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Mathematical study of eco-epidemiological system

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Abstract

In this paper, a mathematical model consisting of the prey- predator involving infectious disease in prey population, is proposed and analyzed. And this disease passed from a prey to predator through attacking of predator to prey. The model represented mathematically by the set of nonlinear differential equations. The existence, uniqueness and boundedness of the solution of this model are investigated. The local and global stability conditions of all possible equilibrium points are established. The occurrence of local bifurcation (such as saddle-node, transcritical and pitchfork) a long with Hopf bifurcation near each of the equilibrium points are discussed. Finally, numerical simulation is used to study the global dynamics of this model.

Keywords: eco-epidemiological model, SI epidemics disease, prey-predator model, stability analysis, Hopf bifurcation.

1. Introduction:

We consider the growth of two interdependent populations. Given two species, interdependence might arises due to the existence of the interaction between them. The most important models of this type are known prey-predator models. Mathematical biologists have been working on merging two major areas of interest Ecology [1-3] and Epidemiology [4] for a long time. Diseases that affect the prey in particular may affect the entire prey-predator system [5-7]. The main objective of these models, which known as eco-epidemiological models, is to investigate the effect of the disease on the dynamical behavior of the prey-predator systems. Prey-predator-pathogen models have been a topic of significant interest since the early1980s. Anderson and May [8] in 1982 constructed the way of merging ecological prey-predator models, which were initiated by Lotka and Volterra, and the epidemiological models that introduced by Kermack and McKendrick. Prey-predator interactions have fascinated mathematical biologists for a long time. Eco-epidemiology is comparatively a new branch in mathematical biology which simultaneously considers the ecological and epidemiological processes [9]. Hadeler and Freedman [10] introduced an eco-epidemiological model regarding prey-predator interactions with both prey and predator subject to disease. Further, it is well known that in nature there is no species lived alone rather than that there are hundreds or thousands of species interact with each other in any given environment. On the other hand densely populated areas are a good incubator for the spread of infectious diseases. Therefore, there is an increasing opportunity for the spread of diseases among the communities interacting with each other. However, many diseases are transmitted in the species not only through contact, but also directly from environment, such as, influenza, bird flu and others see for example [11-15]. However during the last four decades the ideas oriented to study the dynamical behavior of eco-epidemiological models, which represented by mathematical models merging of the two phenomena, that is means the demographics of interacting species and an epidemic evolution in different environment. In this paper a consideration to prey-predator model where the prey population infected by some infectious disease and these disease passed from a prey to predator through attacking or predation process. While the disease transmitted within the same species by contact, between susceptible individuals and infected individuals. In this paper a prey-predator model involving **SI** infection disease in both the prey and predator species is proposed and analyzed.

2. Mathematical model:

In this section an eco-epidemiological model is proposed for study. The model consists of a prey, whose total population density at time T is denoted by $N(T)$, interacting with predator whose total population density at time T is denoted by $P(T)$. The following assumptions are adopted in formulating the basic eco-epidemiological model:

1. There is an SI epidemic disease in prey population divides the prey population into two classes namely $S(T)$ that represents the density of susceptible prey species at time T and $I(T)$ which represents the density of infected prey species at time T. Therefore at any time T, we have $N(T)=S(T)+I(T)$.

2. The disease is transmitted from a prey to predator during attacking of predator to prey, which divides the predator population into two classes namely $P_1(T)$ that represents the density of susceptible predator species at time T and $P_2(T)$ which represents the density of infected predator species at time T. Therefore at any time T, we have $P(T) = P_1(T) + P_2(T)$.
3. The susceptible prey is capable of reproducing in logistic fashion with carrying capacity $k > 0$, intrinsic growth $r > 0$.
4. The disease transmitted within the same species by contact with an infected individual at infection rates $\lambda_1 > 0$ and $\lambda_2 > 0$ for the prey and predator respectively.
5. In the absence of the prey the susceptible and infected predator decay exponentially with death rate $\gamma_2 > 0$.
6. The disease may causes mortality with a constant mortality rates $\gamma_1 > 0$ and $\gamma_3 > 0$ for prey and predator species respectively.
7. The susceptible predator consumes the susceptible and infected prey according to Lotka-Volterra type of functional response at constant consumption rates $c_1 > 0$ and $c_2 > 0$ for susceptible and infected respectively, while the infected predator can't attack the prey directly due to the its weakness.

