) = \frac{\left(\sum_{i=1}^{n}|x_i-a|\right)^n e^{\frac{-\sum_{i=1}^{n}|x_i-a|}{\theta}}}{\theta^{n+1}\Gamma(n)} = \frac{T^n e^{\frac{-T}{\theta}}}{\theta^{n+1}\Gamma(n)}$$

This posterior density is recognized as the density of the Inverse Gamma (IG) distribution: $\theta \sim \text{IG}(n, T)$

3. Bayesian Estimators under Squared – Log Error Loss Function.

Brown (1968) proposed a symmetric loss function for scale parameter estimation, that is called squared log error loss function is:

$$l(\hat{\theta},\theta) = \left(\ln\hat{\theta} - \ln\theta\right)^2 = \left(\ln\frac{\hat{\theta}}{\theta}\right)^2 \quad (9)$$

Which is balanced, with lim $l(\hat{\theta},\theta) \to \infty\ as\ \hat{\theta} \to 0\ or\ \infty.$

A balanced loss function takes both error of estimation and goodness of fit into account but the unbalanced loss function considers only error of estimation. This loss function is convex for:

$\frac{\hat{\theta}}{\theta} \le e$, and concave otherwise, but its risk function has a unique minimum with respect to $\hat{\theta}$ [4].

The risk function under the squared log error loss function (9) is given by

$$R(\hat{\theta}-\theta) = E\left[L(\hat{\theta},\theta)\right] = \int_0^\infty (\ln\hat{\theta}-\ln\theta)^2 h(\theta|x_1 \ldots\ldots\ldots x_n)d\theta$$

$$= (\ln\hat{\theta})^2 - 2(ln\hat{\theta})\, E(\ln\theta|x) + E((\ln\theta)^2|x)$$

$$\frac{\partial Rsik}{\partial\hat{\theta}} = 2(\ln\hat{\theta})\frac{1}{\hat{\theta}} - \frac{2}{\hat{\theta}}E((\ln\theta)|x)$$

By letting $\frac{\partial R(\hat{\theta}-\theta)}{\partial\hat{\theta}} = 0$

The risk function of squared log error loss function (9) has minimum w.r.t.:

$$\hat{\theta}_{SL} = \mathrm{Exp}\left[E(ln\theta|\underline{x})\right] \qquad (10)$$

Now, $E(ln\theta|\underline{x}) = \int_0^\infty \ln\theta\, h(\theta|\underline{x})d\theta$

$$E(ln\theta|\underline{x}) = \int_0^\infty \ln\theta \frac{(\sum_{i=1}^n |x_i-a|)^n e^{\frac{-\sum_{i=1}^n|x_i-a|}{\theta}}}{\theta^{n+1}\Gamma(n)} d\theta$$

$$= \frac{(\sum_{i=1}^n|x_i-a|)^n}{\Gamma(n)}\int_0^\infty \ln\theta \frac{e^{\frac{-\sum_{i=1}^n|x_i-a|}{\theta}}}{\theta^{n+1}} d\theta$$

$$= \int_0^\infty \ln\theta \frac{e^{\frac{-\sum_{i=1}^n|x_i-a|}{\theta}}}{\theta^{n+1}\Gamma(n)} d\theta$$

$let\ y = \frac{\sum_{i=1}^n|x_i-a|}{\theta}$, which implies that, $\theta = \frac{\sum_{i=1}^n|x_i-a|}{y}$, $d\theta = -\frac{\sum_{i=1}^n|x_i-a|}{y^2}dy$

