Boundary value problems for hybrid differential equations

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Abstract

This note is motived from some papers treating the hybrid differential equations. An existence theorem for this equation is proved. Some fundamental differential inequalities for hybrid differential equatins are also established which are utilized to prove the existence of extremal solutions. Necessary tools are considered and the comparison principle is proved which will be useful for further study of qualitative behavior of solutions.

1. Introduction

Quadratic perturbations of nonlinear differential equations have attracted much attention. These are called hybrid differential equations. there have been many works on the theory of hybrid differential equations, and we refer the readers to the articles [6-12]. The existence theory for such hybrid equations can be developed using hybrid fixed point theory. see [6-8]. The theory of differential inequalities for hybrid differential equations is crucial in the qualitative study of nonlinear differential equations. It is known that the differential inequalities play a significant role in the study of extremal solutions for nonlinear differential equations. In the present paper, we establish the existence and uniqueness results and some fundamental inequalities for boundary value problems for hybrid differential equations.

Dhage and Lakshmikantham [8] discussed the following first order hybrid differential equation

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) \quad \text{a.e} \quad t \in J=[0,T] \\ x(t_0) = x_0 \in IR \end{cases}$$

where $f \in C(J \times IR, IR \setminus \{0\})$ and $g \in c(J \times IR, IR)$. They established the existence, uniqueness results and some fundamental differential inequalities for hybrid differential equations initiating the study of theory of such systems and proved utilizing the theory of inequalities, its existence of extremal solutions and a comparison results.

Benchohra and al. [2] are discussed the following boundary value problems for differential equations with fractional order

$$\begin{cases} {}^cD^{\alpha} y(t) = f(t,y(t)) \quad \text{a.e} \quad t \in J=[0,T] \text{ , } 0 \langle \alpha \langle 1 \\ ay(0) + by(T) = c \end{cases}$$

where ${}^cD^{\alpha}$ is the Caputo fractional derivative, $f:[0,T] \times IR \to IR$, is a continuous function, a, b, c are real constants with $a + b \neq 0$.

They presented existence results for this problem and some results one based on Banach fixed point theorem and another one based on Schaefer's fixed point theorem.

2. Boundary value problems for hybrid differential equations with fractional order

In this section, we introduce notations, definitions, and preliminaries facts which are used throughout this paper. By $X = C(J, IR)$ we denote the Banach space of all continuous functions from $\mathrm{J} = [0, \mathrm{T}]$ into IR with the norm $\|y\| = \{|y(t)|, t \in J\}$

Let $c(J \times IR, IR)$ denote the class of functions $g : J \times IR \to IR$ such that

(i) the map $t \mapsto g(t, x)$ is mesurable for each $x \in IR$, and

(ii) the map $x \mapsto g(t, x)$ is continuous for each $t \in IR$.

The class $c(J \times IR, IR)$ is called the Carathéodory class of functions on $J \times IR$ which are Lebesgue integrable when bounded by a Lebesgue integrable function on J .

By $L^1(J, IR)$ denote the space of Lebesgue integrable real-valued functions on J equipped with the norm $\|\cdot\|_{L_1}$ defined by $\|x\|_{L_1} = \int_0^T |x(s)| ds$

We consider boundary value problems for hybrid differential equations (BVPHDE of short)

$$\begin{cases} \dfrac{d}{dt}\left[\dfrac{x(t)}{f(t, x(t))}\right] = g(t, x(t)) \qquad \text{a.e} \qquad t \in \mathrm{J} = [0, T] \\ a\dfrac{x(0)}{f(0, x(0))} + b\dfrac{x(T)}{f(T, x(T))} = c \end{cases} \qquad (2.1)$$

where $f \in C(J \times IR, IR \setminus \{0\})$, $g \in c(J \times IR, IR)$ and a, b, c are real constants with $a + b \neq 0$

By a solution of the BVPHDE(2.1) we mean a function $x \in AC(J, IR)$ such that

(i) the function $t \mapsto \dfrac{x}{f(t, x)}$ is absolutely continuous for each $x \in IR$, and

(ii) x satisfies the equations in (2.1).

Where $AC(J, IR)$ is the space of absolutely continuous real-valued functions defined on J .

The theory of strict and nonstrict differential inequalities related to the ODEs and hybrid differential equations is available in the literature (see |[8,17]). It is known that differential inequalities are useful for proving the existence of extremal solutions of the ODEs and hybrid differential equations defined on J.

3. Existence result

In this section, we prove the existence results for the boundary value problems for hybrid differential equations with fractional order (2.1) on the closed and bounded interval $J = [0, T]$ under mixed Lipschitz and Carathéodory conditions on the nonlinearities involved in it.

We defined the multiplication in X by

$(xy)(t) = x(t)y(t)$, for $x, y \in X$

Clearly $X = C(J; IR)$ is a Banach algebra with respect to above norm and multiplication in it.

