On Mathematical Model of Traffic Control

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Abstract
This paper presents the formation of platoons and their velocity-size distribution in freeway traffic using the stochastic approach. This is aimed at reducing traffic congestions, maintaining safety, increasing highway capacity, limiting the speed of a car by the spacing between it and the car directly ahead and above all reducing motor accidents. Traffic systems shall also be modeled and applied to traffic problems of different kinds to find their solutions.
Keywords: *Models, Traffics, Platoons, Multi-Agent Systems (MAS)*

1. Introduction

Mathematical modeling is the process of creating a mathematical representation of some phenomenon in order to gain a better understanding of that phenomenon. It is a process that attempts to match observation with symbolic statements. During the process of building a mathematical model, the modeler will decide what factors are relevant to the problem and what factors can be de-emphasized. Once a model has been developed and used to answer questions, it should be critically examined and often modified to obtain a more accurate reflection of the observed reality of that phenomenon. In this way, mathematical modeling is an evolving process; as new insight is gained, the process begins again as additional factors are considered. Generally the success of a model depends on how easily it can be used and how accurate are its predictions.

One can think of mathematical modeling as an activity that allows a mathematician to be a chemist, an ecologist, an economist, a physiologist etc instead of undertaking experiments on mathematical representations of the real-world. Also, mathematical modeling is the use of mathematics to describe the real-world phenomena, investigate important questions about the observed world, explain real world phenomenon, test ideas and make predictions about the real-world. The real-world refers to engineering, physics, physiology, ecology, wild-life, management, chemistry, economics, etc. There are relevant and exhaustive materials on this work such as [2, 3, 4, 6, 8, 10, 11 and 12] just to mention few.

Definition 1.0 [5]

Mathematical Model is the representation of the essential aspects of an existing system (or a system to be constructed) which presents knowledge of that system in usable form.

It usually describes a system by a set of variables and a set of equations that establish relationships between the variables. The values of the variables can be practically anything; real or integer numbers, Boolean values or strings, for example. The variables represent some properties of the system, system outputs can be measured in the form of signals, timing data, counters, event occurrence (yes/no). The actual model is the set of functions that describe the relationships between the different variables.

It is important to note that Traffic is the movement of motorized vehicles, unmotorized vehicles and pedestrians on roads. Traffic Flow is the pattern of the way people move through an area or road network, or a measure of the density of traffic. Traffic Jam means the number of vehicles blocking one another until they can scarcely move, that is a situation when all road traffic is stationary or very slow. Traffic Congestion is a condition on any network as use increases and is characterized by slower speeds, longer trip times, and increased queuing. Also, pedestrian is a person traveling in foot, whether walking or running, in fact in some communities those travelling using roller skate boards, and similar devices are considered to be pedestrians. In modern times, the term mostly refers to someone walking on road or footpath.

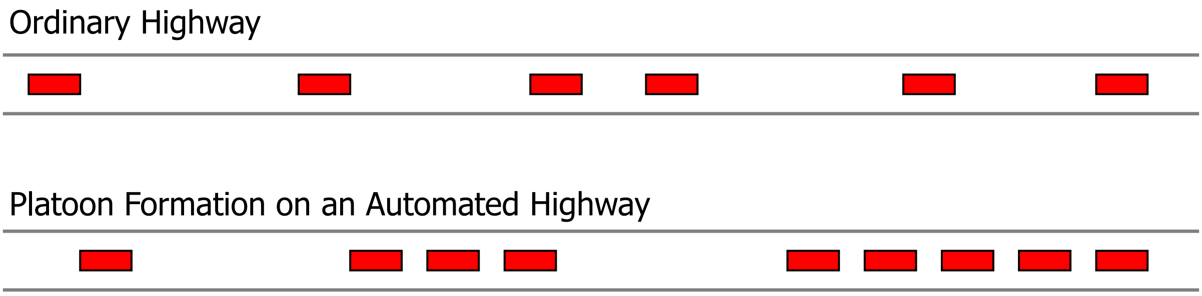
It is noteworthy that in traffic systems, RED Phase means STOP, GREEN Phase means GO, and YELLOW Phase means STAND FOR or SLOW-DOWN. We shall now proceed to consider the formation of platoons in traffic flow.

