Some Contractive Mappings On S-Metric Spaces

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Abstract
The present study prove some fixed point results for two self-mappings in a complete S-metric space under some contractive conditions.

Keywords: S-metric spaces, fixed point, nondecreasing map

1 Introduction.
Studies on generalized metric spaces have received serious attention in recent years. One reason for this interest is their potential applicability. Specifically [5, 6] introduced an improved version of the generalized metric space structure, which they called G-metric space and established the Banach contraction principle. For more details on G-metric space, one can refer to the papers [7, 8]. Recently Sedghi et al.[9] have introduced the concept of S-metric space and some properties. Also, in [3, 4] some new properties of S-metric spaces were represented.

In this paper we attain some fixed point results for self-mappings in a complete S-metric space under some contractive conditions in terms of a nondecreasing map \( \phi \).

2 Basic Concepts
In this part we recast the concept of S-metric space introduced by [9] for our goals.

Definition 2.1 Let \( X \) be a nonempty set. We call S-metric on \( X \) is a function \( S: X^3 \to [0, \infty) \) which satisfies the following conditions for each \( x, y, z, a \in X \)

(i) \( S(x, y, z) \geq 0 \),

(ii) \( S(x, y, z) = 0 \) if and only if \( x = y = z \),

(iii) \( S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a) \).

The set \( X \) in which S-metric is defined is called S-metric space.

The examples of such S-metric spaces are:

(a) Let \( X \) be any normed space, then \( S(x, y, z) = \|y\| + \|z\| - 2\|x\| + \|y\| - \|z\| \) is a S-metric on \( X \).

(b) Let \( (X, d) \) be a metric space, then \( S(x, y, z) = d(x, z) + d(y, z) \) is a S-metric on \( X \). This S-metric is called the usual S-metric on \( X \).

(c) Another S-metric on \( (X, d) \) is \( S(x, y, z) = d(x, y) + d(x, z) + d(y, z) \) which is symmetric with respect to the arguments.

The following lemmas have important role in our work (See[9]).

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Lemma 2.1 In a $S$-metric space, we have $S(x,x,y) = S(y,y,x)$.

Lemma 2.2 Let $(X,S)$ be a $S$-metric space. If there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $\lim_{n \to \infty} x_n = x$ and $\lim_{n \to \infty} y_n = y$, then $\lim_{n \to \infty} S(x_n, x_n, y_n) = S(x,y)$.

There exists a natural topology on a $S$-metric spaces, for more details we refer to [3].

Lemma 2.3 (See[3]). Any $S$-metric space is a Hausdorff space.

Definition 2.2 Let $f$ and $g$ be self-mappings of a set $X$. If $g x = f x = w$ for some $x$ in $X$, then $x$ is called a coincidence point of $f$ and $g$, and $w$ is called a point of coincidence of $f$ and $g$.

Theorem 2.1 [1] Let $f$ and $g$ be weakly compatible self-mappings of a set $X$. If $f$ and $g$ have a unique point of coincidence $w = f x = g x$, then $w$ is the unique common fixed point of $f$ and $g$.

3 Main Result

Suppose by [2] a nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ has the following properties (when the power of functions to be understand with respect to the composition operation):

(M1) $\lim_{n \to \infty} \phi^n(t) = 0$, for all $t \in (0, +\infty)$,

(M2) $\phi(t) < t$ for all $t \in (0, +\infty)$,

(M3) $\phi(0) = 0$.

Examples of such functions will appear in what follows. The set of all function $\phi$ is denoted by $\Phi$.

The method of proof of the following theorem is similar to the proof of the respective fact from [10].

Theorem 3.1 Let $X$ be a complete $S$-metric space and a self-map $T$ on $X$ satisfy the following contraction condition:

$$S(T(x), T(y)) \leq \phi(S(x, x, y))$$

(1)

for a $\phi \in \Phi$ and for all $x, y \in X$. Then $T$ has a unique fixed point $u \in X$ and $T$ is continuous at $u$.