Theorem 3.1 [6] Let S be a non-empty, closed convex and bounded subset of the Banach algebra X and let $A: X \to X$ and $B: X \to X$ two operators such that

(a) A is Lipschitzian with a Lipschitz constant α

(b) B is completely continuous,

(c) $x = AxBy \Rightarrow x \in S$ for all $y \in S$, and

(d) $M\Psi(r)\langle r$ where $M = \|B(S)\| = \sup\{B(x) : x \in S\}$

then the operator equation $\mathrm{AxBx} = \mathrm{x}$ has a solution in S

We make the following assumptions:

(H_0) The function $x \mapsto \dfrac{x}{f(t,x)}$ is increasing in IR almost every where for $t \in J$.

(H_1) There exists a constant $L\rangle 0$ such that $|f(t,x) - f(t,y)| \le L|x-y|$

for all $t \in J$ and $x, y \in IR$.

(H_2) There exists a function $h \in L^1(J, IR)$ such that $|g(t,x)| \le h(t)$ a.e $t \in J$.

For all $x \in J$

The following lemma is useful in what follows.

Lemma 3.1 Assume that hypothesis (H_0) holds and a, b, c are real constants with $a + b \neq 0$ .Then for any $h \in L^1(J, IR)$ The function $x \in C(J, IR)$ is a solution of the BVPHDE

$$\begin{cases} \dfrac{d}{dt}\left[\dfrac{x(t)}{f(t,x(t))}\right] = h(t) \quad \text{a.e} \quad t \in J = [0,T] \\ a\dfrac{x(0)}{f(0,x(0))} + b\dfrac{x(T)}{f(T,x(T))} = c \end{cases} \tag{3.1}$$

if and only if x satisfies the hybrid integral equation

$$x(t) = [f(t,x(t))]\left(\int_0^t h(s)ds - \frac{1}{a+b}\left(b\int_0^T h(s)ds - c\right)\right) \tag{3.2}$$

Proof Assume that x is a solution of the problem (3.2). By definition, $\frac{x(t)}{f(t,x(t))}$ is continuous, then by differentiation we obtain the first equation in (3.1). Again, substituting $t=0$ and $t=T$ in (3.2) we have

$$\frac{x(0)}{f(0,x(0))}=\frac{-1}{a+b}\left(b\int_0^T h(s)\,ds-c\right)$$

$$\frac{x(T)}{f(T,x(T))}=\int_0^T h(s)\,ds-\frac{1}{a+b}\left(b\int_0^T h(s)\,ds-c\right)$$

Then

$$a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=\frac{-ab}{a+b}\int_0^T h(s)ds+\frac{ac}{a+b}+b\int_0^T h(s)ds-\frac{b^2}{a+b}\int_0^T h(s)ds+\frac{bc}{a+b}$$

this implies that

$$a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=c$$

Conversely, Let $h\in L^1(J,IR)$ By definition, $\frac{x(t)}{f(t,x(t))}$ is absolutely continuous, and so, almost every differentiable,

whence $\frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right)$ is Lebesgue integrable on J. Applying integration to $\frac{d}{dt}\left(\frac{x(t)}{f(t,x(t))}\right)=h(t)$ from 0 to t , we obtain

$$\frac{x(t)}{f(t,x(t))}=\frac{x(0)}{f(0,x(0))}+\int_0^t h(s)ds$$

Then

$$b\frac{x(T)}{f(T,x(T))}=b\frac{x(0)}{f(0,x(0))}+\int_0^T h(s)ds$$

Thus
$$a\frac{x(0)}{f(0,x(0))}+b\frac{x(T)}{f(T,x(T))}=(a+b)\frac{x(0)}{f(0,x(0))}+b\int_0^T h(s)ds$$

implies that

$$\frac{x(0)}{f(0,x(0))}=\frac{1}{a+b}\left(c-b\int_0^T h(s)ds\right)$$

consequently

$$x(t)=\left[f(t,x(t))\right]\left(\int_0^t h(s)ds-\frac{1}{a+b}\left(b\int_0^T h(s)ds-c\right)\right)$$

Theorem 3.2 Assume that hypothesis $(H_0)\_(H_2)$ hold and a, b, c are real constants with $a+b\neq 0$
Further if

$$L\left(\|h\|_{L^1}\left(1+\frac{|b|}{a+b}\right)+\frac{|c|}{a+b}\right)\ \langle\ 1 \qquad (3.3)$$

then the hybrid differential equation (2.1) has a solution defined on J .