Considering the above basic assumptions the prey-predator model can be represented in the following set of differential equations.

$$
\begin{aligned}
\frac{dS}{dT} &= rS\left(1 - \frac{S+I}{k}\right) - c_1 S P_1 - \lambda_1 S I = F_1(S, I, P_1, P_2) \\
\frac{dI}{dT} &= \lambda_1 S I - c_2 I P_1 - \gamma_1 I = F_2(S, I, P_1, P_2) \\
\frac{dP_1}{dT} &= -\lambda_2 P_1 P_2 + e_1 c_1 S P_1 + (1-m) e_2 c_2 I P_1 - \gamma_2 P_1 = F_3(S, I, P_1, P_2) \\
\frac{dP_2}{dT} &= \lambda_2 P_1 P_2 + m e_2 c_2 I P_1 - \gamma_2 P_2 - \gamma_3 P_2 = F_4(S, I, P_1, P_2)
\end{aligned}
\tag{1}
$$

with $S(0) > 0\,; I(0) > 0\,; P_1(0) > 0\,; P_2(0) > 0\,;\ 0 < e_i < 1\,; i = 1,2$ represent the conversion rates constants and $0 \le m \le 1$ represents the infection rate of susceptible predator that predation the infected prey. Cleary, system (1) included (12) parameters, which make the analysis difficult. So, in order to simplify the system the number of parameters is reduced by using the following dimensionless variables.

$$t = rT,\ \ x = \frac{S}{k},\ y = \frac{I}{k},\ z = \frac{c_1}{r} P_1,\ \ w = \frac{\lambda_2}{r} P_2$$

Thus we obtain:

$$
\begin{aligned}
\frac{dx}{dt} &= x(1 - x - (1 + a_1) y - z) = f_1(x, y, z, w) \\
\frac{dy}{dt} &= y(a_1 x - a_2 z - b_1) = f_2(x, y, z, w) \\
\frac{dz}{dt} &= z(-w + b_2 x + d_1 y - d_2) = f_3(x, y, z, w) \\
\frac{dw}{dt} &= n_1 z w + n_2 y z - (d_2 + l) w = f_4(x, y, z, w)
\end{aligned}
\tag{2}
$$

Where:

$$a_1 = \frac{\lambda_1}{r}k,\ a_2 = \frac{c_2}{c_1},\ b_1 = \frac{\gamma_1}{r},\ b_2 = \frac{e_1 c_1}{r}k,\ d_1 = (1-m)\frac{e_2 c_2}{r}k$$

$$d_2 = \frac{\gamma_2}{r},\ n_1 = \frac{\lambda_2}{c_1},\ n_2 = m\frac{e_2 c_2 \lambda_2}{c_1 r}k,\ l = \frac{\gamma_3}{r}$$

represent the dimensionless parameters of the system (2). Moreover the initial condition of system (2) may be taken as any point in the region R_+^4. The interaction functions in the right hand side of system (2) are continuously differentiable function on R_+^4, hence they are Lipschitizian. Therefore the solution of system (2) exists and is unique. Further, all the solutions of system (2) with non-negative initial condition are uniformly bounded as shown in the following theorem.

