On s*g-a-Proper Functions

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Abstract

In this paper we introduce a new class of functions in topological spaces, namely, s*g-a-proper functions. Also, we study the basic properties and characterizations of these functions. One of the most important of equivalent definitions to the s*g-a-proper functions gives by using s*g-a-limit points of nets. Moreover we define and study s*g-a-perfect functions and s*g-a-compact functions in topological spaces and we study the relation between s*g-a-proper functions and each of proper functions, s*g-a-perfect functions, closed functions, s*g-a-closed functions and s*g-a-compact functions and we give an example when the converse may not be true.

Key words: s*g-a-proper functions, s*g-a-perfect functions, s*g-a-closed functions, s*g-a-compact functions, s*g-a-limit points, compactly s*g-a-closed sets and s*g-a-K-spaces.

Introduction

Levine, N. [6] introduced the concept of semi open sets. Also, Khan, M. and et.al. [5] introduced and investigated s*g-open sets by using the concept of semi-closed sets. Mahmood, S. and Tareq, J. [7] we introduced and study s*g-open sets and we can prove that the family of all s*g-a-open subsets of a topological space (X, τ) from a topology on X which is finer than τ. The purpose of this paper is to introduce a new class of functions, namely, s*g-a-proper functions. We give the definition by depending on the definition of s*g-a-closed functions. Also, we give useful characterizations of s*g-a-proper functions. The second equivalent definition to s*g-a-proper functions by using s*g-a-limit points of nets is more interesting than the first equivalent definition. Moreover we study the relation between s*g-a-proper functions and certain types of functions such as proper functions, s*g-a-perfect functions, closed functions, s*g-a-closed functions and s*g-a-compact functions and we give an example when the converse may not be true. Recall that a subset A of a topological space (X, τ) is called a semi-open set if there exists an open subset U of X such that U ⊆ A ⊆ cl(U) [6]. The complement of a semi-open set is said to be semi-closed [6]. An s*g-open set is also called g-open [9], s*-open [2] and w-open [8].

1. Preliminaries

1.1 Definition [5]: A subset A of a topological space (X, τ) is called s*g-open if F ⊆ A° whenever F ⊆ A and F is semi-closed in X. The complement of an s*g-open set is defined to be s*g-closed.

1.2 Definition [5]: Let (X, τ) be a topological space and A ⊆ X. Then:

i) The s*g-closure of A, denoted by A°s*g is the intersection of all s*g-closed subsets of X which contains A.

ii) The s*g-interior of A, denoted by As*g° is the union of all s*g-open subsets of X which are contained in A.

1.3 Definition[7]: A subset A of a topological space (X, τ) is called an s*g-a-open set if A ⊆ A°s*g. The complement of an s*g-a-open set is defined to be s*g-a-closed. The family of all s*g-a-open subsets of X is denoted by τs*g-a.

1.4 Definition [7]: A subset A of a topological space (X, τ) is called an s*g-a-neighborhood of a point x in X if there exists an s*g-a-open set U in X such that x ∈ U ⊆ A. The family of all s*g-a-neighborhoods of a point x ∈ X is denoted by Ns*g-a(x).

1.5 Proposition [7]: Let (X, τ) be a topological space and B be a subset of X. Then B is s*g-a-closed in X if and only if B°s*g ⊆ B.

1.6 Definition [7]: Let (X, τ) be a topological space and A ⊆ X. Then the s*g-a-closure of A, denoted by A°s*g-a is the intersection of all s*g-a-closed subsets of X which contains A.
1.7 Theorem [7]: Let \((X, \tau)\) be a topological space and \(A, B \subseteq X\). Then:

i) \(A \subseteq \overline{A^{s\alpha}} \subseteq \overline{A}\)

ii) \(\overline{A^{s\alpha}}\) is an \(s\alpha\)-closed set in \(X\).

iii) If \(A \subseteq B\), then \(\overline{A^{s\alpha}} \subseteq \overline{B^{s\alpha}}\).

iv) \(A\) is \(s\alpha\)-closed if \(\overline{A^{s\alpha}} = A\).

v) \(\overline{A^{s\alpha} \cap A^{s\alpha}} = \overline{A^{s\alpha}}\).

vi) \(x \in \overline{A^{s\alpha}}\) if \(\forall U\) an \(s\alpha\)-open set \(U\) containing \(x\), \(U \cap A \neq \emptyset\).

1.8 Proposition: Let \((X, \tau)\) be a topological space and \(Y\) be an open subspace of \(X\). If \(A\) is an \(s\alpha\)-closed set in \(X\) then \(A \cap Y\) is an \(s\alpha\)-closed set in \(Y\).

1.9 Proposition: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces. If \(A \subseteq X\) and \(B \subseteq Y\). Then if \(A \times B\) is an \(s\alpha\)-closed set in \(X \times Y\), then \(A\) and \(B\) are \(s\alpha\)-closed sets in \(X\) and \(Y\) respectively.

Proof: It is obvious.

1.10 Definition [7]: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces. A function \(f : X \rightarrow Y\) is called \(s\alpha\)-irresolute if the inverse image of every \(s\alpha\)-open subset of \(Y\) is an \(s\alpha\)-open subset of \(X\).

1.11 Proposition [7]: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces. A function \(f : X \rightarrow Y\) is \(s\alpha\)-irresolute if the inverse image of every \(s\alpha\)-closed subset of \(Y\) is an \(s\alpha\)-closed subset of \(X\).

1.12 Definition [4]: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces. A function \(f : X \rightarrow Y\) is called compact if the inverse image of every compact set in \(Y\) is a compact set in \(X\).

1.13 Definition: A family \(\{U_a\}_{a \in \Lambda}\) of \(s\alpha\)-open sets in a topological space \((X, \tau)\) is called an \(s\alpha\)-open cover of a subset \(A\) of \(X\) if \(A \subseteq \bigcup_{a \in \Lambda} U_a\).

1.14 Definition: A topological space \((X, \tau)\) is called an \(s\alpha\)-compact space if every \(s\alpha\)-open cover of \(X\) has a finite subcover.

1.15 Definition: A subset \(A\) of a topological space \((X, \tau)\) is called \(s\alpha\)-compact if every cover of \(A\) by \(s\alpha\)-open subsets of \(X\) has a finite subcover.

1.16 Proposition: Every \(s\alpha\)-compact space is a compact space.

The converse of proposition (1.16) is not true in general as shown by the following example:

1.17 Example: Let \(X\) be any infinite set and \(p \in X\), then \(\tau = \{X, \emptyset, \{p\}\}\) is a topology on \(X\). Notice that \((X, \tau)\) is a compact space. However, it is not an \(s\alpha\)-compact space, because \(\{\{p, x\} : x \in X\}\) is an \(s\alpha\)-open cover of \(X\) which has no finite subcover.

1.18 Proposition: The \(s\alpha\)-irresolute image of an \(s\alpha\)-compact space is \(s\alpha\)-compact.

Proof: It is obvious.

1.19 Definition: A subset \(F\) of a topological space \((X, \tau)\) is called compactly \(s\alpha\)-closed if \(F \cap K\) is a compact set in \(X\) for each \(s\alpha\)-compact set \(K\) in \(X\).

Clearly every \(s\alpha\)-closed subset of a topological space \((X, \tau)\) is compactly \(s\alpha\)-closed. But the converse is not true in general as shown by the following example:
1.20 Example: Let \( X = \{a, b, c\} \) and \( \tau = \{\phi, X, \{a\}\} \) be a topology on \( X \). Therefore the sets in \( \{X, \phi, \{b\}, \{c\}, \{b, c\}\} \) are \( s^g\alpha \)-closed in \( X \). Thus \( \{a\} \) is a compactly \( s^g\alpha \)-closed set in \( X \), but is not \( s^g\alpha \)-closed.