Proof. Choose $x_0 \in X$ and suppose that $x_n = T(x_{n-1})$ for $n \in \mathbb{N}$. Assuming $x_n \neq x_{n-1}$ we will show that $\{x_n\}$ is a Cauchy sequence. For $n \in \mathbb{N}$ we get

$$S(x_n, x_n, x_{n+1}) = S(T(x_{n-1}), T(x_{n-1}), T(x_n))$$

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\[ \leq \phi(S(x_{n-1}, x_{n-1}, x_n)) \]  
\[ ... \]  
\[ \leq \phi^m(S(x_0, x_0, x_1)) \]

Let \( \varepsilon > 0 \) be given. By (M1) and (M2) we have, \( \lim_{n \to +\infty} \phi^n(S(x_0, x_0, x_1)) = 0 \) and \( \phi(\varepsilon) < \varepsilon \), then there exists \( n_0 \) such that

\[ \phi^n(S(x_0, x_0, x_1)) < \frac{1}{2}(\varepsilon - \phi(\varepsilon)) \quad \forall n \geq n_0. \]

Therefore by (2)

\[ S(x_n, x_n, x_{n+1}) < \frac{1}{2}(\varepsilon - \phi(\varepsilon)) \quad \forall n \geq n_0. \]

Applying the induction on \( m \) we can assert that

\[ S(x_n, x_n, x_m) < \varepsilon \quad \text{for all} \quad m \geq n \geq n_0. \]

Since \( \varepsilon - \phi(\varepsilon) < \varepsilon \), and by (3), holds for \( m = k \). By (iii) and Lemma 2.1 for \( m = k + 1 \), we have

\[ S(x_n, x_n, x_{k+1}) \leq 2S(x_n, x_n, x_{n+1}) + S(x_{k+1}, x_{k+1}, x_{n+1}) \]

\[ = 2S(x_n, x_n, x_{n+1}) + S(x_{n+1}, x_{n+1}, x_{k+1}) \]

\[ \leq \varepsilon - \phi(\varepsilon) + \phi(S(x_n, x_n, x_k)) \]

\[ \leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon. \]

Therefore \( \{x_n\} \) is a Cauchy sequence.

Since \( X \) is complete then \( \{x_n\} \) convergent to some \( u \in X \). By (iii) and Lemma 2.1, for \( n \in \mathbb{N} \) we have

\[ S(u, u, T(u)) \leq 2S(u, u, x_{n+1}) + S(T(u), T(u), x_{n+1}) \]

\[ = 2S(u, u, x_{n+1}) + S(x_{n+1}, x_{n+1}, T(u)) \]

\[ = 2S(u, u, x_{n+1}) + S(T(x_n), T(x_n), T(u)) \]

\[ \leq 2S(u, u, x_{n+1}) + \phi(S(x_n, x_n, u)) \]

\[ < 2S(u, u, x_{n+1}) + S(x_n, x_n, u) \]
By letting $n \to \infty$ we have $S(u,u,T(u)) = 0$, hence by (ii) we have $T(u) = u$. Therefore $u$ is a fixed point of $T$. To prove the uniqueness suppose that $v$ is another fixed point of $T$. By (1) and (M2) we have

$$S(u,u,v) = S(T(u),T(u),T(v))$$

$$\leq \phi(S(u,u,v))$$

$$< S(u,u,v).$$

Then $u = v$. To prove the continuity of $T$ at $u$, let $\{y_n\}$ be a sequence that converges to $u$. For $n \in \mathbb{N}$ we get

$$S(u,u,T(y_n)) = S(T(u),T(u),T(y_n))$$

$$\leq \phi(S(u,u,y_n))$$

$$< S(u,u,y_n).$$

Letting $n \to \infty$, we have $\lim_{n \to \infty} S(u,u,T(y_n)) = 0$. Therefore $T(y_n)$ converges to $u = T(u)$.