***Theorem* (1):** All the trajectories of system (2), which initiate in R_+^4 are uniformly bounded.

Proof: From the first equation of system (2) we obtain that

$$\frac{dx}{dt} \le x(1-x)$$

Clearly by solving the above differential inequality we get

$$\lim_{t\to\infty}\sup x(t) \le 1$$

Define the function $G(t) = x(t) + y(t) + z(t) + w(t)$ and then by taking its time derivative along the solution of system (2), gives

$$\frac{dG}{dt} \le x - (b_1 y + d_2 z + (d_2 + l)w) \le x - \mu(y + z + w)$$

where $\mu = \min\{b_1, d_2\}$, then we get

$$\frac{dG}{dt} = (1+\mu)x - \mu G \le (1+\mu) - \mu G$$

Now, by using Gronwall lemma, it obtains that:

$$0 < G(t) \le G(0)e^{-\mu t} - \frac{1+\mu}{\mu}\left(e^{-\mu t} - 1\right)$$

Thus $G(t) \le \frac{1+\mu}{\mu}$ as $t \to \infty$ that is independent of the initial conditions and hence the proof is complete. ■

3. Existence of equilibrium points:

It is observed that, system (2) has at most seven biologically feasible equilibrium points, namely E_i, $i = 0,1,2,3,4,5,6$. The existence conditions for each of these equilibrium points are discussed in the following:

The vanishing equilibrium point $E_0 = (0,0,0,0)$ always exists.

The axial equilibrium point $E_1 = (1,0,0,0)$ always exists.

The predator free equilibrium point $E_2 = (x_2, y_2, 0, 0)$, where:

$$x_2 = \frac{b_1}{a_1} \text{ and } y_2 = \frac{a_1 - b_1}{a_1(1+a_1)} \tag{3a}$$

exists uniquely in the interior of the first quadrant of *xy*-plane under the following necessary and sufficient condition:

$$a_1 > b_1 \tag{3b}$$

The disease free equilibrium point $E_3 = (x_3, 0, z_3, 0)$, where:

$$x_3 = \frac{d_2}{b_2} \text{ and } z_3 = \frac{b_2 - d_2}{b_2} \tag{4a}$$

exists uniquely in the interior of the first quadrant of *xz*-plane under the following necessary and sufficient condition:

$$b_2 > d_2 \tag{4b}$$

The infected prey free equilibrium point $E_4 = (x_4, 0, z_4, w_4)$, where:

$$x_4 = 1 - \frac{d_2 + l}{n_1},\ z_4 = \frac{d_2 + l}{n_1} \text{ and } w_4 = b_2 - \left(\frac{b_2(d_2 + l)}{n_1} + d_2\right) \tag{5a}$$

exists uniquely in the interior of the first octant of *xzw*-space under the following necessary and sufficient condition:

$$n_1 > d_2 + l + \frac{n_1 d_2}{b_2} \tag{5b}$$

The infected predator free equilibrium point $E_5 = (x_5, y_5, z_5, 0)$, where:

$$\begin{aligned} x_5 &= \frac{d_1(a_2 + b_1) - a_2 d_2(1 + a_1)}{d_1(a_1 + a_2) - a_2 b_2(1 + a_1)} \\ y_5 &= \frac{d_2(a_1 + a_2) - b_2(a_2 + b_1)}{d_1(a_1 + a_2) - a_2 b_2(1 + a_1)} \\ z_5 &= \frac{d_1(a_1 - b_1) + (1 + a_1)(b_1 b_2 - a_1 d_2)}{d_1(a_1 + a_2) - a_2 b_2(1 + a_1)} \end{aligned} \tag{6a}$$

exists uniquely in the interior of the first octant of *xyz*-space under the following necessary and sufficient conditions:

$$\left.\begin{aligned} & n_2 = 0 \\ & \frac{d_1}{a_2(1 + a_1)} > \frac{d_2}{a_2 + b_1} > \frac{b_2}{a_1 + a_2} \\ & a_1 d_1 > b_1 d_1 + (1 + a_1)(a_1 d_2 - b_1 b_2) \end{aligned}\right\} \tag{6b}$$

or

$$\left.\begin{aligned} & n_2 = 0 \\ & \frac{d_1}{a_2(1 + a_1)} < \frac{d_2}{a_2 + b_1} < \frac{b_2}{a_1 + a_2} \\ & a_1 d_1 + (1 + a_1)(b_1 b_2 - a_1 d_2) < b_1 d_1 \end{aligned}\right\} \tag{6c}$$