1.21 Definition: A topological space \((X, \tau)\) is called an \( s^g\alpha \)-\( K \)-space if every compactly \( s^g\alpha \)-closed subset of \( X \) is \( s^g\alpha \)-closed.

1.22 Definition: Let \((x_d)_{d \in D}\) be a net in a topological space \((X, \tau)\). Then \((x_d)_{d \in D}\) \( s^g\alpha \)-converges to \( x \in X \) (written \( x_d \xrightarrow{s^g\alpha} x \)) if for each \( s^g\alpha \)-neighborhood \( U \) of \( x \), there is some \( d_0 \in D \) such that \( d \geq d_0 \) implies \( x_d \in U \). This is sometimes said \((x_d)_{d \in D}\) \( s^g\alpha \)-converges to \( x \) if \((x_d)_{d \in D}\) is eventually in every \( s^g\alpha \)-neighborhood of \( x \). The point \( x \) is called an \( s^g\alpha \)-limit point of \((x_d)_{d \in D}\).

1.23 Proposition: Let \((X, \tau)\) be a topological space and \( A \subseteq X \). If \( x \) is a point of \( X \), then \( x \in \overline{A}^{s^g\alpha} \) if and only if there exists a net \((x_d)_{d \in D}\) in \( A \) such that \( x_d \xrightarrow{s^g\alpha} x \).

Proof: \( \Leftarrow \) Suppose that \( \exists \) a net \((x_d)_{d \in D}\) in \( A \) such that \( x_d \xrightarrow{s^g\alpha} x \). To prove that \( x \in \overline{A}^{s^g\alpha} \). Let \( U \in N_{s^g\alpha}(x) \), since \( x_d \xrightarrow{s^g\alpha} x \) \( \Rightarrow \exists d_0 \in D \) such that \( x_d \in U \), \( \forall d \geq d_0 \). But \( x_d \in A \), \( \forall d \in D \) \( \Rightarrow \) \( U \cap A \neq \phi \), \( \forall U \in N_{s^g\alpha}(x) \). Hence by theorem ((1.7),(vi)), we get \( x \in \overline{A}^{s^g\alpha} \).

Conversely, suppose that \( x \in \overline{A}^{s^g\alpha} \). To prove that \( \exists \) a net \((x_d)_{d \in D}\) in \( A \) such that \( x_d \xrightarrow{s^g\alpha} x \). Since \( x \in \overline{A}^{s^g\alpha} \), then by theorem ((1.7),(vi)), we get \( N \cap A \neq \phi \), \( \forall N \in N_{s^g\alpha}(x) \). Hence \( D = N_{s^g\alpha}(x) \) is a directed set by inclusion. Since \( N \cap A \neq \phi \), \( \forall N \in N_{s^g\alpha}(x) \) \( \Rightarrow \exists x_N \in N \cap A \). Define \( x : N_{s^g\alpha}(x) \to A \) by: \( x(N) = x_N \), \( \forall N \in N_{s^g\alpha}(x) \). Thus \((x_N)_{N \in N_{s^g\alpha}(x)} \) is a net in \( A \). To prove that \( x_N \xrightarrow{s^g\alpha} x \). Let \( U \in N_{s^g\alpha}(x) \) to find \( d_0 \in D \) such that \( x_d \in U \), \( \forall d \geq d_0 \). Let \( d_0 = U \) \( \Rightarrow \forall d \geq d_0 \) \( \Rightarrow d = M \in N_{s^g\alpha}(x) \) i.e. \( M \cap A \subseteq U \Rightarrow x_d = x(M) = x_M \in M \cap A \subseteq M \subseteq U \Rightarrow x_M \in U \Rightarrow x_d \in U \), \( \forall d \geq d_0 \). Thus \( x_N \xrightarrow{s^g\alpha} x \).

1.24 Proposition: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces. A function \( f : X \to Y \) is \( s^g\alpha \)-irresolute iff whenever \((x_d)_{d \in D}\) is a net in \( X \) such that \( x_d \xrightarrow{s^g\alpha} x \), then \( f(x_d) \xrightarrow{s^g\alpha} f(x) \) in \( Y \).

Proof: It is obvious.

1.25 Definition [3]: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces, and \( f : X \to Y \) be a function. Then \( f \) is called a proper function if:

i) \( f \) is a continuous function.

ii) \( f \times I_Z : X \times Z \to Y \times Z \) is closed for every topological space \( Z \).

2. Properties of \( s^g\alpha \)-Closed Functions

In this section we introduce a new definition (to the best of our knowledge), namely, \( s^g\alpha \)-closed functions which is weaker than closed functions, and prove some of the results which relate to this concept. Also, we explain the relationship between an \( s^g\alpha \)-closed function and an \( s^g\alpha \)-compact function.

2.1 Definition: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces. A function \( f : X \to Y \) is called an \( s^g\alpha \)-closed (resp. \( s^g\alpha \)-open) function if the image of every closed (resp. open) subset of \( X \) is an \( s^g\alpha \)-closed (resp. \( s^g\alpha \)-open) set in \( Y \).

2.2 Examples:

i) Let \( f : (\mathbb{R}, \mu) \to (\mathbb{R}, \mu) \) be a function which is defined by: \( f(x) = 0 \), \( \forall x \in \mathbb{R} \). Then \( f \) is an \( s^g\alpha \)-closed function.
ii) If $F$ is an $s*g$-$\alpha$-closed (not closed) set in $X$, then the inclusion function $\iota_F : F \to X$ is $s*g$-$\alpha$-closed, but is not a closed function.

Since every closed set is an $s*g$-$\alpha$-closed set, then we have the following proposition.

2.3 Proposition: Every closed function is an $s*g$-$\alpha$-closed function.

The converse of proposition (2.3) may not be true in general as shown by the following example.

2.4 Example: Let $X = \{a, b, c, d\}$ and $Y = \{x, y, z\}$ be sets and let $\tau = \{\emptyset, X, \{a, b, c\}, \{b, c\}, \{a\}\}$ and $\tau' = \{\emptyset, Y, \{x\}\}$ be topologies on $X$ and $Y$, respectively. So the sets in $\{X, \emptyset, \{d\}, \{a, d\}, \{b, c, d\}\}$ are closed in $X$. Also, the sets in $\{Y, \emptyset, \{z\}, \{y\}\}$ are $s*g$-$\alpha$-closed sets in $Y$. Define the function $f : X \to Y$ by: $f(a) = f(c) = z$, $f(b) = x$ and $f(d) = y$. Notice that $f$ is an $s*g$-$\alpha$-closed function. But $f$ is not a closed function, since $\{d\}$ is a closed set in $X$, but $f(\{d\}) = \{y\}$ is not a closed set in $Y$.

2.5 Theorem: Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces. A function $f : X \to Y$ is $s*g$-$\alpha$-closed if and only if for each subset $B$ of $Y$ and each open subset $U$ of $X$ containing $f^{-1}(B)$, there exists an $s*g$-$\alpha$-open set $V$ in $Y$ containing $B$ such that $f^{-1}(V) \subseteq U$.

Proof: Suppose that $B$ is an arbitrary subset of $Y$ and $U$ is an arbitrary open subset of $X$ containing $f^{-1}(B)$.

Put $V = Y - f(X - U)$. Then by definition (2.1), $V$ is an $s*g$-$\alpha$-open set in $Y$. Since $f^{-1}(B) \subseteq U$,

$\Rightarrow X - U \subseteq f^{-1}(Y - B)$

$\Rightarrow f(X - U) \subseteq Y - B$.

