Minimax Estimation of the Scale Parameter of Laplace Distribution under Squared-Log Error Loss Function

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Abstract
In this paper, we obtained Minimax estimator of the scale parameter \( \theta \) for the Laplace distribution under the Squared log error loss function by applying the theorem of Lehmann [1950], and compared it with Minimax estimator under Quadratic loss function in addition of Maximum Likelihood Estimator according to Monte-Carlo simulation study. The performance of these estimators is compared depending on the mean squared errors (MSE’s).

Keywords: Minimax estimator, Laplace distribution, Bayes estimator, Squared-log error loss function, Jeffery prior, Mean squared error.

1. Introduction
The classical Laplace distribution with mean zero and variance \( \sigma^2 \) was introduced by Laplace in 1774. The distribution is symmetrical and Leptokurtic[1]. Also called the double exponential [2].

This distribution has been used for modeling data that have heavier tails than those of the normal distribution and analyze engineering, financial, industrial, environmental, and biological data (Kotz et al., 2001). [3], it used for modeling data that have heavier tails than those of the normal distribution.

The probability density function of a Laplace distributed random variable is given by [3]:

\[
 f(x|\alpha, \theta) = \frac{1}{2\theta} e^{\frac{-|x-\alpha|}{\theta}} \quad -\infty < x < \infty \tag{1}
\]

Where \( \alpha \in (-\infty, \infty) \) and \( \theta > 0 \) are location and scale parameters, respectively.

The cumulative distribution function is given by:

\[
 F(x|\alpha, \theta) = \begin{cases} 
 1 - \frac{1}{2} e^{\frac{\alpha-x}{\theta}} & \text{for } x \geq \alpha \\
 \frac{1}{2} e^{\frac{x-\alpha}{\theta}} & \text{for } x < \alpha 
\end{cases}
\]

With moment–generating function

\[
 M_x(t) = \frac{e^{it\alpha}}{1 - e^{it\theta}}
\]

We can obtain the Maximum likelihood estimator for the scale parameter \( \theta \), as follows:

Let \( X_1, \ldots, X_n \) be a random sample from density (1) (when \( \alpha \) is known). The likelihood function is given by

\[
 L(x; \alpha, \theta) = \left( \frac{1}{2\theta} \right)^n e^{-\frac{\sum|x_i-\alpha|}{\theta}}
\]

By taking the log and differentiating partially, with respect to \( \theta \)

\[
 \frac{\partial \ln L(x; \alpha, \theta)}{\partial \theta} = -\frac{n}{\theta} + \frac{\sum|x_i-\alpha|}{\theta^2} = 0 \tag{2}
\]
The MLE of $\theta$ is the solution of equation (2) after equating the first derivative to zero:

$$\hat{\theta}_{\text{MLE}} = \frac{1}{n} \sum_{i=1}^{n} |x_i - a| = \frac{T}{n}, \text{ where } T = \sum_{i=1}^{n} |x_i - a|$$  \hspace{1cm} (3)

2. Bayes Estimator using Jeffery Prior Information \[5]\[6]

Let us assume that, $\theta$ has non-information density, defined as $g \propto \sqrt{I(\theta)}$

Where $I(\theta)$ represented Fisher information which defined as follows:

$$I(\theta) = -nE\left[\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right], \text{ hence,}$$

$$g(\theta) = k \sqrt{-nE\left[\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right]}$$ \hspace{1cm} (5)

$$\ln f(x;\theta) = -\ln(2\theta) - \frac{|x-a|}{\theta}$$

$$\frac{\partial \ln f(x;\theta)}{\partial \theta} = -\frac{1}{2\theta} - \frac{|x-a|}{\theta^2}$$

$$\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2} = \frac{1}{2\theta^2} - \frac{2|x-a|}{\theta^3}$$

$$E\left[\frac{\partial^2 \ln f(x;\theta)}{\partial \theta^2}\right] = \frac{1}{2\theta^2} - \frac{2}{\theta^3} E[|x-a|] = -\frac{3}{2\theta^2}$$ \hspace{1cm} (6)

After Substitution (6) into (4), we get

$$g(\theta) = k \sqrt{-n\left(-\frac{3}{2\theta^2}\right)} = k \frac{3n}{\theta}$$ \hspace{1cm} (7)