2. Stochastic Model Platoon Formation in Traffic Flow

"Stochastic" means being or having a random variable. A stochastic model is a tool for estimating probability distributions of potential outcomes by allowing for random variation in one or more of the input over time. The random variation is usually based on fluctuations observed in historical data for a selected period using standard time-series techniques. Distributions of potential outcomes are derived from a large number of simulations (stochastic projections) which reflect the random variation in the input(s). The model used for the formation of platoon in Traffic Flow is Stochastic Model due because it is appropriate for determining the delay to pedestrian because the cars arrive at regular intervals; therefore stochastic model refers to as a realistic model.

2.1 Model Description [9]

This section deals with description of stochastic model of platoon formation in traffic flow.



(Figure 1)

The benefit of the figure above is to increase safety and high capacity. The approach that we shall use here is as follows; we shall take Traffic as a MAS (Multi-Agent System) and Car as an Agent. We shall take v_i as velocity drawn from velocity distribution $P_o(v)$ and R_i as risk factor. For convenience, we apply the following passing rules.

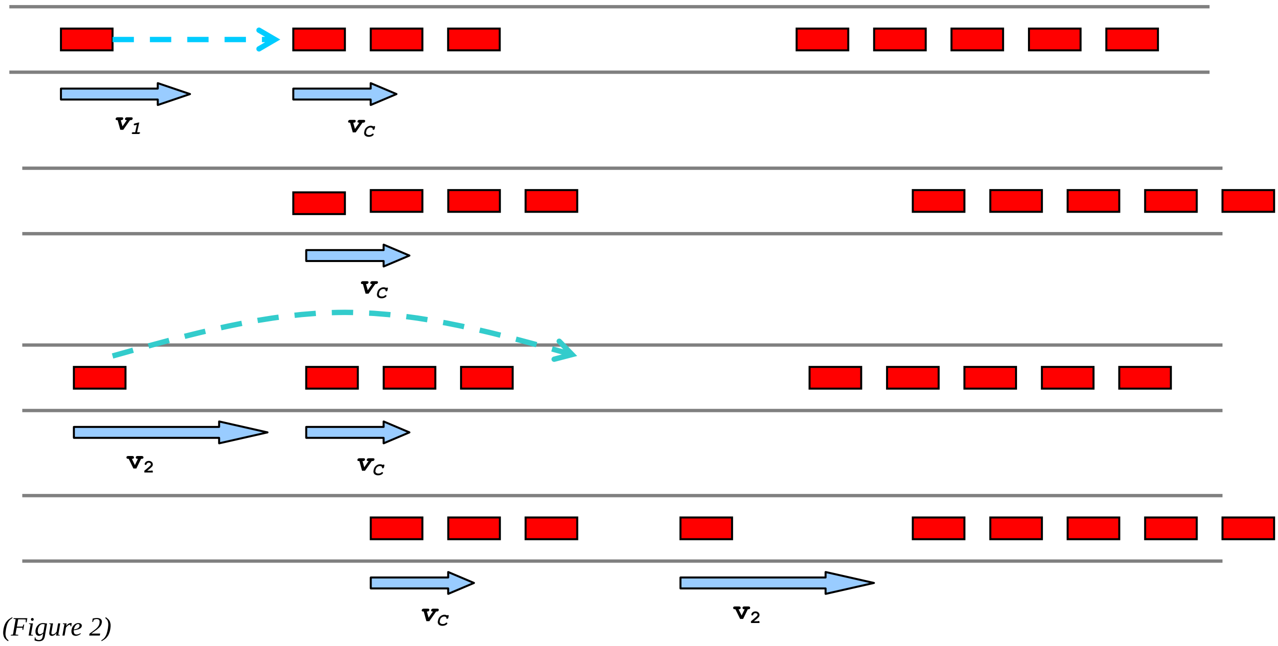
(a) When a fast car (velocity v_i) approaches a platoon (velocity v_c), it maintains its speed and passes the platoon with probability W or slows down and joins platoon with probability $1-W$.

(b) Passing probability W is given by

$$W(V_i - V_c) = \Theta(V_i - V_c - V_{av}R_i) \quad (1)$$

$\Theta(X)$ is a step function and R is the same for all agents (cars).

The diagram below also describes the formation of platoon.