But $f(X - U) = B \subseteq Y$ if $f(X - U) \subseteq Y$.

Therefore, we obtain $f(F) = Y - V$ and hence $f(F)$ is an $s*g$-$\alpha$-closed set in $Y$. This shows that $f$ is an $s*g$-$\alpha$-closed function.

2.6 Proposition: Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces. A function $f : X \to Y$ is $s*g$-$\alpha$-closed if and only if $f(A)$ is $s*g$-$\alpha$-closed for each $A \subseteq X$.

Proof: Suppose that $f : X \to Y$ is an $s*g$-$\alpha$-closed function. Since $f(A) \subseteq f(A)$ and $\overline{A}$ is a closed set in $X$, then $f(\overline{A})$ is $s*g$-$\alpha$-closed in $Y$. Therefore $\overline{f(A)} = f(\overline{A})$ is $s*g$-$\alpha$-closed for each $A \subseteq X$. Conversely, assume that $f(A)$ is $s*g$-$\alpha$-closed for each $A \subseteq X$. Let $F$ be a closed subset of $X$, thus by hypothesis $f(F)$ is $s*g$-$\alpha$-closed in $Y$. Therefore $f(F)$ is an $s*g$-$\alpha$-closed function.

2.7 Proposition: Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces. A bijective function $f : X \to Y$ is an $s*g$-$\alpha$-closed function if and only if $f$ is an $s*g$-$\alpha$-open function.

Proof: Let $f : X \to Y$ be a bijective $s*g$-$\alpha$-closed function and $U$ be an open subset of $X$, thus $U^c$ is closed. Since $f$ is $s*g$-$\alpha$-closed, then $f(U^c)$ is $s*g$-$\alpha$-closed in $Y$, thus $(f(U^c))^c = f(U)$ is $s*g$-$\alpha$-open. Since $f$ is a bijective function, then $(f(U))^c = f(U)$, hence $f(U)$ is an $s*g$-$\alpha$-open set in $Y$. Therefore $f$ is an $s*g$-$\alpha$-open function.

Conversely, let $f : X \to Y$ be a bijective $s*g$-$\alpha$-open function and $F$ be a closed subset of $X$, thus $F^c$ is open. Since $f$ is $s*g$-$\alpha$-open, then $f(F^c)$ is $s*g$-$\alpha$-open in $Y$, thus $(f(F^c))^c = f(F)$ is $s*g$-$\alpha$-closed. Therefore $f$ is an $s*g$-$\alpha$-closed function.

2.8 Proposition: Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces and $f : X \to Y$ be a function. If $f(\overline{A}) = f(\overline{A})$...
for each $A \subseteq X$, then $f$ is a continuous $s^g\alpha$-closed function.

**Proof:** To prove that $f : X \to Y$ is an $s^g\alpha$-closed function. Let $F$ be a closed subset of $X$, then $\overline{F} = F$. By hypothesis $f(\overline{F}) = f(F) = f(F)\subseteq Y$, hence $f(F)$ is an $s^g\alpha$-closed set in $Y$. Therefore $g : X \to Y$ is an $s^g\alpha$-closed function. Now, to prove that $f$ is a continuous function. Since $f(\overline{A}) = f(\overline{A}) \subseteq f(\overline{A})$ for each $A \subseteq X$, thus by ([10], theorem (7.2)), $f : X \to Y$ is a continuous function.

2.9 Theorem: Let $(X, \tau)$, $(Y, \tau')$ and $(Z, \tau')$ be three topological spaces and $f : X \to Y$, $g : Y \to Z$ be two functions. Then:

i) If $f$ is closed and $g$ is $s^g\alpha$-closed, then $g \circ f$ is $s^g\alpha$-closed.

ii) If $g \circ f$ is $s^g\alpha$-closed and $f$ is continuous and onto, then $g$ is $s^g\alpha$-closed.

iii) If $g \circ f$ is $s^g\alpha$-closed and $g$ is one-to-one and $s^g\alpha$-irresolute, then $f$ is $s^g\alpha$-closed.

**Proof:**

i) To prove that $g \circ f : X \to Z$ is an $s^g\alpha$-closed function. Let $F$ be a closed subset of $X$. Since $f$ is closed, then $f(F)$ is a closed set in $Y$. But $g$ is an $s^g\alpha$-closed function, then $g(f(F))$ is an $s^g\alpha$-closed set in $Z$. Therefore $g \circ f : X \to Z$ is an $s^g\alpha$-closed function.

ii) To prove that $g : Y \to Z$ is an $s^g\alpha$-closed function. Let $F$ be a closed subset of $Y$. Since $f \circ g$ is $s^g\alpha$-closed, then $g \circ f$ is $s^g\alpha$-closed in $Z$. Since $f \circ g$ is onto, then $g \circ f$ is an $s^g\alpha$-closed set in $Z$.

iii) To prove that $f : X \to Y$ is an $s^g\alpha$-closed function. Let $F$ be a closed subset of $X$. Since $g \circ f$ is $s^g\alpha$-closed, then $g \circ f$ is $s^g\alpha$-closed in $Z$. Since $g$ is $s^g\alpha$-irresolute, then $g \circ f$ is an $s^g\alpha$-closed set in $Y$. Therefore $f : X \to Y$ is an $s^g\alpha$-closed function.

2.10 Corollary: Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces. If $f : X \to Y$ is an $s^g\alpha$-closed function, then the restriction of $f$ to a closed subset $F$ of $X$ is an $s^g\alpha$-closed function of $F$ into $Y$.

**Proof:** Since $F$ is a closed set in $X$, then the inclusion function $i_F : F \to X$ is a closed function. Since $f \circ i_F : Z \to Y$ is an $s^g\alpha$-closed function, then $f \circ i_F : F \to Y$ is an $s^g\alpha$-closed function. But $f \circ i_F = f | F$, thus the restriction function $f | F : F \to Y$ is an $s^g\alpha$-closed function.

2.11 Proposition: Let $(X, \tau)$ and $(Y, \tau')$ be topological spaces, and $f : X \to Y$ be an $s^g\alpha$-closed function. Then for each open subset $T$ of $Y$, the function $f_T : f^{-1}(T) \to T$ which agrees with $f$ on $f^{-1}(T)$ is also $s^g\alpha$-closed.

**Proof:** Let $F$ be a closed subset of $f^{-1}(T)$, then there is a closed subset $F_1$ of $X$ such that $F = F_1 \cap f^{-1}(T)$. Since $f_1(F) = f(F_1) \cap T$ and $f(F_1)$ is $s^g\alpha$-closed in $Y$ and $T$ is an open subset of $Y$, then by proposition (1.8), $f_1(F_1) \cap T$ is an $s^g\alpha$-closed set in $T$. Thus $f_T$ is an $s^g\alpha$-closed function.

2.12 Remark: If $f : X \to Y$ is an $s^g\alpha$-closed function and $T \subseteq Y$ is not an open set. Then $f_T : f^{-1}(T) \to T$ is not necessarily an $s^g\alpha$-closed function as the following example shows.

2.13 Example: In example (2.4), let $T = \{y, z\}$, notice that $T$ is not open in $Y$ and $\tau_T = \{\emptyset, T\}$, then $f^{-1}(T) = \{a, c, d\}$ and $\tau_{f^{-1}(T)} = \{f^{-1}(T), f, \{a\}, \{c\}, \{a, c\}\}$. Define the function $f_T : f^{-1}(T) \to T$ by:
f_1(x) = f(x), \forall x \in f^{-1}(T). Notice that the subset \{d\} of f^{-1}(T) is closed in f^{-1}(T), but f_T(\{d\}) = \{y\} is not an s*g-a-closed set in T, since \((\bar{(y)})_{fT} = T \not\subset \{y\}. Thus f_T is not an s*g-a-closed function.