The positive equilibrium point $E_6 = (x_6, y_6, z_6, w_6)$, where:

$$\begin{aligned} & w_6 = \frac{n_2 y_6 z_6}{d_2 + l - n_1 z_6},\ z_6 = \frac{a_1 x_6 - b_1}{a_2}, \\ & y_6 = \frac{(d_2 + l - n_1 z_6)(b_2 x_6 - d_2)}{z_6(d_1 n_1 + n_2) - d_1(d_2 + l)} \end{aligned} \tag{7a}$$

while x_6 represents a positive root of the following second order polynomial equation

$$A_1 x^2 + A_2 x + A_3 = 0 \tag{7b}$$

here

$$\begin{aligned} A_1 &= a_1[a_2 n_1 b_2(1 + a_1) - (d_1 n_1 + n_2)(a_1 + a_2)] \\ A_2 &= (n_1 d_1 + n_2)[a_2(a_1 + b_1) + 2a_1 b_1] + d_1 a_2(d_2 + l)(a_2 + a_1) \\ &\quad - a_2(1 + a_1)[a_2 b_2(d_2 + l) + n_1(a_1 d_2 + b_1 b_2)] \end{aligned}$$

$$A_3 = a_2 d_2 (1+a_1)[a_2(d_2+l)+n_1 b_1] - (a_2+b_1)[b_1(n_1 d_1 + n_2) + d_1 a_2 (d_2+l)]$$

Consequently, straightforward computation shows that E_6 exists uniquely in the $Int.R_+^4$ if and only if the following conditions are hold.

$$\begin{array}{l} a_1 x_6 > b_1 \\ d_2 + l > n_1 z_6 \\ \left.\begin{array}{l} b_2 x_6 > d_2 \text{ with } n_2 z_6 > d_1(d_2 + l - n_1 z_6) \\ \text{OR} \\ b_2 x_6 < d_2 \text{ with } n_2 z_6 < d_1(d_2 + l - n_1 z_6) \end{array}\right\} \\ \left.\begin{array}{l} A_1 > 0 \text{ with } A_3 < 0 \\ \text{OR} \\ A_1 < 0 \text{ with } A_3 > 0 \end{array}\right\} \end{array} \tag{7c}$$

4. Local stability analysis of system (2):

In this section, the local stability analysis of system (2) around each of the above equilibrium points are discussed through computing the Jacobian matrix $J(x, y, z, w)$ of system (2) at each of them which given by:

$$J = \begin{pmatrix} 1-2x-(1+a_1)y-z & -x(1+a_1) & -x & 0 \\ a_1 y & a_1 x - a_2 z - b_1 & -a_2 y & 0 \\ b_2 z & d_1 z & b_2 x + d_1 y - w - d_2 & -z \\ 0 & n_2 z & n_1 w + n_2 y & n_1 z - (d_2 + l) \end{pmatrix} \tag{8}$$

The Local stability analysis at E_0:

The Jacobian matrix of system (2) at E_0 can be written as:

$$J_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -b_1 & 0 & 0 \\ 0 & 0 & -d_2 & 0 \\ 0 & 0 & 0 & -(d_2+l) \end{pmatrix} \tag{9}$$

Clearly, J_0 has three negative eigenvalues $\lambda_y = -b_1$, $\lambda_z = -d_2$, $\lambda_w = -(d_2 + l)$ and one positive eigenvalue in the *x*-direction $(\lambda_x = 1)$, so the equilibrium point E_0 is unstable saddle point.