**Corollary 3.1** Let $T$ be a self map on complete $S$-metric space $(X,S)$ satisfying on following contraction condition for $\phi \in \Phi$ and all $x, y \in X$ and for some $m$:

$$S(T^m(x),T^m(x),T^m(y)) \leq \phi(S(x,x,y)),$$

then $T$ has a unique fixed point.

**Proof.** By Theorem 3.2 we deduce that $T^m$ has a fixed point (say, $u$). Since

$$T(u) = T(T^m(u)) = T^{m+1}(u) = T^m(T(u),$$

therefore $T(u)$ is also a fixed point for $T^m$. By uniqueness of $u$, we have $T(u) = u$.

**Corollary 3.2** Let $T$ be a self map on a complete $S$-metric space $(X,S)$. Suppose there is $k \in [0,1)$ such that $T$ satisfies the following two contraction conditions for all $x, y \in X$:

$$S(T(x),T(x),T(y)) \leq kS(x,x,y),$$

(5)
\[ S(T(x), T(x), T(y)) \leq \frac{S(x, x, y)}{1 + S(x, x, y)}, \quad (6) \]

then \( T \) has a unique fixed point (say, \( u \)) and \( T \) is continuous at \( u \).

**Proof.** For (5) define \( \phi : [0, \infty) \to [0, \infty) \) by \( \phi(t) = kt \) and for (6) define \( \phi(t) = \frac{t}{1 + t} \). It’s clear that \( \phi \) is nondecreasing function with \( \lim_{t \to 0} \phi'(t) = 0 \) for all \( t > 0 \). Since (1) is holds, the result follows from Theorem 3.2.

In this paper we prove following theorem:

**Theorem 3.2** Let \( X \) be a \( S \)-metric space. Suppose the maps \( f, g : X \to X \) satisfy:

\[ S(fx, fx, fy) \leq \phi(\max\{S(gx, gx, gy), G(gx, gx, fx), G(gy, gy, fy)\}) \quad (7) \]

for all \( x, y \in X \). If \( f(X) \subseteq g(X) \) and \( g(X) \) is a closed subspace of \( X \), then \( f \) and \( g \) have a unique point of coincidence in \( X \). Moreover, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.

**Proof.** Suppose \( f \) and \( g \) satisfy inequality (7). Let \( x_0 \) be an arbitrary point in \( X \). Since \( f(X) \subseteq g(X) \), choose \( x_i \in X \) such that \( f(x_i) = g(x_i) \). Continuing this process, we produce a sequence \( \{x_n\} \) in \( X \) such that \( f(x_n) = g(x_{n+1}) \) for all \( n \in N \). For \( n \in N \cup \{0\} \), we have

\[
S(gx_n, gx_n, gx_{n+1}) = S(fx_{n-1}, fx_{n-1}, fx_n) \\
\leq \phi(\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gx_n, gx_n, fx_n)\}).
\]

Since

\[ S(gx_n, gx_n, fx_n) = S(gx_n, gx_n, gx_{n+1}) \]

and

\[ \phi(S(gx_n, gx_n, fx_n)) < S(gx_n, gx_n, gx_{n+1}) \]

we have

\[
\max\{S(gx_{n-1}, gx_{n-1}, gx_n), S(gx_{n-1}, gx_{n-1}, fx_{n-1}), S(gx_n, gx_n, fx_n)\}
\]

\[ = S(gx_{n-1}, gx_{n-1}, gx_n). \]
Thus for $n \in \mathbb{N}$, we have
\[
S(gx_n, gx_n, gx_{n+1}) \leq \phi(S(gx_{n-1}, gx_{n-1}, gx_n))
\]
\[
\leq \phi^2(S(gx_{n-2}, gx_{n-2}, gx_{n-1}))
\]
\[
\vdots
\]
\[
\leq \phi^n S((gx_0, gx_0, gx_1)).
\]