Hence,

$$E(ln\theta|\underline{x}) = \frac{(\sum_{i=1}^n|x_i-a|)^n}{\Gamma(n)}\int_0^\infty \frac{[\ln(\sum_{i=1}^n|x_i-a|)-\ln(y)]\boldsymbol{e^{-y}}}{\frac{(\sum_{i=1}^n|x_i-a|)^{n+1}}{y^{n+1}}}\left(\frac{-(\sum_{i=1}^n|x_i-a|)}{y^2}\right)dy$$

$$= \frac{\ln(\sum_{i=1}^n|x_i-a|)}{\Gamma(n)}\int_0^\infty y^{n-1}e^{-y}dy - \int_0^\infty \frac{\ln(y)y^{n-1}e^{-y}}{\Gamma(n)}dy$$

$$= \ln(\sum_{i=1}^n|x_i-a|) - \frac{\boldsymbol{\Gamma'(n)}}{\boldsymbol{\Gamma(n)}}$$

$$E(ln\theta|\underline{x}) = \ln(\sum_{i=1}^n|x_i-a|) - \psi(n) \qquad (11)$$

Such that $\psi(n) = \frac{\Gamma'(n)}{\Gamma(n)}$, where $\psi(n)$ is digamma function

Substituting (11) in (10), we get the Bayes estimator of parameter θ under Squared-Log error loss function with Jeffrey prior information as:

$$\hat{\theta}_{SL} = Exp[\ln(\sum_{i=1}^n|x_i-a|) - \psi(n)]$$

$$\hat{\theta}_{SL} = \frac{\sum_{i=1}^n|x_i-a|}{e^{\psi(n)}}$$

4. Bayesian Estimators under Quadratic Loss Function [9].

De Groot (1970) [10] discussed different types of loss function and obtained the Bayes estimates under these loss function which is a non-negative symmetric and continuous loss function [16], which is defined as:

$$L(\hat{\theta},\theta) = \left(\frac{\hat{\theta}-\theta}{\theta}\right)^2 = \left(1-\frac{\hat{\theta}}{\theta}\right)^2 \qquad (12)$$

Now, the risk function under the Quadratic Loss function (QLF) is given by:

$$R_Q(\hat{\theta}, \theta) = E\left(1 - \frac{\hat{\theta}}{\theta}\right)^2 \quad (13)$$

Taking the partial derivative for $R_Q(\hat{\theta}, \theta)$ with respect to $\hat{\theta}$ and setting it equal to zero yields, $\hat{\theta}_Q$ which is minimizes the risk function of QLF .

$$\hat{\theta}_Q = \frac{E(\frac{1}{\theta})}{E(\frac{1}{\theta^2})} \quad (14)$$

Recall that: $\theta \sim IG(n, T)$

Then, we can easily prove that, $(\frac{1}{\theta}) \sim \Gamma(n, T^{-1})$, with,

$$E\left(\frac{1}{\theta}\right) = \frac{n}{T} \quad , ver\left(\frac{1}{\theta}\right) = \frac{n}{T^2}$$

$$\Rightarrow E\left(\frac{1}{\theta^2}\right) = \frac{n(n+1)}{T^2} = \frac{n(n+1)}{\left(\sum_{i=1}^{n}|x_i - a|\right)^2}$$

After Substituting, we get the Bayes estimator of parameter θ under Quadratic loss function with Jeffrey's prior information as:

$$\hat{\theta}_Q = \frac{\sum_{i=1}^{n}|x_i - a|}{(n+1)}$$

5. Minimax Estimators

The derivation of Minimax estimators depends primarily on a theorem due to Lehmann which can be stated as follows:

Lehmann's Theorem: [11]

Let τ = {Fθ; θεΘ} be a family of distribution functions and D a class of estimators of θ. Suppose that d* ε D is a Bayes' estimator against a prior distribution ξ*(θ) on the parameter space Θ and the risk function R(d*, θ) = constant on Θ; then d* is a minimax estimator of θ.

The main results are contained in the following Theorem.

Theorem 5.1

Let x = (x_1, x_2, . . . , x_n) be n independently and identically distributed random variables drawn from the density (1), then

$$\hat{\theta}_{SL} = \frac{\sum_{i=1}^{n}|x_i - a|}{e^{\psi(n)}} \quad (15)$$

Is the minimax estimator of the parameter θ under the Squared log error loss (9).