Proof

We defined a subset S of X by $S=\{x\in X\ /\ \|x\|\leq N\}$

where, $N=\dfrac{F_0\left(\|h\|_{L^1}\left(1+\frac{|b|}{a+b}\right)+\frac{|c|}{a+b}\right)}{1-L\left(\|h\|_{L^1}\left(1+\frac{|b|}{a+b}\right)+\frac{|c|}{a+b}\right)}$ and $F_0=\sup_{t\in J}|f(t,0)|$

It is clear that S satisfies hypothesis of theorem 4.1 . By a application of Lemma 3.1, the equation (2.1) is equivalent to the nonlinear hybrid integral equation

$$x(t)=\left[f(t,x(t))\right]\left(\int_0^t g(s,x(s))ds-\frac{1}{a+b}\left(b\int_0^T g(s,x(s))ds-c\right)\right) \qquad (3.4)$$

Define two operators by $A:X\to X$ and $B:S\to X$ by

$$Ax(t)=f(t,x(t)),\quad t\in J$$

$$Bx(t)=\int_0^t g(s,x(s))ds-\frac{1}{a+b}\left(b\int_0^T g(s,x(s))ds-c\right) \qquad (3.4)$$

Then the hybrid integral equation (3.4) is transformed into the operator equation as

$$x(t)=Ax(t)Bx(t)\quad ,\quad t\in J$$

We shall show that the operators A and B satisfy all the conditions of Theorem 3.1.

Claim 1, let $x,y\in X$ then by hypothesis (H_1),

$$|Ax(t)-Ay(t)|=|f(t,x(t))-f(t,y(t))|\leq L|x(t)-y(t)|\leq\|x-y\|$$

for all $t\in J$. Taking supremum over t, we obtain

$$\|Ax-Ay\|\leq L\|x-y\|$$

for all $x,y\in X$

Claim 2, we show that B is continuous in S .

Let $(x_n)_n$ be a sequence in S converging to a point $x \in X$. Then by Lebesgue dominated convergence theorem , $\lim_{n\to+\infty}\int_0^t g(s,x_n(s))ds = \int_0^t \lim_{n\to+\infty} g(s,x_n(s))ds$

and $\lim_{n\to+\infty} b\int_0^T g(s,x_n(s))ds = b\int_0^T \lim_{n\to+\infty} g(s,x_n(s))ds$

then

$$\begin{aligned}\lim_{n\to+\infty} B_n x(t) &= \lim_{n\to+\infty}\left[\int_0^t g(s,x_n(s))\,ds - \frac{1}{a+b}\left(b\int_0^T g(s,x_n(s))\,ds - c\right)\right]\\ &= \lim_{n\to+\infty}\int_0^t g(s,x_n(s))\,ds - \lim_{n\to+\infty}\frac{1}{a+b}\left(b\int_0^T g(s,x_n(s))\,ds - c\right)\\ &= \int_0^t g(s,x(s))\,ds - \frac{1}{a+b}\left(b\int_0^T g(s,x(s))\,ds - c\right)\\ &= \mathrm{Bx}(t)\end{aligned}$$

for all $t \in J$. This shows that B is a continuous operator on S

Claim 3, B is compact operator on S .

First, we show that B(S) is a uniformly bounded set in X .

Let $x \in X$ then by hypothesis (H_2) for all $t \in J$

$$\begin{aligned}Bx(t) &\leq \int_0^t |g(s,x(s))|\,ds + \frac{1}{a+b}\left(b\int_0^T |g(s,x(s))|\,ds + |c|\right)\\ &\leq \int_0^t |h(s)|\,ds + \frac{|b|}{|a+b|}\left(b\int_0^T |h(s)|\,ds + \frac{|c|}{|a+b|}\right)\\ &\leq \|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\end{aligned}$$

Thus $\|Bx\| \leq \|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}$ for all $x \in S$

This shows that B is uniformly bounded on S .

Next, we show that $B(S)$ is an equi-continuous set on X .

We set $p(t) = \int_0^t h(s)ds$.

Let $t_1, t_2 \in J$, then for any $x \in S$

$$\begin{aligned}\left|Bx(t_1)-Bx(t_2)\right| &= \left|\int_0^{t_1} g(s,x(s))\,ds - \int_0^{t_2} g(s,x(s))\,ds\right| \\ &\le \left|\int_{t_2}^{t_1} \left|g(s,x(s))\right| ds\right| \\ &\le \left|p(t_1)-p(t_2)\right|\end{aligned}$$

Since p is continuous on compact J , it is uniformly continuous. Hence

$$\forall \varepsilon > 0, \quad \exists \eta > 0 \quad : \quad |t_1 - t_2| < \eta \quad \Rightarrow \quad \left|Bx(t_1)-Bx(t_2)\right| < \varepsilon$$

for all $t_1, t_2 \in J$ and for all $x \in X$

This shows that $B(S)$ is an equi-continuous set in X .

Then by Arzela-Ascoli theorem, B is a continuous and compact operator on S .