The Local stability analysis at E_1:

The Jacobian matrix of system (2) at E_1 can be written as:

$$J_1 = \begin{pmatrix} -1 & -(1+a_1) & -1 & 0 \\ 0 & a_1 - b_1 & 0 & 0 \\ 0 & 0 & b_2 - d_2 & 0 \\ 0 & 0 & 0 & -(d_2+l) \end{pmatrix} \tag{10a}$$

Clearly, J_1 has the following eigenvalues:

$$\lambda_{1x} = -1,\ \lambda_{1y} = a_1 - b_1,\ \lambda_{1z} = b_2 - d_2,\ \lambda_{1w} = -(d_2 + l)$$

Therefore all the eigenvalues have negative real parts and hence the equilibrium point E_1 is locally asymptotically stable in the $Int.R_+^4$ provided that the following conditions are satisfied:

$$b_1 > a_1 \tag{10b}$$

$$d_2 > b_2 \tag{10c}$$

The Local stability analysis at E_2:

The Jacobian matrix of system (2) at E_2 can be written as:

$$J_2 = \begin{pmatrix} -x_2 & -x_2(1+a_1) & -x_2 & 0 \\ \frac{a_1(1-x_2)}{1+a_1} & 0 & \frac{a_2(x_2-1)}{1+a_1} & 0 \\ 0 & 0 & b_2x_2 + \frac{d_1(1-x_2)}{1+a_1} - d_2 & 0 \\ 0 & 0 & \frac{n_2(1-x_2)}{1+a_1} & -(d_2+l) \end{pmatrix} \tag{11a}$$

The characteristic equation of this Jacobian matrix is given by:

$$\left[\lambda_2^2 + x_2\lambda_2 + x_2(a_1 - b_1)\right] \times \left[\left(b_2x_2 + \frac{d_1(1-x_2)}{1+a_1} - d_2 - \lambda_{2z}\right)\left(-d_2 - l - \lambda_{2w}\right)\right] = 0$$

We obtain that:

$$\left.\begin{aligned} \lambda_{2x,y} &= \frac{-x_2 \pm \sqrt{x_2^2 - 4(a_1-b_1)x_2}}{2}, \\ \lambda_{2z} &= b_2x_2 + \frac{d_1(1-x_2)}{1+a_1} - d_2, \ \lambda_{2w} = -(d_2+l) \end{aligned}\right\} \tag{11b}$$

Therefore all the eigenvalues have negative real parts and hence the equilibrium point E_2 is locally asymptotically stable in the $Int.R_+^4$ provided that the following condition is satisfied:

$$d_2 > b_2x_2 + \frac{d_1(1-x_2)}{1+a_1} \tag{11c}$$

The Local stability analysis at E_3:

The Jacobian matrix of system (2) at E_3 can be written as:

$$J_3 = \begin{pmatrix} -x & -x_3(1+a_1) & -x_3 & 0 \\ 0 & (a_1+a_2)x_3 - (a_2+b_1) & 0 & 0 \\ b_2(1-x_3) & d_1(1-x_3) & 0 & x_3 - 1 \\ 0 & n_2(1-x_3) & 0 & n_1(1-x_3) - (d_2+l) \end{pmatrix} \tag{12a}$$

The characteristic equation of this Jacobian matrix is given by:

$$\left[(a_1+a_2)x_3 - (a_2+b_1) - \lambda_{3y}\right]\left[n_1(1-x_3) - (d_2+l) - \lambda_{3w}\right] \times \left[\lambda_3^2 + x_3\lambda_3 + x_3(b_2 - d_2)\right] = 0$$

We obtain that:

$$\left.\begin{aligned}\lambda_{3x,z} &= \frac{-x_3 \pm \sqrt{x_3^2 - 4(b_2 - d_2)x_3}}{2}, \\ \lambda_{3y} &= x_3(a_1 + a_2) - (a_2 + b_1),\ \lambda_{3w} = n_1(1 - x_3) - (d_2 + l)\end{aligned}\right\} \tag{12b}$$