Theorem 5.2

Let X = (x_1, x_2, . . . , x_n) be n independently and identically distributed random variables drawn from the density (1), then

$$\hat{\theta}_Q = \frac{\sum_{i=1}^{n}|x_i - a|}{n+c} \quad (16)$$

Is the minimax estimator of the parameter θ under the quadratic loss function (12).

To prove the theorem 5.1 and 5.2, we shall use Lehmann's theorem, by showing that, the risk function of $\hat{\theta}$ is a constant.

Firstly, we have to prove the theorem (5.1). The Risk function of the estimator $\hat{\theta}_{SL}$ is:

$$R_{\hat{\theta}_{SL}}(\theta) = E\left[L(\theta|\hat{\theta}_{SL})\right]$$

$$= \int_0^{\infty} (\ln\hat{\theta} - \ln\theta)^2\, h(\theta|\underline{x})dx$$

$$= E\left(\ln\hat{\theta}\right)^2 - 2(\ln\theta)E\left(\ln\hat{\theta}\right) + (\ln\theta)^2 \quad (17)$$

According to (15), we have

$\ln\hat{\theta} = \ln(\sum_{i=1}^{n}|x_i - a|) - \psi(n)$

$$R_{\hat{\theta}_{SL}}(\theta) = E\left[\ln\left(\sum_{i=1}^{n}|x_i - a|\right) - \psi(n)\right]^2 - 2(\ln\theta)\,E\left[\ln\left(\sum_{i=1}^{n}|x_i - a|\right) - \psi(n)\right] + (\ln\theta)^2$$

$$= E[\ln(\sum_{i=1}^{n}|x_i - a|)]^2 - 2\psi(n)E(\ln(\sum_{i=1}^{n}|x_i - a|)) + \left(\psi(n)\right)^2 -$$

$$2(\ln\theta)\,E(\ln(\sum_{i=1}^{n}|x_i - a|)) + 2(\ln\theta)\psi(n) + (\ln\theta)^2 \quad (18)$$

From the conclusion $\sum_{i=1}^{n}|x_i - a| \sim \Gamma(n, \theta^{-1})$, we have

$E[\ln(\sum_{i=1}^{n}|x_i - a|)] = \psi(n) + \ln\theta \quad and \quad var[\ln(\sum_{i=1}^{n}|x_i - a|)] = \psi'(n)$ [8][7].

Thus, after substituting in (18), we get

$$R_{\hat{\theta}_{SL}}(\theta) =$$

$$\psi'(n) + (\psi(n) + \ln\theta)^2 - 2\psi(n)(\psi(n) + \ln\theta) + \left(\psi(n)\right)^2 - 2(\ln\theta)(\psi(n) + \ln\theta) +$$

$$2(\ln\theta)\psi(n) + (\ln\theta)^2$$

$$R_{\hat{\theta}_{SL}}(\theta) = \psi'(n) + \left(\psi(n)\right)^2 + 2\psi(n)(\ln\theta) + (\ln\theta)^2 - 2\left(\psi(n)\right)^2 - 2\psi(n)(\ln\theta) + \left(\psi(n)\right)^2 - 2\psi(n)(\ln\theta)$$

$$- 2(\ln\theta)^2 + 2\psi(n)(\ln\theta) + (\ln\theta)^2$$

$R_{\hat{\theta}_{SL}}(\theta) = \psi'(n)$, which is a constant.

So, according to the Lehmann's theorem, it follows that:

$\hat{\theta}_{SL} = \frac{\sum_{i=1}^{n}|x_i - a|}{e^{\psi(n)}}$ is minimax estimator for the scale parameter θ of Laplace distribution under the squared – log error loss function.