Now, Combining the prior (7) with the likelihood function, we have the posterior distribution of $\theta$ with Jeffery’s prior information given by $h(\theta|x)$:

$$h(\theta|x) = \frac{g(\theta)L(\theta|x_1, x_2, ..., x_n)}{\int_{0}^{\infty} g(\theta)L(\theta|x_1, x_2, ..., x_n) d\theta}$$

$$h(\theta|x) = \frac{1}{\Gamma(n)2^{n}e^{\frac{-T}{2}}} \left(\frac{\sum_{i=1}^{n} |x_i-a|}{\theta}\right)^{n-1} \theta^{n-1} e^{-\sum_{i=1}^{n} |x_i-a|/\theta} d\theta$$ \hspace{1cm} (8)

On simplification, we have

$$h(\theta|x) = \left(\frac{\sum_{i=1}^{n} |x_i-a|}{\theta}\right)^{n-1} e^{-\sum_{i=1}^{n} |x_i-a|/\theta} \frac{\theta^{n-1}}{\Gamma(n)} = \frac{T^{n-\frac{T}{2}}}{\theta^{n+1}\Gamma(n)}$$

This posterior density is recognized as the density of the Inverse Gamma (IG) distribution: $\theta \sim IG(n, T)$


Brown (1968) proposed a symmetric loss function for scale parameter estimation, that is called squared log error loss function is:

$$l(\theta, \hat{\theta}) = (\ln \hat{\theta} - \ln \theta)^2 = (\ln \frac{\hat{\theta}}{\theta})^2$$ \hspace{1cm} (9)

Which is balanced, with lim $l(\hat{\theta}, \theta) \to \infty$ as $\theta \to 0 \text{ or } \infty$. A balanced loss function takes both error of estimation and goodness of fit into account but the unbalanced loss function considers only error of estimation. This loss function is convex for:

$$\frac{\theta}{\hat{\theta}} \leq e, \text{ and concave otherwise, but its risk function has a unique minimum with respect to } \hat{\theta} \text{ [4].}$$
The risk function under the squared log error loss function (9) is given by
\[
R(\hat{\theta} - \theta) = E[L(\hat{\theta}, \theta)] = \int_0^\infty (\ln \hat{\theta} - \ln \theta)^2 h(\theta|x_1, \ldots, x_n) d\theta
\]
\[
= (\ln \hat{\theta})^2 - 2(\ln \hat{\theta}) E(\ln \theta|x) + E((\ln \theta)^2|x)
\]
\[
\frac{\partial \text{Risk}}{\partial \theta} = 2(\ln \hat{\theta}) \frac{1}{\theta} - \frac{2}{\theta} E((\ln \theta)|x)
\]
By letting \( \frac{\partial R(\hat{\theta} - \theta)}{\partial \theta} = 0 \)

The risk function of \( \hat{\theta}_{SL} \) (9) has minimum w.r.t:
\[
\hat{\theta}_{SL} = \text{Exp}[E(\ln \theta|x)]
\]  
(10)
Now, \( E(\ln \theta|x) = \int_0^\infty \ln \theta h(\theta|x) d\theta \)

\[
E(\ln \theta|x) = \int_0^\infty \ln \theta \frac{(\sum_{i=1}^n|x_i-a|)^n e^{-\frac{\sum_{i=1}^n|x_i-a|}{\theta}}}{\theta^{n+1} \Gamma(n)} d\theta
\]
\[
= \int_0^\infty \ln \theta \frac{e^{\frac{\sum_{i=1}^n|x_i-a|}{\theta}}}{\theta^{n+1} \Gamma(n)} d\theta
\]
\[let \ y = \frac{\sum_{i=1}^n|x_i-a|}{\theta}, \ which \ implies \ that, \ \theta = \frac{\sum_{i=1}^n|x_i-a|}{y}, \ \ d\theta = -\frac{\sum_{i=1}^n|x_i-a|}{y^2} dy\]

Hence,
\[
E(\ln \theta|x) = \frac{\sum_{i=1}^n|x_i-a|}{\Gamma(n)} \int_0^\infty \frac{\ln(\sum_{i=1}^n|x_i-a|) - \ln(y)e^{-y}}{(\sum_{i=1}^n|x_i-a|)^{n+1}y^{n+1}} dy
\]
\[
= \frac{\ln(\sum_{i=1}^n|x_i-a|)}{\Gamma(n)} - \frac{\Gamma(n)}{\Gamma(n)} - \psi(n)
\]
\[
E(\ln \theta|x) = \ln(\sum_{i=1}^n|x_i-a|) - \psi(n)
\]  
(11)
Such that \( \psi(n) = \frac{\Gamma(n)}{\Gamma(n)} \), where \( \psi(n) \) is digamma function

Substituting (11) in (10), we get the Bayes estimator of parameter \( \theta \) under Squared-Log error loss function with Jeffrey prior information as:
\[
\hat{\theta}_{SL} = \text{Exp}[\ln(\sum_{i=1}^n|x_i-a|) - \psi(n)]
\]
\[
\hat{\theta}_{SL} = \frac{\sum_{i=1}^n|x_i-a|}{e^{\psi(n)}}
\]