The product of two s*g-a-closed functions is not necessarily an s*g-a-closed function as shown by the following example:

2.14 Example: Let f_1 : (\mathcal{R}, \mu) \to (\mathcal{R}, \mu) be a function which is defined by: f_1(x) = 0, \forall x \in \mathcal{R}. And let I_{g_1} : (\mathcal{R}, \mu) \to (\mathcal{R}, \mu) be a function which is defined by: I_{g_1}(x) = x, \forall x \in \mathcal{R} where I_{g_1} is the identity function on \mathcal{R}. Clearly f_1 and I_{g_1} are s*g-a-closed functions, but f_1 \times I_{g_1} : \mathcal{R} \times \mathcal{R} \to \mathcal{R} \times \mathcal{R} such that (f_1 \times I_{g_1})(x, y) = (0, y) for each (x, y) \in \mathcal{R} \times \mathcal{R} is not an s*g-a-closed function, since the set A = \{(x, y) \in \mathcal{R} \times \mathcal{R} : x = y\} is closed in \mathcal{R} \times \mathcal{R}, but (f_1 \times I_{g_1})(A) = \{(0) \times \mathcal{R}/\{0\}\} is not s*g-a-closed in \mathcal{R} \times \mathcal{R}.

2.15 Theorem: Let f_1 : X_1 \to Y_1 and f_2 : X_2 \to Y_2 be two functions. If f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 is s*g-a-closed, then f_1 and f_2 are also s*g-a-closed functions.

Proof: Suppose that f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2 is an s*g-a-closed function. To prove that f_1 : X_1 \to Y_1 is s*g-a-closed. Let F be a closed subset of X_1, to prove that f_1(F) is an s*g-a-closed set in Y_1. Suppose that G = f_1(F) ⇒ F \times X_2 is a closed set in X_1 \times X_2. Since f_1 \times f_2 is s*g-a-closed, then (f_1 \times f_2)(F \times X_2) = f_1(F) \times f_2(X_2) = G \times f_2(X_2) is s*g-a-closed in Y_1 \times Y_2. By proposition (1.9), we have: G \times f_2(X_2) \subseteq \bar{G} \times \bar{f_2}(X_2). But by proposition (1.5), G = f_1(F) is an s*g-a-closed set in Y_1. Thus f_1 is an s*g-a-closed function. By the same way we can prove that f_2 is an s*g-a-closed function. Thus f_1 and f_2 are s*g-a-closed functions.

2.16 Definition: Let (X, \tau) and (Y, \tau') be topological spaces. A function f : X \to Y is called s*g-a-compact if the inverse image of every s*g-compact set in Y is a compact set in X.

2.17 Proposition: Let (X, \tau), (Y, \tau') and (Z, \tau'') be three topological spaces and f : X \to Y, g : Y \to Z be two functions. Then:

i) If f is compact and g is s*g-a-compact, then g \circ f is s*g-a-compact.

ii) If g \circ f is s*g-a-compact and f is continuous and onto, then g is s*g-a-compact.

iii) If g \circ f is s*g-a-compact and g is s*g-a-irresolute and one-to-one, then f is s*g-a-compact.

Proof: The proof is similar of theorem (2.9).

2.18 Remark: s*g-a-closed function and s*g-a-compact function are in general independent. Consider the following examples:

2.19 Examples:(i) Let X = Y = \{a, b, c\} and \tau = \{\emptyset, X, \{a, c\}\} and \tau' = \{\emptyset, Y, \{b\}\}, and let f : (X, \tau) \to (Y, \tau') be a function which is defined by: f(a) = f(c) = a and f(b) = b. Since X and Y are finite spaces, then f^{-1}(K) is a compact set in X for each s*g-a-compact subset K of Y. Hence f is an s*g-a-compact function, but f is not an s*g-a-closed function, since \{b\} is a closed set in X, but f(\{b\}) = \{b\} is not an s*g-a-closed set in Y, since \bar{\{b\}} = Y \not\subset \{b\}.

(ii) Let (\mathcal{R}, \mu) be the usual topological space and let f : (\mathcal{R}, \mu) \to (\mathcal{R}, \mu) be a function which is defined by: f(x) = 0, \forall x \in \mathcal{R}. Then f is an s*g-a-closed function, but f is not an s*g-a-compact function, since \{0\} is an s*g-a-compact set in \mathcal{R}, but f^{-1}(\{0\}) = \mathcal{R} is not compact in \mathcal{R}.

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### 2.20 Proposition: Let \((X, \tau)\) be a topological space and \((Y, \tau')\) be an \(s^*g\alpha\)-K-space. Then every continuous \(s^*g\alpha\)-compact function \(f : X \to Y\) is an \(s^*g\alpha\)-closed function.

**Proof:** Let \(F\) be a closed set in \(X\), to prove that \(f(F)\) is an \(s^*g\alpha\)-closed set in \(Y\). Let \(K\) be an \(s^*g\alpha\)-compact set in \(Y\). Since \(f\) is an \(s^*g\alpha\)-compact function, then \(f^{-1}(K)\) is a compact set in \(X\). Since \(F \cap f^{-1}(K)\) is a compact set in \(X\) and \(f\) is continuous, then by \((10, \text{ theorem (17.7)})\), \((F \cap f^{-1}(K))\) is a compact set in \(Y\). But \(f(F \cap f^{-1}(K)) = f(F)\cap f(K)\), thus \(f(F)\cap f(K)\) is a compact set in \(Y\). Therefore by definition \((1.19)\), \(f(F)\) is a compactly \(s^*g\alpha\)-closed set in \(Y\). Since \(Y\) is an \(s^*g\alpha\)-K-space, then by definition \((1.21)\), \(f(F)\) is an \(s^*g\alpha\)-closed set in \(Y\). Hence \(f\) is an \(s^*g\alpha\)-closed function.

### 2.21 Proposition: Any one-to-one \(s^*g\alpha\)-closed function is an \(s^*g\alpha\)-compact function.

**Proof:** Let \(f : (X, \tau) \to (Y, \tau')\) be a one-to-one \(s^*g\alpha\)-closed function and \(K\) be an \(s^*g\alpha\)-compact set in \(Y\). To prove that \(f^{-1}(K)\) is a compact set in \(X\). Let \(\{U_{\alpha}\}_{\alpha \in \Lambda}\) be any open cover of \(f^{-1}(K)\), then \(f^{-1}(K) \subseteq \bigcup_{\alpha \in \Lambda} U_{\alpha}\) and \(U_{\alpha}\) is an open set in \(X\) for each \(\alpha \in \Lambda\). Hence \(\bigcap_{\alpha \in \Lambda} U_{\alpha} \subseteq X - f^{-1}(K)\), there is \(\bigcap_{\alpha \in \Lambda} U_{\alpha} \subseteq f^{-1}(Y - K)\). Since \(f\) is a one-to-one function, then \(\bigcap_{\alpha \in \Lambda} f(U_{\alpha}^c) = f\left(\bigcap_{\alpha \in \Lambda} U_{\alpha}^c\right) \subseteq f(f^{-1}(Y - K)) \subseteq Y - K\). Hence \(f^{-1}(K) \subseteq \bigcup_{\alpha \in \Lambda} Y - f(U_{\alpha}^c)\).