Now, we have to prove the theorem (5.2). The Risk function of the estimator $\hat{\theta}_Q$ is:

$$R_{\hat{\theta}_Q}(\theta) = E[\,L(\theta \mid \hat{\theta}_Q)\,] = E\left(\frac{\theta - \hat{\theta}_Q}{\theta}\right)^2$$

$$= \frac{1}{\theta^2}\left[\theta^2 - 2\theta E\left(\hat{\theta}_Q\right) + E\left(\hat{\theta}_Q\right)^2\right]$$

$$= \frac{1}{\theta^2}\left[\theta^2 - \frac{2\theta}{(n+1)}E(\sum_{i=1}^{n}|x_i - a|) + \frac{1}{(n+1)^2}E(\sum_{i=1}^{n}|x_i - a|)^2\right] \quad (19)$$

Recall that, $\sum_{i=1}^{n}|x_i - a| \sim \Gamma(n, \theta^{-1})$, hence,

$E(\sum_{i=1}^{n}|x_i - a|) = n\theta \quad and \quad var(\sum_{i=1}^{n}|x_i - a|) = n\theta^2$

Thus, after substituting in (19), we get

$$R_{\hat{\theta}_Q}(\theta) = \frac{1}{\theta^2}\left[\theta^2 - \frac{2\theta}{(n+1)}n\theta + \frac{1}{(n+1)^2}n\theta^2\right]$$

$$= 1 - \frac{2n}{(n+1)} + \frac{n}{(n+1)^2}$$, which is a constant.

So according to the Lehmann's theorem, it follows that:

$\hat{\theta}_Q = \frac{\sum_{i=1}^{n}|x_i - a|}{n+1}$, is the minimax estimator of the parameter θ of the Laplace distribution under the quadratic loss function.

6. Simulation Results

In our simulation study, we generated I = 5000 samples of size n = 5, 10, 20, 50 and 100 from Laplace distribution to represent small, moderate and large sample size with the scale parameter θ =0.5,1, 1.5, 3,and 4. In this section, Monte – Carlo simulation study is performed to compare the methods of estimation by using mean square Errors (MSE’s) as an index for precision to compare the efficiency of each of estimators, where:

$$MSE(\hat{\theta}) = \frac{\sum_{i=1}^{I}\left(\hat{\theta}_i \; - \theta\right)^2}{I}$$

The results were summarized and tabulated in the following tables for each estimator and for all sample sizes. The expectations and MSE’s for θ are schedule in tables (1, 2, 3, 4 and 5).

7. Discussion

The results of the simulation study for estimating the scale parameter (θ) of Laplace distribution when the location parameter, (α) is known, are summarized and tabulated in tables (1, 2, 3, 4 and 5) which contain the Expected values and MSE's, we have observed that:

- The performance of Bayes estimator under Quadratic loss function with Jeffery prior information is the best estimator, comparing to other estimator for all sample sizes and with all values of the scale parameter, followed by the Maximum Likelihood Estimator.
- It is observed that, MSE's of all estimators of scale parameter is increasing with the increase of the scale parameter value.
- Finally, for all parameter values, an obvious reduction in MSE's is observed with the increase in sample size.

References

[1] Krishnamoorthy, k.(2006);Handbook of statistical distribution with application with application; chapman and Hall/crc.

[2] Abramowitz, M. and Stegun, I. A. (1972), Handbook of Mathematical Function with Formulas ,Grapha, and Mathematical Tables, 9th printing. New York : Dover,pp.930.

[3] Rasheed, H. A., Al-Baldawi, T. H.K., AL.Obedy, N, J.(2014);"Some Bayes Estimator for Laplace Distribution under Different Loss Function" ; Journal of Babylon University,pure and Applied sciences, No.(3)vol.(22), pp.976.

[4] Rasheed, H. A., Al- Noor, N. H., (2012)," Minimax Estimation of the Scale Parameter of the Laplace Distribution under Quadratic Loss Function"; International Journal for Sciences and Technology, Vol. 7 No. 3, pp. 100-108.

[5]Nasir, S.A and AL-Anber, N.J,(2012),A Comparison of the Bayesian and other Methods for Estimation of Reliability Function for Burr-xii Distribution ,Journal of Mathematics and statistics, vol.8.No.1,pp.42-48

[6] Al-Kutubi, H.S. and N.A. Ibrahim, 2009. Bayes estimator for exponential distribution with extension of Jeffery prior information. Malaysian Journal of Mathematical Sciences, vol.3,No,2,pp. 297-313.