(Figure 2)

2.2 Master Equation for Platoon Formation [7]

In terminology of statistical physics, the state of a system is described by a distribution function $F(\vec{x}(t))$ which obeys the continuity equation analogous to the Boltzmann equation from kinetic theory of gases. The vector $\vec{x}(t)$ denotes the state of an individual vehicle, i.e., its position, velocity, lane number, etc. But we will assume that the system is spatially homogeneous, and that the state of each vehicle is described by its velocity and the size of the cluster it belongs to. For the sake of simplicity, we will consider the case of one-lane traffic with passing.

Consider a segment of a freeway with on and off ramps "homogeneous" located along the freeway. Initially, isolated cars are randomly distributed on the freeway according to some density ρ_0 . Each car moves with its velocity drawn from a given (continuous) distribution $P_0(v)$ (we will measure the velocity, distance and time in units of initial average velocity $v_0 = \int_0^\infty dvP_0(v)$, inverse density, and the average collision time $1/v_0\rho_0$ respectively). When a fast car (cluster) is blocked in front by a slow cluster, it slows down and joins the cluster with probability $1-W$. With probability W , it maintains its speed and passes the cluster. Note that the probability for a car to undertake a passing maneuver depends on the velocity difference of the fast (overtaking) and slow (blocking) clusters. In the multi-agent terminology, it means that an agent is more likely to join a cluster with velocity close to its own.

To incorporate this assumption into the model we adopt the following simple functional form for the passing probability,

$$W(v-v^{'})=\Theta(v-v^{'}-v_R) \tag{2}$$

Where $\Theta(X)$ is the step function, and v_R is a parameter of a model that determines the balance between an agent desire for safety (not passing and joining platoon) and minimizing its travel time (passing and keeping its inherent velocity). If the no passing limit, $v_R \to \infty$ fast cars would cluster behind slower ones, which will bring the system to a congested phase at sufficiently high densities. In the opposite limit, $v_R \to 0$ passing occurs very frequently as each car tends to optimize its own performance by maintaining its inherent velocity. In this paper, we assume that v_R is the same for all agents, the only source of the heterogeneity (disorder) is the initial distribution $P_0(v)$.

Let $P_m(v,t)$ be the time-dependent density of cluster of size m, moving with velocity v, so that $mP_m(v)$ gives the probability that a car picked at random moves with velocity v in a Cluster of size m. Also, let $U(v,v^{'})$ be the rate at which cars (clusters) with velocity $v^{'}$ are joining the cars (clusters) with velocity v. For the Boltzmann model from the kinetic theory of gases one has

$$U(v,v^{'})=\left|v-v^{'}\right|(1-W(\left|v-v^{'}\right|)) \tag{3}$$

The term $\left|v-v^{'}\right|$ is simply the rate at which clusters with velocity $v^{'}$ and v approach each other, and $(1-W(|v-v'|))$ is the probability that they will merge, as explained earlier.

Hence, the Master equation governing the time evolution of the joint size-velocity distribution reads

$$\frac{\partial P_m(v)}{\partial t}=\overbrace{-P_m(v)\sum_k\int_0^\infty dv^{'}P_k(v^{'})U}^{loss\ due\ to\ collisions}+\overbrace{\sum_{k+j=m}\int_v^\infty dv^{'}P_k(v^{'})P_j(v)U}^{merging\ of\ smaller\ platoons}+\underbrace{\gamma\left[(m+1)P_{m+1}(v)-mP_m(v)\right]}_{outflow\ of\ cars}+\underbrace{\gamma P_0(v)\delta_m}_{\inf low\ of\ cars},\ m=1,2,... \tag{4}$$

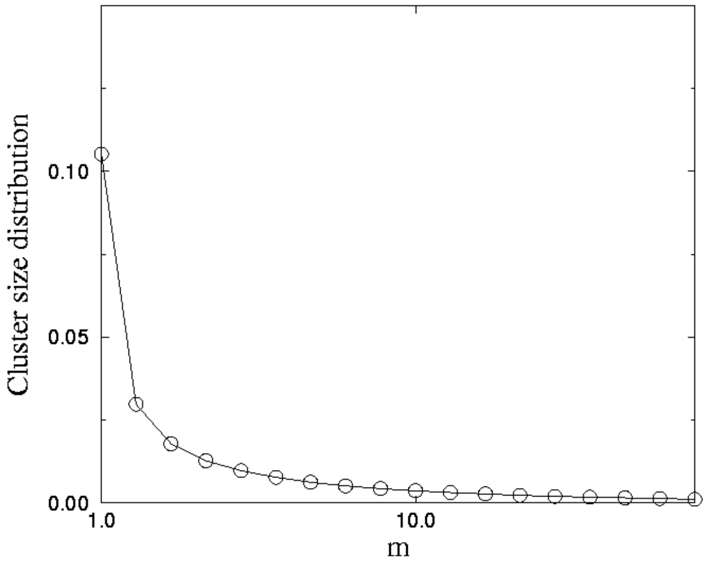
From (4), it is important to note that inflow and outflow drives the system into a steady state. The first term in (4) describes the loss in the cluster density due to collision with other clusters, while the second term describes the gain in the cluster density due to the "merging" of two smaller clusters. Also, the terms "proportional to" describes the change in cluster density due to the inflow and outflow of vehicles on the freeway, can verify that (4) conserves the car density,