Since \(f\) is an \(s^*g\alpha\)-closed function and \(U_{\alpha}^c\) is a closed set in \(X\) for each \(\alpha \in \Lambda\), then \(f(U_{\alpha}^c)\) is an \(s^*g\alpha\)-closed set in \(Y\) for each \(\alpha \in \Lambda\). Thus \(Y - f(U_{\alpha}^c)\) is an \(s^*g\alpha\)-open cover of \(K\). Since \(K\) is an \(s^*g\alpha\)-compact, then \(\exists \{Y - f(U_{\alpha}^c)\}_{\alpha \in \Lambda}^n\) is a finite subcover of \(\{Y - f(U_{\alpha}^c)\}_{\alpha \in \Lambda}\), i.e. \(K \subseteq \bigcup_{i=1}^{n} Y - f(U_{\alpha_i}^c)\) \(\Rightarrow f^{-1}(K) \subseteq \bigcup_{i=1}^{n} f(U_{\alpha_i}^c)\) is a finite subcover of \(\{U_{\alpha_i}\}_{\alpha \in \Lambda}\). Hence \(f^{-1}(K)\) is a compact set in \(X\). Thus \(f : X \to Y\) is an \(s^*g\alpha\)-compact function.

### 2.22 Corollary: Let \((X, \tau)\) be a topological space and \((Y, \tau')\) be an \(s^*g\alpha\)-K-space. Then a one-to-one continuous function \(f : X \to Y\) is an \(s^*g\alpha\)-closed function if and only if \(f\) is an \(s^*g\alpha\)-compact function.

**Proof:** It is obvious.

### 2.23 Definition: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces. A function \(f : X \to Y\) is called an \(s^*g\alpha\)-homeomorphism if:

i) \(f\) is bijective.

ii) \(f\) is continuous.

iii) \(f\) is \(s^*g\alpha\)-closed (resp. \(s^*g\alpha\)-open).

### 3. Properties of \(s^*g\alpha\)-Proper Functions

In this section we introduce a new definition (to the best of our knowledge), namely, \(s^*g\alpha\)-proper functions. Also, we study the basic properties and characterizations of these functions. Moreover we study the relation between \(s^*g\alpha\)-proper functions and certain types of functions such as proper functions, \(s^*g\alpha\)-perfect functions, closed functions, \(s^*g\alpha\)-closed functions and \(s^*g\alpha\)-compact functions.

### 3.1 Definition: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces, and \(f : X \to Y\) be a function. Then \(f\) is called an \(s^*g\alpha\)-proper function if:

i) \(f\) is a continuous function.

ii) \(f \times I_Z : X \times Z \to Y \times Z\) is \(s^*g\alpha\)-closed for every topological space \(Z\).

### 3.2 Examples:

i) Let \(f : (\mathfrak{R}, \mu) \to (\mathfrak{R}, \mu)\) be a function which is defined by: \(f(x) = 0, \forall x \in \mathfrak{R}\). Notice that \(f\) is an \(s^*g\alpha\)-
closed function, but \( f \) is not \( s^g\alpha \)-proper, since for the usual topological space \((\mathbb{R},\mu)\), the function \( f \times 1_{\mathbb{R}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R} \) such that \( (f \times 1_{\mathbb{R}})(x,y) = (0,y) \) for each \((x,y) \in \mathbb{R} \times \mathbb{R}\) is not an \( s^g\alpha \)-closed function.

ii) An inclusion function \( t_F : F \to X \) is \( s^g\alpha \)-proper if and only if \( F \) is an \( s^g\alpha \)-closed set in \( X \).

Since every closed function is an \( s^g\alpha \)-closed function, then we have the following proposition:

3.3 Proposition: Every proper function is an \( s^g\alpha \)-proper function.

The converse of proposition (3.3) may not be true in general as shown by the following example:

3.4 Example: Let \( X = Y = \{a, b, c\} \) and let \( \tau = \{\phi, [a], [a, c], X\} \) and \( \tau' = \{\phi, [a], Y\} \) be topologies on \( X \) and \( Y \), respectively. Define the function \( f : X \to Y \) by: \( f(a) = a \), \( f(b) = b \) and \( f(c) = c \). Therefore \( f \) is an \( s^g\alpha \)-proper function, but \( f \) is not a proper function, since \( f \) is not a closed function.

3.5 Proposition: Every \( s^g\alpha \)-proper function is an \( s^g\alpha \)-closed function.

Proof: Let \( f : X \to Y \) be an \( s^g\alpha \)-proper function, then the function \( f \times 1_{Z} : X \times Z \to Y \times Z \) is \( s^g\alpha \)-closed for each topological space \( Z \). Let \( Z = \{t\} \), then \( X \times Z = X \times \{t\} \subseteq X \) and \( Y \times Z = Y \times \{t\} \subseteq Y \) and we can replace \( f \times 1_{Z} \) by \( f \). Thus \( f : X \to Y \) is an \( s^g\alpha \)-closed function.

3.6 Remark: The converse of proposition (3.5) may not be true in general. Observe that in examples ((3.2),(ii)) \( f : (\mathbb{R},\mu) \to (\mathbb{R},\mu) \) is an \( s^g\alpha \)-closed function, but is not an \( s^g\alpha \)-proper function.

3.7 Theorem: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces, and \( f : X \to Y \) be a continuous, one-to-one function. Then \( f \) is an \( s^g\alpha \)-proper function if and only if \( f \) is an \( s^g\alpha \)-closed function.

Proof: \( \Rightarrow \) By proposition (3.5).

Conversely, assume that \( f : X \to Y \) is an \( s^g\alpha \)-closed function. To prove that \( f \) is \( s^g\alpha \)-proper i.e. to prove that \( h = f \times 1_{Z} : X \times Z \to Y \times Z \) is \( s^g\alpha \)-closed for every topological space \( Z \). Let \( C \) be any closed set in \( X \times Z \). To prove that \( h(C) = D \) is an \( s^g\alpha \)-closed set in \( Y \times Z \). Let \( (y, s) \in D \Rightarrow h^{-1}(y, s) \in h^{-1}(D) \Rightarrow (f \times 1_{Z})^{-1}(y, s) \in h^{-1}(D) \Rightarrow f^{-1}(y) \times \{s\} \subseteq C \), where \( C \) is an open set in \( X \times Z \). Since \( f \) is a one-to-one \( s^g\alpha \)-closed function, then by proposition (2.21), \( f^{-1}(y) \) is a compact set in \( X \). Hence by (10), theorem (17.6)) there are open sets \( U \) and \( V \) in \( Z \) such that \( f^{-1}(y) \times \{s\} \subseteq U \times V \subseteq C \) \Rightarrow \( f^{-1}(y) \subseteq U \text{ and } \{s\} \subseteq V \text{. Since } f \text{ and } I_{Z} \text{ are } s^g\alpha \text{-closed, then by theorem } (2.5), \text{ there are } s^g\alpha \text{-open sets } U' \text{ in } Y \text{ and } V' \text{ in } Z \text{ such that } \{y\} \subseteq U', \{s\} \subseteq V', \text{ and } f^{-1}(U') \subseteq U \text{ and } I_{Z}^{-1}(V') \subseteq V \Rightarrow (y, s) \subseteq U' \times V' \subseteq C \Rightarrow D \) is an \( s^g\alpha \)-open set in \( Y \times Z \Rightarrow D \) is an \( s^g\alpha \)-closed in \( Y \times Z \). Hence \( f \times 1_{Z} : X \times Z \to Y \times Z \) is an \( s^g\alpha \)-closed function. Thus \( f : X \to Y \) is an \( s^g\alpha \)-proper function.