Claim 4, The hypothesis (c) of theorem 3.1 is satisfied

Let $x \in X$ and $y \in Y$ be arbitrary such that $x = AxBy$ .Then,

$$\begin{aligned}|x(t)| &= |Ax(t)||By(t)| \\ &\le \left|f(t,(x(t)))\right|\left|\int_0^t g(s,x(s))\,ds - \frac{1}{a+b}\left(b\int_0^T g(s,x(s))\,ds - c\right)\right| \\ &\le \left[\left|f(t,(x(t)))-f(t,0)\right| + \left|f(t,0)\right|\right]\left(\int_0^T h(s)ds + \frac{|b|}{|a+b|}\int_0^T h(s)ds + \frac{|c|}{|a+b|}\right) \\ &\le \left[Lx(t)+F_0\right]\left(\|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\right)\end{aligned}$$

and so,

$$|x(t)| - L\left(\|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\right)|x(t)| \le F_0\left(\|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\right)$$

which implies $|x(t)| \le \dfrac{F_0\left(\|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\right)}{1-L\left(\|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\right)}$

Taking supremum over t, $\|x\| \le \dfrac{F_0\left(\|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\right)}{1-L\left(\|h\|_{L^1}\left(1+\frac{|b|}{|a+b|}\right)+\frac{|c|}{|a+b|}\right)}$

Then $x \in S$ and the hypothesis (c) of theorem 3.1 is satisfied.

Finally, we have

$$M = \|B(S)\| = \{Bx : x \in S\} \leq \|h\|_{L^1}\left(1 + \left|\frac{b}{a+b}\right|\right) + \frac{|c|}{|a+b|}$$

and so, $$\alpha M \leq \left(\|h\|_{L^1}\left(1 + \frac{|b|}{|a+b|}\right) + \frac{|c|}{|a+b|}\right) \langle\ 1$$

Thus, all the conditions of Theorem (3.1) are satisfied and hence the operator equation $x = AxBx$ has a solution in S. As a result, the BVPHDEF (2.1) has a solution defined on J. This completes the proof.

4. Hybrid differential inequalities

We discuss a fundamental result relative to strict inequalities for the BVPHDE (2.1).

Theorem 4.1 [8] Assume that hypothesis (H_0) holds. Suppose that there exist a functions $y, z : [0,T] \to IR$ that are locally Holder continuous such that

$$\frac{d}{dt}\left(\frac{y(t)}{f(t, y(t))}\right) \leq g(t, y(t)) \quad \text{a.e} \quad t \in J$$

$$\frac{d}{dt}\left(\frac{z(t)}{f(t, z(t))}\right) \geq g(t, z(t)) \quad \text{a.e} \quad t \in J$$

one of the inequalities being strict,Then

$y(0)\langle z(0)$ implies $y(t)\langle z(t)$ for all $t \in J$.

Theorem 4.2 Assume that hypotheses (H_0) holds and a, b, c are real constants with $a + b \neq 0$. Suppose that there exist a functions $y, z : [0,T] \to IR$ that are locally Holder continuous such that

$$\frac{d}{dt}\left(\frac{y(t)}{f(t, y(t))}\right) \leq g(t, y(t)) \quad \text{a.e} \quad t \in J \qquad (4.1)$$

$$\frac{d}{dt}\left(\frac{z(t)}{f(t, z(t))}\right) \geq g(t, z(t)) \quad \text{a.e} \quad t \in J \qquad (4.2)$$

one of the inequalities being strict, and if $a\rangle 0$, $b\langle 0$ and $y(T)\langle z(T)$ Then

$$a\frac{y(0)}{f(0, y(0))} + b\frac{y(T)}{f(T, y(T))} \ \langle\ a\frac{z(0)}{f(0, z(0))} + b\frac{z(T)}{f(T, z(T))} \qquad (4.3)$$

Implies $$y(t)\langle z(t) \qquad (4.4)$$

for all $t \in J$.

Proof

We have $a\frac{y(0)}{f(0,y(0))}+b\frac{y(T)}{f(T,y(T))}\ \langle\ a\frac{z(0)}{f(0,z(0))}+b\frac{z(T)}{f(T,z(T))}$

This implies $a\left(\frac{y(0)}{f(0,y(0))}-\frac{z(0)}{f(0,z(0))}\right)\langle\ b\left(\frac{z(T)}{f(T,z(T))}-\frac{y(T)}{f(T,y(T))}\right)$

Since $b\langle 0$ and $y(T)\langle z(T)$ hypothesis (H_0) then $\frac{z(T)}{f(T,z(T))}-\frac{y(T)}{f(T,y(T))}\ \rangle\ 0$

This shows that $a\left(\frac{y(0)}{f(0,y(0))}-\frac{z(0)}{f(0,z(0))}\right)\ \langle\ 0$ since $a\rangle 0$,

and by hypothesis (H_0) we have $y(0)\langle z(0)$.

Hence by application of theorem 4.1 yields that $y(t)\langle z(t)$

Theorem 4.3

Assume that the conditions of Theorem 4.2. hold with inequalities (4.1) and (4.2).

Suppose that there exists a real number $ M > 0 $ such that

$$g(t,x_1)-g(t,x_2)\le M\left(\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\right)\qquad \text{a.e } t\in J \qquad (4.5)$$

for all $x_1,x_2\in IR$ with $x_1\ge x_2$. Then

$$a\frac{y(0)}{f(0,y(0))}+b\frac{y(T)}{f(T,y(T))}\ \langle\ a\frac{z(0)}{f(0,z(0))}+b\frac{z(T)}{f(T,z(T))}$$

implies $y(t)\langle z(t)$ for all $t\in J$.