$$\sum_k k\int_0^\infty dvP_k(v,t)=1 \tag{5}$$

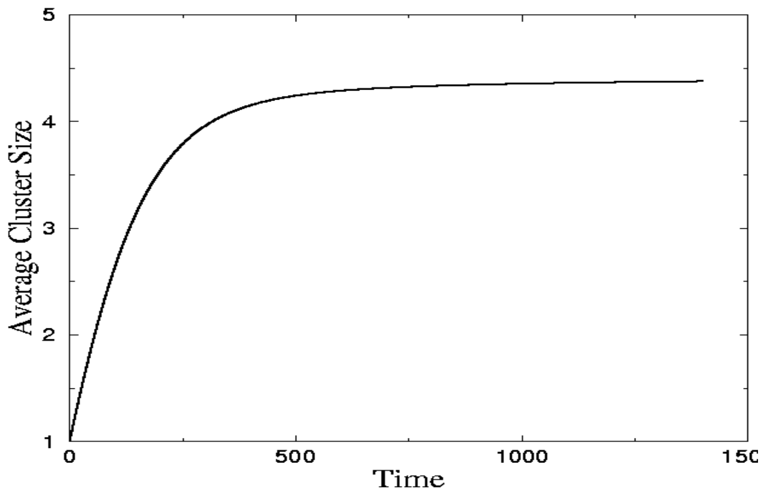
The Master equation (4) is a complicated system of coupled integro-differential equation, [1] studied the system (4) with velocity –independent collision rates (Maxwell Model), $U(v,v^{'})=cons\tan t$ which they showed to be analytically tractable. With a velocity-dependent kernel (2), however, (4) cannot be solved analytically.

Below are some preliminary results obtained by numerical methods. Plotting the steady-state cluster-size distribution $P_m = \int_0^\infty dv P_m(v, t \to \infty)$ for an initial uniform velocity distribution and with $\gamma = 0.001, v_R = 0.3,$ one can see that the initial $\delta - like$ distribution has spread out. Note also, that P_m is not normalized since the total cluster density is not constant, in contrast to the car density. In fact, the cluster density is the inverse of the average cluster size;

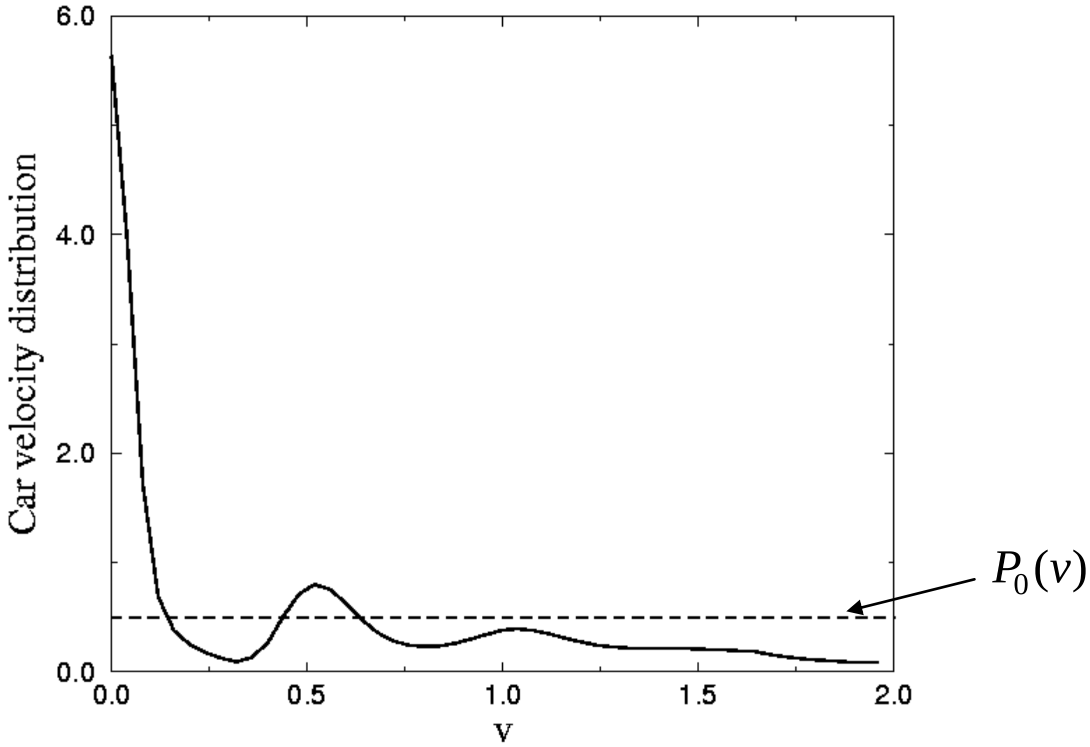
$$\langle m \rangle = \sum_k k P_k(t) / \sum_k P_k(t) = 1 \Big/ \sum_k P_k(t) \qquad (6)$$