Given $\varepsilon > 0$. Since $\lim_{n \to \infty} \phi^n (S(gx_0, gx_0, gx_1)) = 0$ and $\frac{1}{3} (\varepsilon - \phi(\varepsilon)) > 0$, there is an integer $k_0$ such that
\[
\phi^n (gx_0, gx_1, x_1) < \frac{1}{3} (\varepsilon - \phi(\varepsilon)) \quad \text{for all} \quad n \geq k_0.
\]

Hence
\[
S(gx_n, gx_n, gx_{n+1}) < \frac{1}{3} (\varepsilon - \phi(\varepsilon)) \quad \text{for all} \quad n \geq k_0.
\] (8)

For $k, n \in \mathbb{N}$ with $k > n$, we claim:
\[
S(gx_n, gx_n, gx_k) < \varepsilon \quad \text{for all} \quad k \geq n \geq k_0.
\] (9)

By induction on $k$ we prove inequality (9). Inequality (9) holds for $k = n + 1$ by using inequality (8) and the fact that $\frac{1}{3} (\varepsilon - \phi(\varepsilon)) < \varepsilon$. Assume inequality (9) holds for $k = m$, that is,
\[
G(gx_n, gx_n, gx_m) < \varepsilon \quad \text{for all} \quad m \geq n \geq k_0.
\] (10)

For $k = m + 1$, we have
\[
S(gx_n, gx_n, gx_{m+1}) \leq 2S(gx_n, gx_n, gx_{m+1}) + S(gx_{m+1}, gx_{m+1}, g_{m+1})
\]

From inequality (7), we have
\[
S(gx_{m+1}, gx_{m+1}, gx_{m+1}) = S(fx_n, fx_n, fx_m)
\]
\[
\leq \phi(\max \{S(gx_n, gx_n, gx_m), S(gx_n, gx_n, fx_n), S(gx_m, gx_m, fx_m)\}).
\]

If
\[
\max \{S(gx_n, gx_n, gx_m), S(gx_n, gx_n, fx_n), S(gx_m, gx_m, fx_m)\} = S(gx_n, gx_n, gx_m)
\]
then
\[
S(gx_n, gx_n, gx_{m+1}) \leq 2S(gx_n, gx_n, gx_{m+1}) + \phi(S(gx_n, gx_n, g_{m+1})
\]

By inequalities (8) and (10), we get
\[
G(gx_n, gx_n, gx_{m+1}) < \frac{2}{3} (\varepsilon - \phi(\varepsilon)) + \phi(\varepsilon) < \varepsilon
\]
If
\[
\max \{ S(g_{x_n}, g_{x_{n+1}}, g_{x_1}), S(g_{x_n}, g_{x_{n+1}}, f_{x_n}), S(g_{x_n}, g_{x_{n+1}}, f_{x_n}) \} = S(g_{x_n}, g_{x_{n+1}}, f_{x_n}).
\]

Then
\[
S(g_{x_n}, g_{x_{n+1}}, g_{x_{m+1}}) \leq 2S(g_{x_n}, g_{x_{m}}, g_{x_{n+1}}) + \phi(S(g_{x_n}, g_{x_{m}}, f_{x_m})) < 3S(g_{x_n}, g_{x_{n+1}}, g_{x_{m+1}})
\]

By inequality (8), we get
\[
S(g_{x_n}, g_{x_{n+1}}, g_{x_{m+1}}) < \varepsilon - \phi(\varepsilon) < \varepsilon.
\]

If
\[
\max \{ S(g_{x_n}, g_{x_{n+1}}, g_{x_1}), S(g_{x_n}, g_{x_{n+1}}, f_{x_n}), S(g_{x_n}, g_{x_{n+1}}, f_{x_n}) \} = S(g_{x_m}, g_{x_{m+1}}, f_{x_m}),
\]

then
\[
S(g_{x_n}, g_{x_{n+1}}, g_{x_{m+1}}) \leq 2S(g_{x_n}, g_{x_{m}}, g_{x_{n+1}}) + \phi(S(g_{x_m}, g_{x_{m}}, f_{x_m}))
\]