3.8 Corollary: Every \( s^g\alpha \)-homeomorphism is an \( s^g\alpha \)-proper function.

The converse of corollary (3.8) may not be true in general as shown by the following example:

3.9 Example: Let \( f : ([0,1],\mu') \to (\mathbb{R},\mu) \) be a function which is defined by: \( f(x) = x, \forall x \in [0,1] \) where \( \mu' \) is the relative usual topology on \([0,1]\). Clearly \( f \) is an \( s^g\alpha \)-proper function, but is not \( s^g\alpha \)-homeomorphism.

3.10 Theorem: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces, and \( f : X \to Y \) be a continuous, function. Then the following statements are equivalent:

i) \( f \) is an \( s^g\alpha \)-proper function.

ii) \( f \) is an \( s^g\alpha \)-closed function and \( f^{-1}(y) \) is a compact set in \( X \) for each \( y \in Y \).

iii) If \( (x_{\delta})_{\delta \in D} \) is a net in \( X \) and \( y \in Y \) is an \( s^g\alpha \)-limit point of the net \( (f(x_{\delta}))_{\delta \in D} \), then there is a cluster point \( x \in X \) of \( (x_{\delta})_{\delta \in D} \) such that \( f(x) = y \).
Proof: (i $\rightarrow$ ii). If $f$ is an $s*g$-$\alpha$-proper function, then by proposition (3.5), $f$ is an $s*g$-$\alpha$-closed function. Also, by ([1], theorem (3.1.12)), $f^{-1}(y)$ is a compact set in $X$ for each $y \in Y$.

(ii $\rightarrow$ iii). Let $(x_d)_{d \in D}$ be a net in $X$ and $y \in Y$ be an $s*g$-$\alpha$-limit point of a net $(f(x_d))_{d \in D}$ in $Y$. To prove that there is a cluster point $x \in X$ of $(x_d)_{d \in D}$ such that $f(x) = y$. Claim $f^{-1}(y) \neq \emptyset$, if $f^{-1}(y) = \emptyset \Rightarrow y \notin f(X) \Rightarrow y \in (f(X))^c$, since $X$ is a closed set in $X$ and $f$ is $s*g$-$\alpha$-closed, then $f(X)$ is an $s*g$-$\alpha$-closed set in $Y$. Thus $(f(X))^c$ is an $s*g$-$\alpha$-open set in $Y$. Therefore $(f(x_d))_{d \in D}$ is eventually in $(f(X))^c$. But $f(x_d) \in f(X), \forall d \in D$, then $(f(X)) \cap (f(X))^c \neq \emptyset$, and this is a contradiction. Thus $f^{-1}(y) \neq \emptyset$.

Now, suppose that the statement (iii) is not true, that means, for all $x \in f^{-1}(y)$ there exists an open set $U_x$ in $X$ contains $x$ such that $(x_d)_{d \in D}$ is not frequently in $U_x$. Notice that $f^{-1}(y) = \bigcup_{x \in f^{-1}(y)} U_x$.

Therefore the family $\{U_x : x \in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is a compact set, then there exists $x_1, x_2, \ldots, x_n$ such that $f^{-1}(y) \subseteq \bigcup_{i=1}^{n} U_{x_i} \Rightarrow f^{-1}(y) \cap \bigcup_{i=1}^{n} U_{x_i} = \emptyset \Rightarrow f^{-1}(y) \cap \bigcup_{i=1}^{n} U_{x_i} = \emptyset$. But $(x_d)_{d \in D}$ is not frequently in $U_{x_i}, \forall i = 1 \ldots n$, thus $(x_d)_{d \in D}$ is not frequently in $\bigcup_{i=1}^{n} U_{x_i}$. Since $\bigcup_{i=1}^{n} U_{x_i}$ is an open set in $X$, then $\bigcap_{i=1}^{n} U_{x_i}$ is a closed set in $X$. Thus $f(\bigcap_{i=1}^{n} U_{x_i})$ is an $s*g$-$\alpha$-closed set in $Y$. Claim

$y \notin \bigcap_{i=1}^{n} U_{x_i}$, if $y \in f(\bigcap_{i=1}^{n} U_{x_i})$, then there exists $x \in \bigcap_{i=1}^{n} U_{x_i}$ such that $f(x) = y$, thus $x \notin \bigcup_{i=1}^{n} U_{x_i}$, but $x \in f^{-1}(y)$, therefore $f^{-1}(y)$ is not a subset of $\bigcup_{i=1}^{n} U_{x_i}$, and this is a contradiction. Hence $y \notin \bigcap_{i=1}^{n} U_{x_i}$ and by theorem ((1.7),(vi)), there is an $s*g$-$\alpha$-open set $A$ in $Y$ such that $y \in A$ and $A \cap f(\bigcap_{i=1}^{n} U_{x_i}) = \emptyset \Rightarrow f^{-1}(A) \cap f^{-1}(f(\bigcap_{i=1}^{n} U_{x_i})) = \emptyset \Rightarrow f^{-1}(A) \cap \bigcap_{i=1}^{n} U_{x_i} = \emptyset$. But $(f(x_d))_{d \in D}$ is eventually in $A$, then $(f(x_d))_{d \in D}$ is frequently in $A$, thus $(x_d)_{d \in D}$ is frequently in $f^{-1}(A)$ and then $(x_d)_{d \in D}$ is frequently in $\bigcup_{i=1}^{n} U_{x_i}$, this is a contradiction. Thus there is a cluster point $x \in f^{-1}(y)$ of $(x_d)_{d \in D}$ such that $f(x) = y$.

(iii $\rightarrow$ i). To prove that $f \times I_Z : X \times Z \rightarrow Y \times Z$ is an $s*g$-$\alpha$-closed function for every topological space $Z$. Let $F$ be a closed subset of $X \times Z$ and $(f \times I_Z)(F) = G$. To prove that $G$ is an $s*g$-$\alpha$-closed set in $Y \times Z$. Let

$(y, z) \in \overline{G^{s*g-\alpha}}$, then by proposition (1.23), there exists a net $\{(y_d, z_d)\}_{d \in D}$ in $G$ such that $(y_d, z_d) \rightarrow_{s*g-\alpha} (y, z)$. Thus there is a net $\{(x_d, z_d)\}_{d \in D}$ in $F$ such that $(f \times I_Z)(x_d, z_d) = (y_d, z_d), \forall d \in D$. Since $(f(x_d), I_Z(z_d)) \rightarrow_{s*g-\alpha} (y, z)$, then $f(x_d) \rightarrow_{s*g-\alpha} y$ and $z_d \rightarrow_{s*g-\alpha} z$, hence by hypothesis there is a point $x \in X$ such that $x_d \rightarrow x$ and $f(x) = y$. Since $z_d \rightarrow_{s*g-\alpha} z$, then $z_d \rightarrow z$. Therefore $x_d \rightarrow x$ and $z_d \rightarrow z \Rightarrow (x_d, z_d) \rightarrow (x, z)$. Since $\{(x_d, z_d)\}$ is a net in $F$ and $F$ is closed, thus by ([10], theorem (11.7)), $(x, z) \in \overline{F} = F \Rightarrow (y, z) = (f \times I_Z(x, z), z) \in G$. Thus $\overline{G^{s*g-\alpha}} \subseteq G$. 

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3.11 Corollary: Let \((X, \tau)\) be a topological space and \([p]\) be a space consisting of a single point. Then a function \(f : X \rightarrow [p]\) is \(s^g\)-\(\alpha\)-proper if and only if \(X\) is a compact space.

Proof: It is obvious.

3.12 Definition: If the function \(f : (X, \tau) \rightarrow (Y, \tau')\) is \(s^g\)-\(\alpha\)-proper and \((X, \tau)\) is a \(T_2\)-space, then \(f\) is called an \(s^g\)-\(\alpha\)-perfect function.