Therefore all the eigenvalues have negative real parts and hence the equilibrium point E_3 is locally asymptotically stable in the $Int.R_+^4$ provided that the following conditions are satisfied:

$$x_3 < \frac{(a_2 + b_1)}{(a_1 + a_2)} \text{ and } n_1 < \frac{(d_2 + l)}{(1 - x_3)} \tag{12c}$$

The Local stability analysis at E_4:

The Jacobian matrix of system (2) at E_4 can be written as

$$J_4 = (a_{ij})_{4\times 4} \tag{13a}$$

where:

$$\begin{aligned}&a_{11} = -x_4 < 0,\ a_{12} = -x_4(1 + a_1) < 0,\ a_{13} = -x_4 < 0,\ a_{14} = 0\\ &a_{21} = 0,\ a_{22} = a_1 - (b_1 + z_4(a_1 + a_2)),\ a_{23} = 0,\ a_{24} = 0\\ &a_{31} = b_2 z_4 > 0,\ a_{32} = d_1 z_4 > 0,\ a_{33} = 0,\ a_{34} = -z_4 < 0\\ &a_{41} = 0,\ a_{42} = n_2 z_4 > 0,\ a_{43} = n_1 w_4 > 0,\ a_{44} = 0\end{aligned}$$

Then the characteristic equation of J_4 can be written as:

$$(a_{22} - \lambda_{4y})\left[\lambda_4^3 + L_1\lambda_4^2 + L_2\lambda_4 + L_3\right] = 0 \tag{13b}$$

here

$$L_1 = -a_{11},\ L_2 = -(a_{13}a_{31} + a_{34}a_{43}),\ L_3 = a_{11}a_{34}a_{43}$$

Further, it is easy to verify that $\Delta = L_1 L_2 - L_3 = a_{11}a_{13}a_{31}$. Clearly, the eigenvalue λ_{4y} in *y*-direction has negative real part if and only if the following condition holds.

$$a_1 < b_1 + z_4(a_1 + a_2) \tag{13c}$$

However, $L_i > 0,\ \forall i = 1,3$ and $\Delta > 0$. So, according to Routh-Hawirtiz criterion the equilibrium point E_4 is locally asymptotically stable.

The Local stability analysis at E_5:

The Jacobian matrix of system (2) at E_5 can be written as

$$J_5 = (b_{ij})_{4\times 4} \tag{14a}$$

where:

$$\begin{aligned}&b_{11} = -x_5 < 0,\ b_{12} = -x_5(1 + a_1) < 0,\ b_{13} = -x_5 < 0,\ b_{14} = 0\\ &b_{21} = a_1 y_5 > 0,\ b_{22} = 0,\ b_{23} = -a_2 y_5 < 0,\ b_{24} = 0\\ &b_{31} = b_2 z_5 > 0,\ b_{32} = d_1 z_5 > 0,\ b_{33} = 0,\ b_{34} = -z_5 < 0\\ &b_{41} = 0,\ b_{42} = 0,\ b_{43} = 0,\ b_{44} = n_1 z_5 - (d_2 + l)\end{aligned}$$

Then the characteristic equation of J_5 can be written as:

$$(b_{44} - \lambda_{5w})\left[\lambda_5^3 + D_1\lambda_5^2 + D_2\lambda_5 + D_3\right] = 0 \tag{14b}$$

here:

$$\begin{aligned}&D_1 = -b_{11},\ D_2 = -(b_{12}b_{21} + b_{13}b_{31} + b_{23}b_{32}),\\ &D_3 = b_{11}b_{23}b_{32} - b_{12}b_{23}b_{31} - b_{21}b_{32}b_{13}\end{aligned}$$