Proof

We set $\frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))}=\frac{z(t)}{f(t,z(t))}+\varepsilon e^{2Mt}$,for small $\varepsilon\rangle 0$ and let $Z_\varepsilon(t)=\frac{z_\varepsilon(t)}{f(t,z_\varepsilon(t))}$

$Z(t)=\frac{z(t)}{f(t,z(t))}$ for $t\in J$

So that we have $Z_\varepsilon(t)\ \rangle\ Z(t)\quad\Rightarrow\quad \mathrm{z}_\varepsilon(t)\ \rangle\ \mathrm{z}(t)$

Since $g(t,x_1)-g(t,x_2)\le M\left(\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\right)$ and $\frac{d}{dt}\left(\frac{z(t)}{f(t,z(t))}\right)\ge g(t,z(t))$

for all $t \in J$ one has

$$\begin{aligned}\frac{d}{dt}Z_\varepsilon(t) &= \frac{d}{dt}Z(t)+2M\varepsilon e^{2Mt}\\ &\geq g(t,z(t))+2M\varepsilon e^{2Mt}\\ &\geq g(t,z_\varepsilon(t))-M(Z_\varepsilon-Z)+2M\varepsilon e^{2Mt}\\ &\geq g(t,z_\varepsilon(t))-M\varepsilon e^{2Mt}+2M\varepsilon e^{2Mt}\\ &\rangle g(t,z_\varepsilon(t))\end{aligned}$$

Also, we have $z_\varepsilon(0)\rangle z(0)\rangle y(0)$. Hence, by an application of Theorem (4.1) yields that $y(t)\langle z_\varepsilon(t)$ for all $t \in J$.

By the arbitrariness of $\varepsilon\rangle 0$, taking the limits as $\varepsilon \to 0$, we have $y(t)\langle z(t)$ for all $t \in J$. This completes the proof.

Remark 4.1

Let $f(t,x)=1$ and $g(t,x)=x$ We can easily verify that f and g satisfy the condition (4.3)

5. **Existence of maximal and minimal solutions**

In this section, we shall prove the existence of maximal and minimal solutions for the BVPHDE (2.1) on $J=[0,T]$. We need the following definition in what follows.

Definition 5.1

A solution r of the BVPHDE (2.1) is said to be maximal if for any other solution x to the BVPHDE (2.1) one has $x(t)\leq r(t)$, for all $t \in J$. Similarly, a solution ρ of the BVPHDEF (2.1) is said to be minimal if $\rho(t)\leq x(t)$, for all $t \in J$, where x is any solution of the BVPHDE (2.1) on J.

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the same arguments with appropriate modifications. Given an arbitrary small real number $\varepsilon\rangle 0$, consider the following initial value problem of BVPHDE ,

$$\begin{cases}\dfrac{d}{dt}\left[\dfrac{x(t)}{f(t,x(t))}\right]=g(t,x(t))+\varepsilon \quad \text{a.e} \quad t\in \text{ J}=[0,T]\\ a\dfrac{x(0)}{f(0,x(0))}+b\dfrac{x(T)}{f(T,x(T))}=c+\varepsilon\end{cases} \qquad (5.1)$$

where $f \in C(J\times IR, IR\setminus\{0\})$ and $g\in c(J\times IR, IR)$.

An existence theorem for the BVPHDE (5.1) can be stated as follows.

Theorem 5.1 Assume that hypothesis $(H_0)\_(H_2)$ hold and a, b, c are real constants with $a+b\neq 0$. Suppose that inequality (3.3) holds. Then for every small number $\varepsilon\rangle 0$,the BVPHDE (5.1) has a solution defined on J

Proof By hypothesis, since $L\left(\|h\|_{L^1}\left(1+\frac{|b|}{a+b}\right)+\frac{|c|}{a+b}\right) \langle\ 1$

there exists an $\varepsilon_0 \rangle 0$ such that $L\left(\left(\|h\|_{L^1}+\varepsilon T\right)\left(1+\frac{|b|}{a+b}\right)+\frac{|c|}{a+b}\right) \langle\ 1$

for all $0 \langle \varepsilon \le \varepsilon_0$. Now the rest of the proof is similar to Theorem 3.2

Our main existence theorem for maximal solution for the BVPHDE (2.1) is

Theorem 5.2 Assume that hypothesis $(H_0)\_(H_2)$ hold with the conditions of Theorem 4.2 and a, b, c are real constants with $a+b \neq 0$. Furthermore, if condition (3.3) holds, then the BVPHDE (2.1) has a maximal solution defined on J .