3.13 Corollary: Every \(s^g\)-\(\alpha\)-perfect function is an \(s^g\)-\(\alpha\)-proper function.

3.14 Remark: The converse of corollary (3.13) may not be true in general. Consider the following example:

3.15 Example: Let \(f : (\mathfrak{H}, \tau_{cof.}) \rightarrow (\mathfrak{H}, \tau_{cof.})\) be the identity function, where \(\tau_{cof.}\) be the cofinite topology on \(\mathfrak{H}\). Then \(f\) is an \(s^g\)-\(\alpha\)-homeomorphism and by corollary (3.8), \(f\) is \(s^g\)-\(\alpha\)-proper. Since \((\mathfrak{H}, \tau_{cof.})\) is not a \(T_2\)-space, then \(f\) is not an \(s^g\)-\(\alpha\)-perfect function.

3.16 Theorem: Let \((X, \tau), (Y, \tau')\) and \((Z, \tau'')\) be topological spaces, and \(f : X \rightarrow Y, g : Y \rightarrow Z\) be continuous functions. Then:

i) If \(f\) is proper and \(g\) is \(s^g\)-\(\alpha\)-proper, then \(g \circ f\) is \(s^g\)-\(\alpha\)-proper.

ii) If \(g \circ f\) is \(s^g\)-\(\alpha\)-proper and \(f\) is onto, then \(g\) is \(s^g\)-\(\alpha\)-proper.

iii) If \(g \circ f\) is \(s^g\)-\(\alpha\)-proper and \(g\) is one-to-one and \(s^g\)-\(\alpha\)-irresolute, then \(f\) is \(s^g\)-\(\alpha\)-proper.

Proof:

i) It is clear that \(g \circ f : X \rightarrow Z\) is a continuous function. Let \((x_d)_{dD}\) be a net in \(X\) such that

\[ (g \circ f)(x_d) \xrightarrow{s^g-\alpha} z \in Z. \]

Since \(g\) is an \(s^g\)-\(\alpha\)-proper function and \(g(f(x_d))) \xrightarrow{s^g-\alpha} z\), then by theorem (3.10), there is a point \(y \in Y\) such that \(f(x_d) \rightarrow y\) and \(g(y) = z\). Since \(f\) is a proper function, then by [3], there is a point \(x \in X\) such that \(x_d \rightarrow x\) and \(f(x) = y\). Hence there is \(x \in X\) such that \(x_d \rightarrow x\) and \((g \circ f)(x) = g(f(x)) = g(y) = z\). Thus \(g \circ f : X \rightarrow Z\) is an \(s^g\)-\(\alpha\)-proper function.

ii) Let \((y_d)_{dD}\) be a net in \(Y\) such that \(g(y_d) \xrightarrow{s^g-\alpha} z \in Z\). Since \((y_d)_{dD}\) is a net in \(Y\) and \(f\) is onto, then there is a net \((x_d)_{dD}\) in \(X\) such that \(f(x_d) = y_d, \forall d \in D\). Hence \(g(f(x_d)) = (g \circ f)(x_d) \xrightarrow{s^g-\alpha} z\). Since \(g \circ f\) is \(s^g\)-\(\alpha\)-proper, then by theorem (3.10), there is a point \(x \in X\) such that \(x_d \rightarrow x\) and \((g \circ f)(x) = z\). Since \(f\) is continuous, then by ([10], theorem (11.8)), \(f(x_d) \rightarrow f(x)\). Hence there is a point \(f(x) \in Y\) such that \(y_d \rightarrow f(x)\) and \(g(f(x)) = g(f(x)) = g(y) = z\). Thus \(g : Y \rightarrow Z\) is an \(s^g\)-\(\alpha\)-proper function.

iii) Let \((x_d)_{dD}\) be a net in \(X\) such that \(f(x_d) \xrightarrow{s^g-\alpha} y \in Y\). Since \(g\) is \(s^g\)-\(\alpha\)-irresolute, then by proposition (1.24), \(g(f(x_d)) \xrightarrow{s^g-\alpha} g(y)\). But \(g \circ f\) is \(s^g\)-\(\alpha\)-proper, then by theorem (3.10), there is a point \(x \in X\) such that \(x_d \rightarrow x\) and \((g \circ f)(x) = g(y)\). Since \((g \circ f)(x) = g(f(x)) = g(y)\) and since \(g\) is one-to-one, then \(f(x) = y\). Thus \(f : X \rightarrow Y\) is an \(s^g\)-\(\alpha\)-proper function.

3.17 Corollary: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces. If \(f : X \rightarrow Y\) is an \(s^g\)-\(\alpha\)-proper function, then the restriction of \(f\) to a closed subset \(F\) of \(X\) is an \(s^g\)-\(\alpha\)-proper function of \(F\) into \(Y\).

Proof: Since \(F\) is a closed set in \(X\), then the inclusion function \(\iota_F : F \rightarrow X\) is a proper function. Since \(f : X \rightarrow Y\) is an \(s^g\)-\(\alpha\)-proper function, then by theorem ((3.16),(i)), \(f \circ \iota_F : F \rightarrow Y\) is an \(s^g\)-\(\alpha\)-proper function. But \(f \circ \iota_F = f | F\), thus the restriction function \(f | F : F \rightarrow Y\) is an \(s^g\)-\(\alpha\)-proper function.
3.18 Corollary: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces. If \(f : X \to Y\) is an \(s^g\alpha\)-perfect function, then the restriction of \(f\) to a closed subset \(F\) of \(X\) is an \(s^g\alpha\)-perfect function of \(F\) into \(Y\).

Proof: It is obvious.

3.19 Proposition: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces and \(f : X \to Y\) be an \(s^g\alpha\)-proper function. Then for each open subset \(T\) of \(Y\), the function \(f^{-1}(T) : X \to T\) which agrees with \(f\) on \(f^{-1}(T)\) is also \(s^g\alpha\)-proper.

Proof: Since \(f : X \to Y\) is continuous, then so is \(f^{-1}\). To prove that \(f^{-1} \times I_Z : f^{-1}(T) \times Z \to T \times Z\) is \(s^g\alpha\)-closed for every topological space \(Z\). Since \(f \) is \(s^g\alpha\)-proper, then \(f \times I_Z : X \times Z \to Y \times Z\) is \(s^g\alpha\)-closed for every topological space \(Z\). Since \(f^{-1} \times I_Z = (f \times I_Z)_{\tau Z}\) and \(T \times Z\) is an open subset of \(Y \times Z\), then by proposition (2.11), \(f^{-1} \times I_Z\) is an \(s^g\alpha\)-closed function. Thus \(f^{-1} : f^{-1}(T) \to T\) is an \(s^g\alpha\)-proper function.

3.20 Corollary: Let \((X, \tau)\) and \((Y, \tau')\) be topological spaces and \(f : X \to Y\) be an \(s^g\alpha\)-perfect function. Then for each open subset \(T\) of \(Y\), the function \(f^{-1}(T) : X \to T\) which agrees with \(f\) on \(f^{-1}(T)\) is also \(s^g\alpha\)-perfect.

Proof: It is obvious.

3.21 Proposition: If \(f_1 : X_1 \to Y_1\) is a proper function and \(f_2 : X_2 \to Y_2\) is an \(s^g\alpha\)-proper function. Then \(f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2\) is an \(s^g\alpha\)-proper function.