4. Bayesian Estimators under Quadratic Loss Function [9].
De Groot (1970) [10] discussed different types of loss function and obtained the Bayes estimates under these loss function which is a non-negative symmetric and continuous loss function [16], which is defined as:
\[
L(\hat{\theta}, \theta) = \left( \frac{\hat{\theta} - \theta}{\theta} \right)^2 = (1 - \frac{\hat{\theta}^2}{\theta})
\]  
(12)
Now, the risk function under the Quadratic Loss function (QLF) is given by:
\[
R^Q(\theta, \hat{\theta}) = E\left(1 - \frac{\hat{\theta}}{\theta}\right)^2
\]
(13)
Taking the partial derivative for \(R^Q(\hat{\theta}, \theta)\) with respect to \(\hat{\theta}\) and setting it equal to zero yields, \(\hat{\theta}_Q\) which is minimizes the risk function of QLF.
\[
\hat{\theta}_Q = \frac{\int \hat{\theta} \, d\hat{\theta}}{\int \frac{1}{\hat{\theta}^2} \, d\hat{\theta}}
\]
(14)
Recall that: \(\theta \sim IG(n, T)\)
Then, we can easily prove that, \(\left(\frac{1}{\hat{\theta}}\right) \sim \Gamma(n, T^{-1})\), with,
\[
E\left(\frac{1}{\theta}\right) = \frac{n}{T}, \quad \text{ver} \left(\frac{1}{\theta}\right) = \frac{n}{T^2}
\]
\[
\Rightarrow E\left(\frac{1}{\theta^2}\right) = \frac{n(n+1)}{T^2} = \frac{n(n+1)}{\left(\sum_{x=1}^{n} |x-a|\right)^2}
\]
After Substituting, we get the Bayes estimator of parameter \(\theta\) under Quadratic loss function with Jeffrey's prior information as:
\[
\hat{\theta}_Q = \frac{\sum_{x=1}^{n} |x-a|}{(n+1)}
\]
5. Minimax Estimators
The derivation of Minimax estimators depends primarily on a theorem due to Lehmann which can be stated as follows:
**Lehmann's Theorem:** [11]
Let \(\tau = \{F_\theta; \theta \in \Theta\}\) be a family of distribution functions and \(D\) a class of estimators of \(\theta\). Suppose that \(d^* \in D\) is a Bayes’ estimator against a prior distribution \(\xi^*(\theta)\) on the parameter space \(\Theta\) and the risk function \(R(d^*, \theta) = \text{constant on } \Theta;\) then \(d^*\) is a minimax estimator of \(\theta\).
The main results are contained in the following Theorem.
**Theorem 5.1**
Let \(x = (x_1, x_2, \ldots, x_n)\) be n independently and identically distributed random variables drawn from the density (1), then
\[
\hat{\theta}_{SL} = \frac{\sum_{x=1}^{n} |x-a|}{\sum_{x=1}^{n}}
\]
(15)
Is the minimax estimator of the parameter \(\theta\) under the Squared log error loss (9).
**Theorem 5.2**
Let \(X = (x_1, x_2, \ldots, x_n)\) be n independently and identically distributed random variables drawn from the density (1), then
\[
\hat{\theta}_{Q} = \frac{\sum_{x=1}^{n} |x-a|}{n+c}
\]
(16)
Is the minimax estimator of the parameter \(\theta\) under the quadratic loss function (12).
To prove the theorem 5.1 and 5.2, we shall use Lehmann’s theorem, by showing that, the risk function of \(\hat{\theta}\) is a constant.
Firstly, we have to prove the theorem (5.1). The Risk function of the estimator \(\hat{\theta}_{SL}\) is:
\[
R_{\hat{\theta}_{SL}}(\theta) = E[L(\theta|\hat{\theta}_{SL})]
\]
\[