Proof Let $\{\varepsilon_n\}_0^\infty$ be a decreasing sequence of positive real numbers such that $\lim_{n\to+\infty} \varepsilon_n = 0$, where ε_0 is a positive real number satisfying the inequality

$$L\left(\left(\|h\|_{L^1}+\varepsilon_0 T\right)\left(1+\frac{|b|}{a+b}\right)+\frac{|c|}{a+b}\right) \langle\ 1$$

The number ε_0 exists in view of inequality (3.3). By Theorem 5.1, then there exists a solution $r(t,\varepsilon_n)$ defined on J of the BVPHDE

$$\begin{cases} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = g(t,x(t)) + \varepsilon_n \quad \text{a.e} \quad t \in J=[0,T] \\ a\frac{x(0)}{f(0,x(0))} + b\frac{x(T)}{f(T,x(T))} = c + \varepsilon_n \end{cases} \qquad (5.2)$$

Then for any solution u of the BVPHDE (2.1) satisfies

$$\frac{d}{dt}\left(\frac{u(t)}{f(t,u(t))}\right) \le g(t,u(t)),$$

and any solution of auxiliary problem (5.2) satisfies

$$\frac{d}{dt}\left(\frac{r(t,\varepsilon_n)}{f(t,r(t,\varepsilon_n))}\right) = g(t,r(t,\varepsilon_n)) + \varepsilon_n \rangle g(t,r(t,\varepsilon_n)) ,$$

where $a\frac{u(0)}{f(0,u(0))} + b\frac{u(T)}{f(T,u(T))} = c \le c + \varepsilon_n = a\frac{r(0,\varepsilon_n)}{f(0,r(0,\varepsilon_n))} + b\frac{r(T,\varepsilon_n)}{f(T,r(T,\varepsilon_n))}$.

By Theorem 4.2, we infer that

$$u(t) \le r(t,\varepsilon_n) \qquad (5.3)$$

for all $t \in J$ and $n \in IN$.

Since $c+\varepsilon_2 = a\dfrac{r(0,\varepsilon_2)}{f(0,r(0,\varepsilon_2))}+b\dfrac{r(T,\varepsilon_2)}{f(T,r(T,\varepsilon_2))} \le a\dfrac{r(0,\varepsilon_1)}{f(0,r(0,\varepsilon_1))}+b\dfrac{r(T,\varepsilon_1)}{f(T,r(T,\varepsilon_1))} = c+\varepsilon_1$

then by theorem 4.2, we infer that $r(t,\varepsilon_2) \le r(t,\varepsilon_1)$. therefore, $r(t,\varepsilon_n)$ is a decreasing sequence of positive real numbers, and the limit

$$r(t) = \lim_{n\to\infty} r(t,\varepsilon_n) \qquad (5.4)$$

exists. we show that the convergence in (5.4) is uniform on J. To finish, it is enough to prove that the sequence $r(t,\varepsilon_n)$ is equicontinuous in $C(J,IR)$. Let $t_1,t_2 \in J$ with $t_1 \langle t_2$ be arbitrary. Then,

$$\begin{aligned}
\left|r(t_1,\varepsilon_n)-r(t_2,\varepsilon_n)\right| &= \Big|\left[f(t_1,r(t_1,\varepsilon_n))\right]\Big(\int_0^{t_1}\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds \\
&\text{-}\frac{1}{a+b}\Big(b\int_0^T\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds-c-\varepsilon_n\Big)\Big) \\
&\text{-}\left[f(t_2,r(t_2,\varepsilon_n))\right]\Big(\int_0^{t_2}\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds \\
&\text{-}\frac{1}{a+b}\Big(b\int_0^T\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds-c-\varepsilon_n\Big)\Big)\Big| \\
&= \Big|\left[f(t_1,r(t_1,\varepsilon_n))\right]\Big(\int_0^{t_1}\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds\Big) \\
&\text{-}\left[f(t_2,r(t_2,\varepsilon_n))\right]\Big(\int_0^{t_2}\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds\Big) \\
&\text{-}\Big(f(t_1,r(t_1,\varepsilon_n))-f(t_2,r(t_2,\varepsilon_n))\Big)\text{-}\frac{1}{a+b}\Big(b\int_0^T\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds-c-\varepsilon_n\Big)\Big| \\
&\le \Big|\left[f(t_1,r(t_1,\varepsilon_n))\right]\Big(\int_0^{t_1}\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds\Big) \\
&\text{-}\left[f(t_2,r(t_2,\varepsilon_n))\right]\Big(\int_0^{t_1}\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds\Big)\Big| \\
&+\Big|\Big(f(t_1,r(t_1,\varepsilon_n))-f(t_2,r(t_2,\varepsilon_n))\Big)\text{-}\frac{1}{a+b}\Big(b\int_0^T\left(g(s,r(s,\varepsilon_n))+\varepsilon_n\right)ds-c-\varepsilon_n\Big)\Big|
\end{aligned}$$

$$+\left|\left[f\left(t_2,r\left(t_{21},\varepsilon_n\right)\right)\right]\left(\int_0^{t_1}\left(g\left(s,r\left(s,\varepsilon_n\right)\right)+\varepsilon_n\right)ds\right)\right.$$