Since \( \phi(S(g_{x_m}, g_{x_{m}}, f_{x_m})) < S(g_{x_m}, g_{x_{m}}, f_{x_m}) \) and \( m > n \geq k_0 \), then by (8) we have
\[
S(g_{x_n}, g_{x_{n+1}}, g_{x_{m+1}}) < \varepsilon - \phi(\varepsilon) < \varepsilon.
\]

By induction on \( k \), we conclude that inequality (7) holds for all \( k \geq n \geq k_0 \). So \{ \( g_{x_n} \) \} is a Cauchy sequence in \( g(X) \). Since \( g(X) \) is complete, there is a point \( q \) in \( g(X) \) such that \{ \( g_{x_n} \) \} is convergent to some \( q \). Choose \( p \in X \) such that \( gp = q \). We claim \( fp = gp \). If not, then for \( n \in \mathbb{N} \cup \{0\} \) we have
\[
S(g_{x_n}, g_{x_{n+1}}, fp) = S(f_{x_{n-1}}, f_{x_{n-1}}, fp)
\]
\[
\phi(\max \{ S(g_{x_{n-1}}, g_{x_{n-1}}, gp), S(g_{x_{n-1}}, g_{x_{n-1}}, f_{x_{n-1}}), S(gp, gp, fp) \}) = \phi(\max \{ S(g_{x_{n-1}}, g_{x_{n-1}}, gp), S(g_{x_{n-1}}, g_{x_{n-1}}, f_{x_{n-1}}), S(gp, gp, fp) \})
\]

If
\[
\max \{ S(g_{x_{n-1}}, g_{x_{n-1}}, gp), S(g_{x_{n-1}}, g_{x_{n-1}}, f_{x_{n-1}}), S(gp, gp, fp) \} = S(g_{x_{n-1}}, g_{x_{n-1}}, gp),
\]

then
\[
S(g_{x_n}, g_{x_{n+1}}, fp) \leq \phi(S(g_{x_{n-1}}, g_{x_{n-1}}, gp)) < S(g_{x_{n-1}}, g_{x_{n-1}}, gp).
\]

Letting \( n \to \infty \), we get that \( gp = fp \). If
\[
\max \{ S(g_{x_{n-1}}, g_{x_{n-1}}, gp), S(g_{x_{n-1}}, g_{x_{n-1}}, f_{x_{n-1}}), S(gp, gp, fp) \} = S(g_{x_{n-1}}, g_{x_{n-1}}, f_{x_{n-1}}),
\]

then
\[
S(g_{x_n}, g_{x_{n+1}}, fp) \leq \phi(S(g_{x_{n-1}}, g_{x_{n-1}}, f_{x_{n-1}})) = \phi(S(g_{x_{n-1}}, g_{x_{n-1}}, g_{x_{n}}))
\]

Since \{ \( g_{x_n} \) \} is a Cauchy sequence and \( \phi(S(g_{x_{n-1}}, g_{x_{n-1}}, g_{x_{n}})) < S(g_{x_{n-1}}, g_{x_{n-1}}, g_{x_{n}}) \), by letting

\[
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\]
\[ n \to \infty, \text{ we get } gp = fp. \text{ If } \]
\[ \max\{S(gx_{n-1}, gx_{n-1}, gp), S(gx_{n-1}, fx_{n-1}, fx_{n-1}), S(gp, gp, fp)\} = S(gp, gp, fp), \]
\[ \text{then } S(gx_n, gx_n, fp) \leq \phi(S(gp, gp, fp)). \text{ Letting } n \to \infty \text{ we get } \]
\[ S(gp, gp, fp) \leq \phi(S(gp, gp, fp)) \]