Steady State Cluster–Size Distribution.(Figure 3)



The Average Cluster Size vs Time. *(Figure 4)*



Car Velocity Distribution at steady state *(Figure 5)*

The initial homogeneous distribution (dashed line) as in Figure 5 has evolved into multi-modal distribution, indicating that along with clusters formed behind the slowest cars, there are also faster cluster. This is due to the filtering effect of the velocity-dependent passing probabilities. Note also that, the spread of the car velocities inside a cluster can be considerably greater than V_R. *T*his is because a fast car might slow down several times due to further coalescence of clusters.

2.3 Traffic Congestion [12]

Congestion is relatively easy to recognize-roads filled with cars, trucks and buses, sidewalks filled with pedestrians. The following are causes of congestions.

Category 1; Traffic-Influencing Events

Traffic Incidents- are events that disrupt the normal flow of traffic usually physical impedance in the travel lanes. Events such as vehicular crashes, breakdowns and debris in travel lanes are the most common forms of incidents.

Work Zones- are construction activities on the roadway that result in physical changes to the highway environment. These include reduction in the number or width of travel lanes, lane shifts, lane diversions, and even temporary roadway closures.

Weather- environmental conditions can lead to changes in driver behavior that affect traffic flow. Due to reduced visibility, drivers will usually lower their speeds and increase their headways when precipitation, bright sunlight on the horizon, fog or smoke are present. Snowy or icy roadway surface conditions will also lead to the same effect even after precipitation has ended.

Category 2; Traffic Demand

Fluctuations in Normal Traffic- day-to-day variability in demand leads to some days with higher traffic volumes than others. Varying demand volumes superimposed on a system with fixed capacity also results in variable (i.e, unreliable) travel times, even without any category 1 events occurring.

Special Events- are special case of demand fluctuations where traffic flow in the vicinity of the event will be radically different from “typical” patterns. Special events occasionally can use “surges” in traffic demand that overwhelms the system.

Category 3; Physical Highway Features

Traffic Control Devices- intermittent disruption of traffic flow by control devices such as rail road grade crossings and poorly time timed signal also contribute to congestion and travel time variability.

Physical Bottlenecks (“capacity”)- transportation engineers have long studied and addressed the physical capacity of roadways- the maximum amount of traffic capable of being handled by a given highway section. Capacity is determined by a number of factors; the number and width of lanes and shoulder; merge areas at interchanges; and road alignment (grades and curves). Toll booths may also be thought of as a special case of bottlenecks because they restrict the physical flow of traffic. These lead to an increase in the amount of traffic that can be handled.

3.0 Applications

Traffic System can be modeled to solve traffic problems of different kind. Here we shall derive a model and compare the model with an existing model, we shall also put into consideration the movement of pedestrians.

Problem 1

The traffic lights at a road junction are set to operate with a red phase of length 100 seconds and a green phase of length 60 seconds. The amber phase may be ignored as it can be incorporated into either the red phase or the green phase. Vehicle arrive at the traffic lights turn to green, the vehicle in the queue leave at a rate of one every second. A red and a green cycle is hence of length 160 seconds during which, an average, 40 vehicles arrive. During the green phase a maximum of 60 vehicles may pass through the road junction. Hence, we would expect all the vehicles arriving one cycle to pass through the junction during that cycle.