$$\left.-\left[f\left(t_2,r\left(t_2,\varepsilon_n\right)\right)\right]\left(\int_0^{t_2}\left(g\left(s,r\left(s,\varepsilon_n\right)\right)+\varepsilon_n\right)ds\right)\right|$$

$$\le\left|\left(f\left(t_1,r\left(t_1,\varepsilon_n\right)\right)-f\left(t_2,r\left(t_2,\varepsilon_n\right)\right)\right)\right|\left[\left(\|h\|_{L_1}+\varepsilon_n\right)T+\frac{|b|\left(\|h\|_{L_1}+\varepsilon_n\right)T}{|a+b|}+\frac{|c|+\varepsilon_n}{|a+b|}\right]$$

$$+\mathrm{F}\left[\left|p\left(t_1\right)-p\left(t_2\right)\right|+\left|t_1-t_2\right|\varepsilon_n\right]$$

where $F=\sup_{(t,x)\in J\times[-N,N]}\left|f\left(t,x\right)\right|$ and $p\left(t\right)=\int_0^t h\left(s\right)ds$.

Since f is continuous on compact set $J\times\left[-N,N\right]$, it is uniformly continuous there. Hence,

$$\left|f\left(t_1,r\left(t_1,\varepsilon_n\right)\right)-f\left(t_2,r\left(t_2,\varepsilon_n\right)\right)\right|\to 0 \text{ as } t_1\to t_2$$

uniformly for all $n\in IN$. Similary, since the function p is continuous on compact set J , it is uniformly continuous and hence $\left|p\left(t_1\right)-p\left(t_2\right)\right|\to 0$ as $t_1\to t_2$

Therefore, from the above inequality, it follows that $\left|r\left(t_1,\varepsilon_n\right)-r\left(t_2,\varepsilon_n\right)\right|\to 0$ as $t_1\to t_2$

uniformly for all $n\in IN$. Therefore, $r(t,\varepsilon_n)\to r(t)$ as $n\to\infty$ for all $t\in J$.

Next, we show that the function $r(t)$ is a solution of the BVPHDE (2.1) defined on J . Now, since $r(t,\varepsilon_n)$ is a solution of the BVPHDE (5.2), we have

$$r\left(t,\varepsilon_n\right)=\left[f\left(t,r\left(t,\varepsilon_n\right)\right)\right]\left(\int_0^t g\left(s,r\left(s,\varepsilon_n\right)+\varepsilon_n\right)ds\right.$$
$$\left.-\frac{1}{a+b}\left(b\int_0^T g\left(s,r\left(s,\varepsilon_n\right)+\varepsilon_n\right)ds-c-\varepsilon_n\right)\right)$$

for all $t\in J$. Taking the limit as $n\to\infty$ in the above this equation yields

$$r\left(t\right)=\left[f\left(t,r\left(s\right)\right)\right]\left(\int_0^t g\left(s,r\left(s\right)\right)ds-\frac{1}{a+b}\left(b\int_0^T g\left(s,r\left(s\right)\right)ds-c\right)\right)$$

for $t\in J$. Thus, the function r is a solution of the BVPHDE (2.1) on J . Finally, from inequality (5.3), it follows that $u\left(t\right)\le r\left(t\right)$ for all $t\in J$. Hence, the BVPHDE (2.1) has a maximal solution on J . This completes the proof.

6. Comparison theorems

The main problem of the differential inequalities is to estimate a bound for the solution set for the differential inequality related to the BVPHDEF (2.1). In this section, we prove that the maximal and minimal solutions serve as bounds for the solutions of the related differential inequality to BVPHDEF (2.1) on $J=\left[0,T\right]$.

Theorem 6.1 Assume that hypothesis $(H_0)\_(H_2)$ and condition (3.3) hold and a, b, c are real constants with $a+b\neq 0$. Suppose that there exists a real number M such that

$$g(t,x_1)-g(t,x_2)\leq M\left(\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\right) \qquad \text{a.e } t\in J$$

for all $x_1,x_2\in IR$ with $x_1\geq x_2$.Furthermore, if there exists a function $u\in AC(J,IR)$ such that

$$\frac{d}{dt}\left(\frac{u(t)}{f(t,u(t))}\right)\leq g(t,u(t)) \qquad \text{a.e } t\in J \tag{6.1}$$

$$a\frac{u(0)}{f(0,u(0))}+b\frac{u(T)}{f(T,u(T))}\leq c$$

Then

$$u(t)\leq r(t) \tag{6.2}$$

for all $t\in J$ where r is a maximal solution of the BVPHDE (2.1) on J

Proof Let $\varepsilon\rangle 0$ be arbitrary small. By Theorem 5.2, $r(t,\varepsilon)$ is a maximal solution of the BVPHDE (5.1) and that the limit $r(t)=\lim_{\varepsilon\to 0} r(t,\varepsilon)$

is uniform on J and the function r is a maximal solution of the BVPHDE (2.1) on J. Hence, we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\frac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}\right)=g(t,r(t,\varepsilon))+\varepsilon \qquad \text{a.e } t\in J\\ &a\frac{r(0,\varepsilon)}{f(0,r(0,\varepsilon))}+b\frac{r(T,\varepsilon)}{f(T,r(T,\varepsilon))}=c+\varepsilon\end{aligned} \tag{6.3}$$