[7]Li,L.(2014).Minimax Estimation of the parameter of Exponential Distribution based on Record Values, vol,03.pp.47-53.

[8] Park, S,; Bera, A. K. (2009), Maxmum entropy autoregressive Conditional beteroskedasticity moael, Journal of Econometrics, vol. 150, pp.219-230.

[9] Bhuiyan, M.K, ROYM.K.and Imam M,F.(2007),Minimax Estimation of the Parameter of Rayleigh Distribution, Festschrift in honor Distinguished Professor Mir Masson Ali on the occasion of his retirement,pp.207-212.

[10] De Geoot, M.H.(1970),Otimal Statistical Decisions MCGRAW-Hill Inc,United states of America.

[11] Day S.(2008) Minimax estimation of the parameter of the Rayleigh Distribution under Quadratic loss function. Data sci.J.7,pp23-30.

Table 1: Expected values and MSE's of the parameter of Laplace Distribution with $\theta = 0.5$

n	Estimator / Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{Q}$	$\hat{\theta}_{SL}$
5	Exp.(θ)	0.50168	0.41807	0.55630
	MSE	0.04910	0.04081	0.06354
10	Exp.(θ)	0.50106	0.45550	0.52721
	MSE	0.02488	0.02254	0.02828
20	Exp.(θ)	0.50197	0.47807	0.51482
	MSE	0.01275	0.01205	0.01363
50	Exp.(θ)	0.49997	0.49019	0.50514
	MSE	0.00499	0.00490	0.00512
100	Exp.(θ)	0.50048	0.49553	0.50318
	MSE	0.00249	0.00247	0.00253

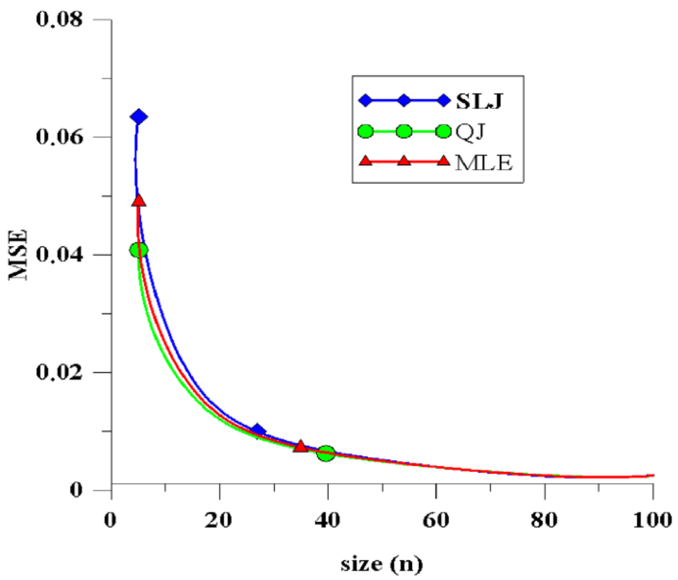

Fig.1: MSE's of the parameter of Laplace Distribution with $\theta = 0.5$

Table 2: Expected values and MSE's of the parameter of Laplace Distribution with $\theta = 1$

n	Criteria \ Estimator	$\hat{\theta}_{MLE}$	$\hat{\theta}_{Q}$	$\hat{\theta}_{SL}$
5	**Exp.(θ)**	1.00336	0.83614	1.11259
	MSE	0.19642	0.16324	0.25417
10	**Exp.(θ)**	1.00212	0.91102	1.05441
	MSE	0.09952	0.09016	0.11313
20	**Exp.(θ)**	1.00394	0.95614	1.02964
	MSE	0.05102	0.04819	0.05453
50	**Exp.(θ)**	0.99999	0.98038	1.01028
	MSE	0.01998	0.01959	0.02050
100	**Exp.(θ)**	1.00097	0.99106	1.00366
	MSE	0.00998	0.00986	0.01013