Since \( \phi(S(gp, gp, fp)) < S(gp, gp, fp) \), we have \( S(gp, gp, fp) < S(gp, gp, fp) \) which is a contradiction. Therefore \( gp = fp \). For uniqueness \( p \), suppose that there exists another \( q \) in \( X \) such that \( fq = gq \). If \( gp \neq gq \), then we have
\[
S(gq, gq, gp) = S(fq, fq, fp)
\]
\[ \phi(\max\{S(gq, gq, gp), S(fq, fq, fp), S(gp, gp, fp)\}). \]

Since \( G(gq, gq, f) = 0 \), \( S(gp, gp, fp) = 0 \), and \( \phi(S(gq, gq, gp)) < S(gq, gq, gp) \), we have \( S(gq, gq, gp) < S(gq, gp, gp) \) which is a contradiction. So \( gp = gq \). From Theorem 2.1, \( f \) and \( g \) have a unique common fixed point.

Theorem 3.2 generalizes Theorems 2.3 and 2.4 in [1] for \( S \)-metric spaces.

**Corollary 3.3** Let \( X \) be a \( S \)-metric space. Suppose the maps \( f, g : X \to X \) satisfy on following inequality:
\[
S(fx, fx, fy) \leq aS(gx, gx, gy) + bS(gx, gx, fx) + cS(gy, gy, fy)
\]
for all \( x, y \in X \), where \( a + b + c < 1 \). If \( f(X) \subseteq g(X) \) and \( g(X) \) is a closed subspace of \( X \), then \( f \) and \( g \) have a unique point of coincidence in \( X \). Moreover, if \( f \) and \( g \) are weakly compatible, then \( f \) and \( g \) have a unique common fixed point.

**Proof.** For \( x, y \in X \), suppose
\[
H(x, x, y) = \max\{S(gx, gx, gy), S(gx, gx, fx), S(gy, gy, fy)\}.
\]
Then
\[
aS(gx, gx, gy) + bS(gx, gx, fx) + cS(gy, gy, fy) \leq (a + b + c)H(x, x, y).
\]
So if,
\[ S(fx, fx, fy) \leq aS(gx, gx, gy) + bS(gx, gx, fx) + cS(gy, gy, fy) \]

then \[ S(fx, fx, fy) \leq (a + b + c)H(x, x, y) \] . Define \( \phi : [0, +\infty) \rightarrow [0, +\infty) \) by \( \phi(t) = (a + b + c)t \) .

Then \( \phi \) is a nondecreasing function. Also, if \( a + b + c < 1 \) then \( \lim_{n \rightarrow +\infty} \phi^n(t) = 0 \) for all \( t > 0 \) . Hence by Theorem 3.2, we get the result.

**Corollary 3.4** Let \( X \) be a \( S \)-metric space. Suppose the maps \( f, g : X \rightarrow X \) satisfy on following inequality:

\[
S(fx, fx, fy) \leq k\max\{S(gx, gx, fx), S(gy, gy, fy)\}
\]

(11) for all \( x, y \in X \), where \( 0 \leq k < 1 \). If \( f(X) \subseteq g(X) \) and \( g(X) \) is a complete subspace of \( X \), then \( f \) and \( g \) have a unique point of coincidence in \( X \) . Moreover, if \( f \) and \( g \) are weakly compatible , then \( f \) and \( g \) have a unique common fixed point.

**Proof.** For all \( x, y \in X \), we let

\[ H(x, x, y) = \max\{S(gx, gx, fx), S(gy, gy, fy)\} \]

if inequality (11) is hold, then \( S(fx, fx, fy) \leq kH(x, x, y) \) . Define \( \phi : [0, +\infty) \rightarrow [0, +\infty) \) by \( \phi(t) = kt \) . Then its clear that \( \phi \) is nondecreasing and \( \lim_{n \rightarrow +\infty} \phi^n(t) = 0 \) for all \( t > 0 \) . The result follows from Theorem 3.2.

**References**


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