Proof: Let \(Z\) be any topological space. We can write \(f_1 \times f_2 \times I_Z\) by the composition of \(I_{X_1} \times f_2 \times I_Z\) and \(f_1 \times I_{X_2} \times I_Z\). Since \(f_1\) is proper, then \(f_1 \times I_{X_2} \times I_Z\) is closed. Since \(f_2\) is \(s^g\alpha\)-proper, then \(I_{X_1} \times f_2 \times I_Z\) is \(s^g\alpha\)-closed, hence by theorem ((2.9), (ii)), \((I_{X_1} \times f_2 \times I_Z) \circ (f_1 \times I_{X_2} \times I_Z)\) is \(s^g\alpha\)-closed. But \(f_1 \times f_2 \times I_Z = (I_{X_1} \times f_2 \times I_Z) \circ (f_1 \times I_{X_2} \times I_Z) \Rightarrow f_1 \times f_2 \times I_Z\) is \(s^g\alpha\)-closed. Thus \(f_1 \times f_2\) is an \(s^g\alpha\)-proper function.

3.22 Theorem: Let \(f_1 : X_1 \to Y_1\) and \(f_2 : X_2 \to Y_2\) be functions such that \(f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2\) is an \(s^g\alpha\)-proper function. Then \(f_1\) and \(f_2\) are \(s^g\alpha\)-proper.

Proof: Let \(Z\) be any topological space. To prove that \(f_2 \times I_Z : X_2 \times Z \to Y_2 \times Z\) is \(s^g\alpha\)-closed. Let \(F\) be a closed set in \(X_2 \times Z\) and \(G = (f_2 \times I_Z)(F)\). To prove that \(G\) is \(s^g\alpha\)-closed in \(Y_2 \times Z\). Since \(X_1 \neq \emptyset\), then \(X_1 \times F\) is closed in \(X_1 \times X_2 \times Z\). Since \(f_1 \times f_2\) is \(s^g\alpha\)-proper, then \((f_1 \times f_2 \times I_Z)(X_1 \times F) = f_1(X_1) \times G \subseteq S^{g\alpha}\) is \(s^g\alpha\)-closed in \(Y_1 \times Y_2 \times Z\). Hence by proposition (1.9), \(f_1(X_1) \times G \subseteq S^{g\alpha}\). But by proposition (1.9), we have \(f_1(X_1) \times G \subseteq S^{g\alpha}\). Hence by proposition (1.5), \(G = (f_2 \times I_Z)(F)\) is an \(s^g\alpha\)-closed set in \(Y_2 \times Z\). Therefore \(f_2 \times I_Z\) is an \(s^g\alpha\)-closed function. Thus \(f_2\) is an \(s^g\alpha\)-proper function. By the same way we can prove that \(f_1\) is an \(s^g\alpha\)-proper function.

3.23 Proposition: If \(X\) is any compact topological space and \(Y\) is any topological space, then the projection \(pr_2 : X \times Y \to Y\) is an \(s^g\alpha\)-proper function.

Proof: \(pr_2\) factorizes into \(X \times Y \rightsquigarrow X \times X \overset{h \times 1}{\longrightarrow} Y \times X \overset{1 \times \alpha f}{\longrightarrow} Y\), where \(h(x, y) = (y, x)\) \(h\) is a homeomorphism, hence \(h\) is proper. Since \(X\) is a compact space, then by corollary (3.11), \(f : X \to \{p\}\) is \(s^g\alpha\)-proper, since \(1_Y : Y \to Y\) is proper, then by proposition (3.21), \(Y \times X \overset{1 \times \alpha f}{\longrightarrow} Y \times \{p\} \equiv Y\) is \(s^g\alpha\)-proper. Therefore by theorem ((3.16), (ii)), \(pr_2 = (1_Y \times f) \circ h\) is an \(s^g\alpha\)-proper function.
Now, we shall explain the relationships between the $s^g$-a-proper functions and the $s^g$-a-compact functions.

**3.24 Proposition:** Every $s^g$-a-proper function is an $s^g$-a-compact function.

**Proof:** Let $f : (X, \tau) \to (Y, \tau')$ be an $s^g$-a-proper function. To prove that $f$ is an $s^g$-a-compact function. Let $K$ be an $s^g$-a-compact subset of $Y$ and let \( \{ U_a \}_{a \in A} \) be any open cover of $f^{-1}(K)$. Since $f$ is an $s^g$-a-proper function, then by theorem (3.10), $f^{-1}(k)$ is a compact set in $X$ for each $k \in K$. But $f^{-1}(k) \subseteq f^{-1}(K) \subseteq \bigcup_{a \in A} U_a$ , thus there exists $n_k$ such that $f^{-1}(k) \subseteq \bigcup_{i=1}^{n_k} U_{a_i}$. Let $U_k = \bigcup_{i=1}^{n_k} U_{a_i}$, thus $f^{-1}(k) \subseteq U_k$.

Notice that for each $k \in K$, $k \in (Y \setminus f(X \setminus U_k))$. Hence $K \subseteq \bigcup_{k \in K} (Y \setminus f(X \setminus U_k))$, but $K$ is an $s^g$-a-compact set in $Y$ and the sets $(Y \setminus f(X \setminus U_k))$ are $s^g$-a-open. Thus there exists $k_1, k_2, \ldots, k_j$ such that $K \subseteq \bigcup_{a=1}^{j} (Y \setminus f(X \setminus U_{k_a}))$. Hence $f^{-1}(K) \subseteq \bigcup_{a=1}^{j} U_{k_a}$. Therefore $f^{-1}(K)$ is a compact set in $X$. Hence the function $f : (X, \tau) \to (Y, \tau')$ is an $s^g$-a-compact function.

The converse of proposition (3.24) may not be true in general. Consider the following example:

**3.25 Example:** Let $f : (\mathfrak{H}, \mu) \to (\mathfrak{H}, \tau)$ be a function from the usual topological space $(\mathfrak{H}, \mu)$ to a topological space $(\mathfrak{H}, \tau)$, where $\tau = \{ \emptyset, \mathfrak{H}, \{0\} \}$ such that $f(x) = x$ for each $x \in \mathfrak{H}$. Then $f$ is not an $s^g$-a-proper function, since $\{0\}$ is a closed set in $(\mathfrak{H}, \mu)$, but $f(\{0\}) = \{0\}$ is not an $s^g$-a-closed set in $(\mathfrak{H}, \tau)$. While $f$ is an $s^g$-a-compact function.

**3.26 Proposition:** Let $f : (X, \tau) \to (Y, \tau')$ be a continuous function such that $Y$ is an $s^g$-a-K-space. Then $f$ is an $s^g$-a-proper function if and only if $f$ is an $s^g$-a-compact function.

**Proof:** $\Rightarrow$ By proposition (3.24), every $s^g$-a-proper function is an $s^g$-a-compact function.

Conversely, since $f$ is an $s^g$-a-compact function and $[y]$ is an $s^g$-a-compact set in $Y$, then by definition (2.16), $f^{-1}(y)$ is a compact set in $X$ for each $y \in Y$. Now, to prove that $f$ is an $s^g$-a-closed function. Let $F$ be a closed set in $X$, to prove that $f(F)$ is an $s^g$-a-closed set in $Y$. Suppose that $K$ is an $s^g$-a-compact set in $Y$, then $f^{-1}(K)$ is a compact set in $X$. But $F \cap f^{-1}(K)$ is a compact set in $X$ and $f$ is continuous, then by ([10], theorem (17.7)), $f(F \cap f^{-1}(K))$ is a compact set in $Y$. Since $f(F \cap f^{-1}(K)) = f(F) \cap K$, then $f(F) \cap K$ is a compact set in $Y$. Therefore by definition (1.19), $f(F)$ is a compactly $s^g$-a-closed set in $Y$. Since $Y$ is an $s^g$-a-K-space, then by definition (1.21), $f(F)$ is an $s^g$-a-closed set in $Y$. Thus by theorem (3.10), $f$ is an $s^g$-a-proper function.

**References**


36-41, 1963.


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