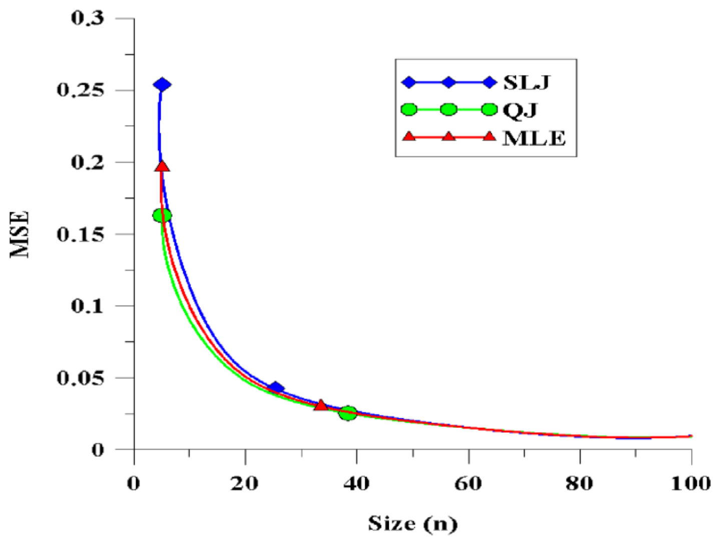

Fig.2: MSE's of the parameter of Laplace Distribution with $\theta = 1$

Table 3: Expected values and MSE's of the parameter of Laplace Distribution with $\theta = 1.5$

n	Estimator / Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{Q}$	$\hat{\theta}_{SL}$
5	**Exp.(θ)**	1.50504	1.52420	1.66889
	MSE	0.41494	0.36730	0.57189
10	**Exp.(θ)**	1.50318	1.36653	1.58162
	MSE	0.22392	0.20287	0.25455
20	**Exp.(θ)**	1.50595	1.43420	1.54446
	MSE	0.11479	0.10842	0.12269
50	**Exp.(θ)**	1.49999	1.47058	1.51541
	MSE	0.04495	0.04407	0.04612
100	**Exp.(θ)**	1.50151	1.48659	1.50954
	MSE	0.02245	0.02219	0.02278

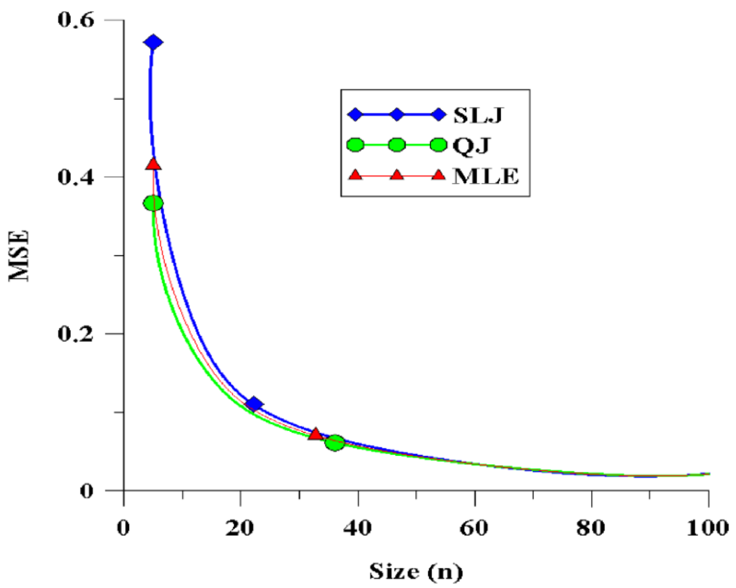

Fig.3: MSE's of the parameter of Laplace Distribution with $\theta = 1.5$

Table 4: Expected values and MSE's of the parameter of Laplace Distribution with $\theta = 3$