They system may be modeled by assuming that the vehicles arrive every 4 seconds precisely i.e. at times 2,6,10, . . ., 154,158; after the start of the red phase, and that the first vehicle in the queue leaves at the start of the green phase, the next one second later, etc.

Solution

It is possible to show that the last seven vehicles arriving a red-green cycle are not subject to any delay. This is achieved by creating a table of the car arrival times and working out these subsequent delays given the length of the red phase and the amount of the time taken for each car to go through the lights once the green phase begins. From the table it becomes clear that car 33 goes through the light at 132 seconds resulting in no delay for the following 7 vehicles.

Table 1

Car arrival times secs (s)	Delay (s)
2	98
6	95
10	92
14	89
18	86
22	83
26	80
30	77
34	74
38	71
42	68
46	65
50	62
54	59
58	56
62	53
66	50
70	47
74	44
78	41
82	38
86	35
90	32
94	29
98	26
102	23
106	20
110	17
114	14
118	11
122	8
126	5
130	2
134	0
138	0
142	0
146	0
150	0
154	0
158	0

Red Phase

Green Phase

Car 33 goes through the lights at 132 seconds resulting in no delay for the following 7 vehicles

Red Phase

From table 1 above, it is possible to find the average delay per vehicle.

$$Average\, delay\ per\ vehicle = \frac{1}{N}\sum Delay$$

$$= \frac{1}{40}\sum Delay = \frac{1650}{40} = 41.25$$

Where N is the total number of vehicles that arrive during phase cycle and the $\sum Delay$ is the sum of all vehicles delayed.

Problem 2

We consider the traffic light model of problem 1 with general values of the parameters

$r_v = lenght\ of\ red\ phase\,(\sec onds)$

$g_v = lenght\ of\ green\ phase\,(\sec onds)$

$a = time\, between\, arrivals\, of\ vehicles\,(\sec onds)$

$d = time\, between\, department\, of\ vehicles\, in\, the\, queue\,(\sec onds)$

Solution

It is possible to devise a formula to give the average delay per vehicle in terms of $r_v, g_v, a, and\ d$

The next question is how can we generate our N without counting one after the other the number of arrived vehicles? Then with assumption, we equate out N to be number of vehicles

$$= \frac{r_v + g_v}{a}$$

$$= \frac{100+60}{4} = 40$$

The $\sum Delay$ can be expressed as the sum of an arithmetic progression

$$S_n = \frac{1}{2}n(A+L)$$

Where $A = a - d$ is the first term and $L = r_v - 1$ is the last term;

$$n = \frac{r_v - 1}{a-d}$$

$$S_n = \frac{(r_v - 1)}{2(a-d)}\left[(a-d)+(r_v - 1)\right]$$

$$= \frac{(r_v - 1)}{2}\left[1 + \frac{(r_v - 1)}{(a-d)}\right]$$

The formula can be re-written as Average Delay per vehicle $= S_n$ and

$$S_n = \frac{a(r_v - 1)}{2(r_v + g_v)}\left[1 + \frac{(r_v - 1)}{(a-d)}\right]$$
$$= (99/80) \bullet [1 + (99/3)] = 42.075$$

Therefore, we put the values into the formula $\frac{1}{N}\sum Delay$, we obtain

$$= (1/40) \bullet 42.075$$
$$= 1.051875(this\, is\, the\, average\, delay\ per\, vehicle)$$

If the system is modeled stochastically rather than deterministically, what effect will this have on the average delay per vehicle?
In this case, the car arrival and departure times are random variable

$$X = arrival\, time$$
$$Y = departure\, time$$

Therefore $X \sim P_0(a)\ and\ Y \sim P_0(d)$

Therefore taking into consideration random variation, if a large number of vehicle arrive at the traffic light in a short period of time, then the average delay per vehicle will increase due to the congestion. However, if every few cars arrive at the lights over a short period of time then the average delay per vehicle will decrease because there is no long standing cue of traffic waiting to go through.

4.0 Conclusion

We have been able to present how platoons are being formed and their velocity-size distribution in freeway traffic using the stochastic master equation approach. We have also been able to reduce motor accidents

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