= \int_0^\infty (\ln \hat{\theta} - \ln \theta)^2 h(\theta|x) \, dx
\]
According to (15), we have
\[
\ln \hat{\theta} = \ln(\sum_{i=1}^{n}|x_i - a|) - \psi(n)
\]
\[
R_{\hat{\theta}, SL}(\theta) = E \left[ \frac{\ln(\sum_{i=1}^{n}|x_i - a|)}{n} \right] - \psi(n) \right]^{2} - 2\ln(\theta)E(\ln(\sum_{i=1}^{n}|x_i - a|)) + (\psi(n))^{2} - 2\ln(\theta)\psi(n) + (\ln \theta)^{2} + 2(\ln(\theta)^{2} + 2\psi(n)(\ln(\theta) + (\psi(n))^{2} - 2\psi(n)(\ln \theta) - 2(\ln(\theta)^{2} + 2\psi(n)(\ln \theta) + (\ln \theta)^{2})
\]
\[
R_{\hat{\theta}, SL}(\theta) = \psi'(n) + (\psi(n) + \ln \theta)^{2} - 2\psi(n)(\psi(n) + \ln \theta) + (\psi(n))^{2} - 2(\ln \theta)\psi(n) + (\ln \theta)^{2} + 2(\ln(\theta)^{2} + 2\psi(n)(\ln \theta) + (\ln \theta)^{2} - 2\psi(n)(\ln \theta) - 2(\ln(\theta)^{2} + 2\psi(n)(\ln \theta) + (\ln \theta)^{2})
\]
\[
R_{\hat{\theta}, SL}(\theta) = \psi'(n) + (\psi(n) + \ln \theta)^{2} - 2\psi(n)(\psi(n) + \ln \theta) + (\psi(n))^{2} - 2(\ln \theta)\psi(n) + (\ln \theta)^{2} + 2(\ln(\theta)^{2} + 2\psi(n)(\ln \theta) + (\ln \theta)^{2} - 2\psi(n)(\ln \theta) - 2(\ln(\theta)^{2} + 2\psi(n)(\ln \theta) + (\ln \theta)^{2})
\]
So, according to the Lehmann's theorem, it follows that:
\[
\hat{\theta}_{SL} = \frac{\sum_{i=1}^{n}|x_i - a|}{n} \text{ is minimax estimator for the scale parameter } \theta \text{ of Laplace distribution under the squared–log error loss function.}
\]
Now, we have to prove the theorem (5.2). The Risk function of the estimator \( \hat{\theta}_{Q} \) is:
\[
R_{\theta, Q}(\theta) = E[L(\theta | \hat{\theta}_{Q})] = E\left[\frac{\theta - \hat{\theta}_{Q}}{\theta}\right]^{2}
\]
\[
= \frac{1}{\theta^{2}} \left[ \theta^{2} - 2\theta E(\hat{\theta}_{Q}) + E(\hat{\theta}_{Q})^{2} \right]
\]
\[
= \frac{1}{\theta^{2}} \left[ \theta^{2} - \frac{2\theta}{(n+1)}E(\sum_{i=1}^{n}|x_i - a|) + \frac{1}{(n+1)^{2}}E(\sum_{i=1}^{n}|x_i - a|)^{2} \right]
\]
Recall that, \( \sum_{i=1}^{n}|x_i - a| \sim \Gamma(n, \theta^{-1}) \), hence,
\[
E(\sum_{i=1}^{n}|x_i - a|) = n\theta \text{ and } \text{var}(\sum_{i=1}^{n}|x_i - a|) = n\theta^{2}
\]
Thus, after substituting in (19), we get
\[
R_{\theta, Q}(\theta) = \frac{1}{\theta^{2}} \left[ \theta^{2} - \frac{2\theta}{(n+1)}n\theta + \frac{1}{(n+1)^{2}}n\theta^{2} \right]
\]
\[
= 1 - \frac{2n}{(n+1)} + \frac{n}{(n+1)^{2}} \text{, which is a constant.}
\]
So according to the Lehmann's theorem, it follows that:
\[
\hat{\theta}_{Q} = \frac{\sum_{i=1}^{n}|x_i - a|}{n+1}, \text{ is the minimax estimator of the parameter } \theta \text{ of the Laplace distribution under the quadratic loss function.}
\]
6. Simulation Results
In our simulation study, we generated $I = 5000$ samples of size $n = 5, 10, 20, 50$ and $100$ from Laplace distribution to represent small, moderate and large sample size with the scale parameter $\theta = 0.5, 1, 1.5, 3, and 4$. In this section, Monte Carlo simulation study is performed to compare the methods of estimation by using mean square Errors (MSE’s) as an index for precision to compare the efficiency of each of estimators, where:

$$MSE(\hat{\theta}) = \frac{\sum_{i=1}^{I}(\hat{\theta}_i - \theta)^2}{I}$$

The results were summarized and tabulated in the following tables for each estimator and for all sample sizes. The expectations and MSE’s for $\theta$ are schedule in tables (1, 2, 3, 4 and 5).

7. Discussion

The results of the simulation study for estimating the scale parameter ($\theta$) of Laplace distribution when the location parameter, ($\alpha$) is known, are summarized and tabulated in tables (1, 2, 3, 4 and 5) which contain the Expected values and MSE’s, we have observed that:

- The performance of Bayes estimator under Quadratic loss function with Jeffery prior information is the best estimator, comparing to other estimator for all sample sizes and with all values of the scale parameter, followed by the Maximum Likelihood Estimator.
- It is observed that, MSE's of all estimators of scale parameter is increasing with the increase of the scale parameter value.
- Finally, for all parameter values, an obvious reduction in MSE's is observed with the increase in sample size.

References

[1] Krishnamoorthy, k.(2006); Handbook of statistical distribution with application with application; chapman and Hall/crc.


Table 1: Expected values and MSE’s of the parameter of Laplace Distribution with $\theta = 0.5$

<table>
<thead>
<tr>
<th>n</th>
<th>Estimator Criteria</th>
<th>$\hat{\theta}_{MLE}$</th>
<th>$\hat{\theta}_Q$</th>
<th>$\hat{\theta}_{SL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>$\text{Exp.}(\theta)$</td>
<td>0.50168</td>
<td>0.41807</td>
<td>0.55630</td>
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<tr>
<td></td>
<td>MSE</td>
<td>0.04910</td>
<td>0.04081</td>
<td>0.06354</td>
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<tr>
<td>10</td>
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<td>0.45550</td>
<td>0.52721</td>
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<td></td>
<td>MSE</td>
<td>0.02488</td>
<td>0.02254</td>
<td>0.02828</td>
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<td>20</td>
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<td>0.50197</td>
<td>0.47807</td>
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<tr>
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<td>MSE</td>
<td>0.01275</td>
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<td>50</td>
<td>$\text{Exp.}(\theta)$</td>
<td>0.49997</td>
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<td>0.00499</td>
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<td>0.50048</td>
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<td>MSE</td>
<td>0.00249</td>
<td>0.00247</td>
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Fig.1: MSE’s of the parameter of Laplace Distribution with $\theta = 0.5$
Table 2: Expected values and MSE’s of the parameter of Laplace Distribution with $\theta = 1$

<table>
<thead>
<tr>
<th>n</th>
<th>Estimator Criteria</th>
<th>$\hat{\theta}_{MLE}$</th>
<th>$\hat{\theta}_Q$</th>
<th>$\hat{\theta}_{SL}$</th>
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<tbody>
<tr>
<td>5</td>
<td>Exp($\theta$)</td>
<td>1.00336</td>
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<td>MSE</td>
<td>0.19642</td>
<td>0.16324</td>
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<td>MSE</td>
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<td>0.09016</td>
<td>0.11313</td>
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<td>0.95614</td>
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<td>MSE</td>
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<td>0.04819</td>
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<td>Exp($\theta$)</td>
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<td>0.98038</td>
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<td>MSE</td>
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<td>MSE</td>
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<td>0.00986</td>
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Fig.2: MSE’s of the parameter of Laplace Distribution with $\theta = 1$
Table 3: Expected values and MSE’s of the parameter of Laplace Distribution with $\theta = 1.5$

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<tr>
<th>n</th>
<th>Estimator Criteria</th>
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<th>$\hat{\theta}_Q$</th>
<th>$\hat{\theta}_{SL}$</th>
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<td>MSE</td>
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<td>MSE</td>
<td>0.22392</td>
<td>0.20287</td>
<td>0.25455</td>
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<td>20</td>
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<td>1.50595</td>
<td>1.43420</td>
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<td>MSE</td>
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<td>0.12269</td>
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<td>Exp.$(\theta)$</td>
<td>1.49999</td>
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<td>0.04407</td>
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<td>0.02245</td>
<td>0.02219</td>
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Fig. 3: MSE’s of the parameter of Laplace Distribution with $\theta = 1.5$
Table 4: Expected values and MSE’s of the parameter of Laplace Distribution with $\theta = 3$

<table>
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<th>n</th>
<th>Estimator</th>
<th>$\hat{\theta}_{MLE}$</th>
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<th>$\hat{\theta}_{SL}$</th>
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<td>MSE</td>
<td>0.89569</td>
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Fig.4: MSE’s of the parameter of Laplace Distribution with $\theta = 3$
### Table 5: Expected values and MSE’s of the parameter of Laplace Distribution with $\theta = 4$

<table>
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<tr>
<th>n</th>
<th>Estimator Criteria</th>
<th>$\hat{\theta}_{\text{MLE}}$</th>
<th>$\hat{\theta}_Q$</th>
<th>$\hat{\theta}_{\text{SL}}$</th>
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### Fig.5: MSE’s of the parameter of Laplace Distribution with $\theta = 4$
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