n	Estimator / Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{Q}$	$\hat{\theta}_{SL}$
5	Exp.(θ)	3.01009	2.50841	3.33779
	MSE	1.76775	1.46920	2.28757
10	Exp.(θ)	3.00636	2.73305	3.16324
	MSE	0.89569	0.81146	1.01821
20	Exp.(θ)	3.01183	2.86841	3.08893
	MSE	0.45917	0.43367	0.49074
50	Exp.(θ)	2.99998	2.94116	3.03082
	MSE	0.17981	0.17629	0.18447
100	Exp.(θ)	3.00290	3.03082	3.01908
	MSE	0.08981	0.08755	0.09114

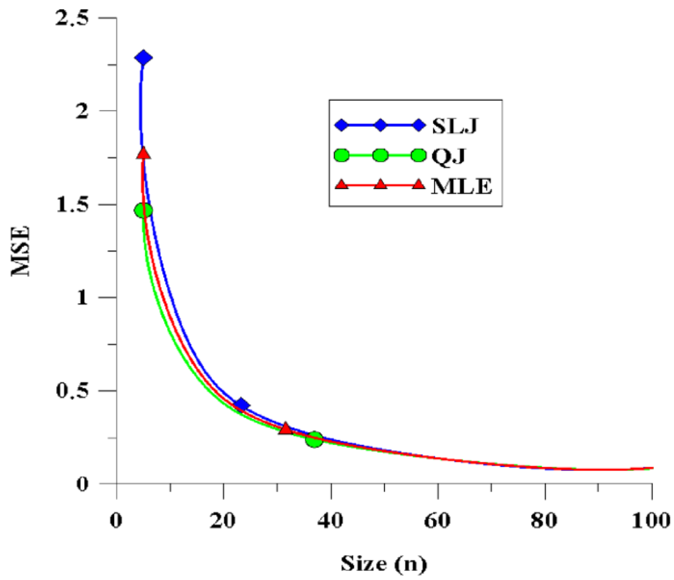

Fig.4: MSE's of the parameter of Laplace Distribution with $\theta = 3$

Table 5: Expected values and MSE's of the parameter of Laplace Distribution with $\theta = 4$

n	Estimator / Criteria	$\hat{\theta}_{MLE}$	$\hat{\theta}_{Q}$	$\hat{\theta}_{SL}$
5	**Exp.(θ)**	4.01345	3.34454	4.45038
	MSE	3.14267	2.61191	4.06680
10	**Exp.(θ)**	4.00848	3.64407	4.21766
	MSE	1.59233	1.44260	1.81015
20	**Exp.(θ)**	4.01577	3.82454	4.11858
	MSE	0.81631	0.77098	0.87244
50	**Exp.(θ)**	3.99998	3.92154	4.04110
	MSE	0.31966	0.31340	0.32793
100	**Exp.(θ)**	4.00387	3.96423	4.02544
	MSE	0.15967	0.15779	0.16202

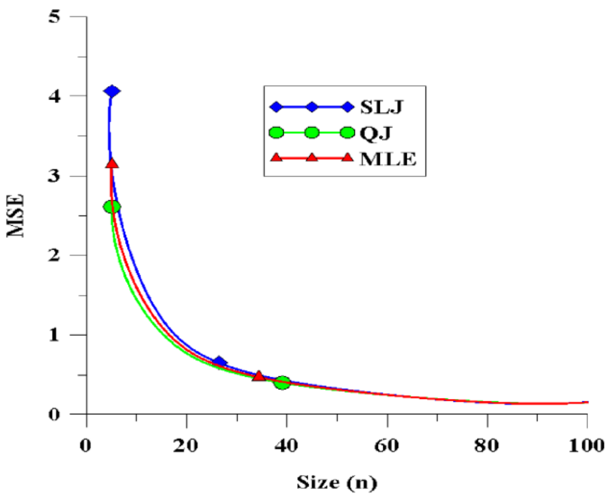

Fig.5: MSE's of the parameter of Laplace Distribution with $\theta = 4$