From the above inequality it follows that

$$\begin{aligned}&\frac{d}{dt}\left(\frac{r(t,\varepsilon)}{f(t,r(t,\varepsilon))}\right)\rangle g(t,r(t,\varepsilon)) \qquad \text{a.e } t\in J\\ &a\frac{r(0,\varepsilon)}{f(0,r(0,\varepsilon))}+b\frac{r(T,\varepsilon)}{f(T,r(T,\varepsilon))}\rangle c\end{aligned} \tag{6.4}$$

Now we apply Theorem 4.3 to the inequalities (6.1) and (6.4) and conclude that

$u(t)\langle r(t,\varepsilon)$ for all $t\in J$. This further in view of limit (6.3) implies that inequality (6.2) holds on J . This completes the proof.

Theorem 6.2 Assume that hypothesis $(H_0)\_(H_2)$ and condition (3.3) hold and a, b, c are real constants with $a+b\neq 0$ and . Suppose that there exists a real number $M\rangle 0$ such that

$$g(t,x_1)-g(t,x_2)\le M\left(\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\right) \qquad \text{a.e } t\in J$$

for all $x_1, x_2 \in IR$ with $x_1 \ge x_2$. Furthermore, if there exists a function $v \in AC(J, IR)$ such that

$$\frac{d}{dt}\left(\frac{v(t)}{f(t,u(t))}\right) \rangle g(t,v(t)) \qquad \text{a.e } t\in J$$

$$a\frac{v(0)}{f(0,v(0))}+b\frac{v(T)}{f(T,v(T))} \rangle c$$

Then

$$\rho(t)\le v(t)$$

for all $t\in J$, where ρ is a minimal solution of the BVPHDE (2.1) on J.

Note that Theorem 6.1 is useful to prove the boundedness and uniqueness of the solutions for the BVPHDE (2.1) on J. A result in this direction is

Theorem 6.3 Assume that hypothesis $(H_0)\_(H_2)$ and condition (3.3) hold and a, b, c are real constants with $a+b\ne 0$. Suppose that there exists a function $G: J\times IR_+ \to IR_+$ such that

$$g(t,x_1)-g(t,x_2)\le G\left(t,\frac{x_1}{f(t,x_1)}-\frac{x_2}{f(t,x_2)}\right) \qquad \text{a.e } t\in J$$

for all $x_1, x_2 \in IR_+$. If identically zero function is the only solution of the differential equation

$$m'(t)=G(t,m(t)) \qquad \text{a.e } t\in J, \qquad \text{am}(0)+\text{bm}(T)=0 \qquad (6.5)$$

Then the BVPHDE (2.1) has a unique solution on J.

Proof

By Theorem 3.2, the BVPHDE (2.1) has a solution defined on J. Suppose that there are two solutions u_1 and u_2 of the BVPHDE (2.1) existing on J with $u_1 \rangle u_2$. Define a function

$m: J\to IR$ by

$$m(t)=\frac{u_1(t)}{f(t,u_1(t))}-\frac{u_2(t)}{f(t,u_2(t))}$$

In view of hypothesis (H_0), we concude that $m(t) \rangle 0$

$$\begin{aligned} m'(t) &= \frac{d}{dt}\left[\frac{u_1(t)}{f(t,u_1(t))}\right]-\frac{d}{dt}\left[\frac{u_2(t)}{f(t,u_2(t))}\right] \\ &\le \text{g}(t,u_1(t))-\text{g}(t,u_2(t)) \end{aligned}$$

$$\le G\left(t, \frac{u_1(t)}{f(t,u_1(t))} - \frac{u_2(t)}{f(t,u_2(t))}\right)$$
$$=G(t,m(t))$$

for almost everywhere $t \in J$, and since $m(0) = \frac{u_1(0)}{f(0,u_1(0))} - \frac{u_2(0)}{f(0,u_2(0))}$ and

$m(T) = \frac{u_1(T)}{f(T,u_1(T))} - \frac{u_2(T)}{f(T,u_2(T))}$ and

$$a\frac{u_1(0)}{f(0,u_1(0))} + \frac{u_1(T)}{f(T,u_1(T))} = a\frac{u_2(0)}{f(0,u_2(0))} + b\frac{u_2(T)}{f(T,u_2(T))}$$ we have

Now, we apply Theorem 6.1 with $f(t,x)=1$ and $c=0$ to get that $m(t) \le 0$ for all $t \in J$, where identically zero function is the only solution of the differential equation (6.5), $m(t) \le 0$ is a contradiction with $m(t) \rangle 0$. Then we can get $u_1 = u_2$. This completes the proof.

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