Further, it is easy to verify that

$$\Delta = D_1 D_2 - D_3 = b_{11}(b_{12}b_{21} + b_{13}b_{31}) + b_{12}b_{23}b_{31} + b_{21}b_{32}b_{13}$$

Not that, according to the element of J_5, it easy to verify that:

$$D_1 = x_5 > 0$$
$$D_3 = x_5 y_5 z_5 (d_1(a_1 + a_2) - a_2 b_2 (1 + a_1))$$
$$\Delta = x_5 [x_5 (a_1(1 + a_1) y_5 + b_2 z_5) + y_5 z_5 (a_2 b_2 (1 + a_1) - a_1 d_1)]$$

Clearly, the eigenvalue λ_{5w} in *w*- direction has negative real part if and only if the following condition holds.

$$n_1 z_5 < d_2 + l \tag{14c}$$

However, according to existence condition (6b) $D_i > 0$, $\forall i = 1,3$ and $\Delta > 0$ if and only if

$$a_2 b_2 (1 + a_1) \geq a_1 d_1 . \tag{14d}$$

So, according to Routh-Hawirtiz criterion the equilibrium point E_5 is locally asymptotically stable. While, according to existence condition (6c) we have $D_3 < 0$. So, according to Routh-Hawirtiz criterion the equilibrium point E_5 is unstable.

The Local stability analysis at E_6:

The Jacobian matrix of system (2) at E_6 can be written as

$$J_6 = \lfloor j_{ij} \rfloor_{4\times 4} \tag{15a}$$

where:

$$j_{11} = 1 - (2x_6 + (1 + a_1) y_6 + z_6),\ j_{12} = -x_6(1 + a_1) < 0,\ j_{13} = -x_6 < 0,\ j_{14} = 0$$
$$j_{21} = a_1 y_6 > 0,\ j_{22} = 0,\ j_{23} = -a_2 y_6 < 0,\ j_{24} = 0$$
$$j_{31} = b_2 z_6 > 0,\ j_{32} = d_1 z_6 > 0,\ j_{33} = 0,\ j_{34} = -z_6 < 0$$
$$j_{41} = 0,\ j_{42} = n_2 z_6 > 0,\ j_{43} = n_1 w_6 + n_2 y_6 > 0,\ j_{44} = n_1 z_6 - (d_2 + l) < 0$$

Then the characteristic equation of J_6 can be written as:

$$\lambda_6^4 + C_1 \lambda_6^3 + C_2 \lambda_6^2 + C_3 \lambda_6 + C_4 = 0 \tag{15b}$$

here:

$$C_1 = -(j_{11} + j_{44})$$
$$C_2 = -(j_{12}j_{21} + j_{13}j_{31} + j_{23}j_{32} + j_{34}j_{43} - j_{11}j_{44})$$
$$C_3 = j_{11}\Gamma_1 + j_{23}\Gamma_2 - j_{12}\Gamma_3 + j_{13}(j_{31}j_{44} - j_{21}j_{32})$$
$$C_4 = \Gamma_2 (j_{13}j_{21} - j_{11}j_{23}) + j_{12}\Gamma_4$$

Consequently:

$$\begin{aligned}\Delta_1 &= C_1 C_2 - C_3 \\ &= j_{11}(j_{13}j_{31} - j_{44}\Gamma_5) + j_{11}j_{12}j_{21} + j_{34}j_{43}j_{44} + j_{23}j_{34}j_{42} + \Gamma_6\end{aligned}$$

and

$$\Delta_2 = C_3(C_1 C_2 - C_3) - C_1^2 C_4 = B_1 + B_2 + B_3 + B_4 + B_5 + B_6$$

here:

$$B_1 = [j_{11}j_{13}j_{31} + j_{11}j_{12}j_{21} + j_{23}j_{34}j_{42} + \Gamma_6][j_{23}\Gamma_2 + j_{12}j_{21}j_{44}]$$
$$\begin{aligned}B_2 = [j_{11}(j_{13}j_{31} - j_{44}\Gamma_5) + j_{34}j_{43}j_{44} + j_{23}j_{34}j_{42} + \Gamma_6] \\ \times [j_{44}j_{13}j_{31} + j_{11}j_{34}j_{43}]\end{aligned}$$
$$\begin{aligned}B_3 = [j_{11}j_{13}j_{31} + j_{12}j_{21}j_{11} + j_{34}j_{43}j_{44} + j_{23}j_{34}j_{42} + \Gamma_6] \\ \times [j_{23}\Gamma_7 - j_{13}j_{21}j_{32}]\end{aligned}$$

$$B_4 = j_{44}\left[j_{23}j_{34}j_{43}\Gamma_2 + 2j_{11}j_{21}j_{34}\Gamma_8 - j_{11}j_{44}j_{12}j_{21}(j_{11} + j_{44})\right]$$
$$B_5 = -(j_{11} + j_{44})\left(j_{23}j_{11}^2 j_{34}j_{42} + j_{44}^2\Gamma_6\right)$$
$$B_6 = j_{21}\left(j_{34}j_{42}j_{13}\left(j_{11}^2 + j_{44}^2\right) + j_{13}j_{31}j_{12}j_{11}j_{44}\right)$$

with:

$$\Gamma_1 = j_{34}j_{43} + j_{23}j_{32} < 0$$
$$\Gamma_2 = j_{32}j_{44} - j_{34}j_{42} = z_6\left(z_6(d_1 n_1 + n_2) - d_1(d_2 + l)\right)$$
$$\Gamma_3 = j_{23}j_{31} - j_{21}j_{44} = -y_6\left[a_2 b_2 z_6 + a_1\left(n_1 z_6 - (d_2 + l)\right)\right]$$
$$\Gamma_4 = j_{21}j_{34}j_{43} + j_{23}j_{31}j_{44} = -y_6 z_6\left[a_1(n_1 w_6 + n_2 y_6) + a_2 b_2\left(n_1 z_6 - (d_2 + l)\right)\right]$$
$$\Gamma_5 = j_{11} + j_{44} = 1 - 2x_6 - (1 + a_1)y_6 - z_6 + n_1 z_6 - (d_2 + l)$$
$$\Gamma_6 = j_{13}j_{21}j_{32} + j_{31}j_{12}j_{23} = x_6 y_6 z_6\left(a_2 b_2(1 + a_1) - a_1 d_1\right)$$
$$\Gamma_7 = j_{11}j_{32} - j_{12}j_{31} = z_6\left(d_1\left(1 - 2x_6 - (1 + a_1)y_6 - z_6\right) + b_2(1 + a_1)x_6\right)$$
$$\Gamma_8 = j_{13}j_{42} - j_{12}j_{43} = x_6\left((1 + a_1)(n_1 w_6 + n_2 y_6) - n_2 z_6\right)$$

Now, according to existence conditions (7c) we have two cases:

Case 1: if the second part of condition (7c) holds then we have $C_i > 0, i = 1,3,4$ and $\Delta_2 > 0$ if and only if the following conditions hold:

$$1 < 2x_6 + (1 + a_1)y_6 + z_6 \tag{15c}$$
$$\frac{a_2 b_2 z_6}{a_1} < d_2 + l - n_1 z_6 < \frac{a_1(n_1 w_6 + n_2 y_6)}{a_2 b_2} \tag{15d}$$
$$n_1 w_6 + n_2 y_6 < \frac{n_2 z_6}{1 + a_1} \tag{15e}$$
$$\frac{b_2(1 + a_1)x_6}{2x_6 + (1 + a_1)y_6 + z_6 - 1} < d_1 \le \frac{a_2 b_2(1 + a_1)}{a_1} \tag{15f}$$

So, according to Routh-Hawirtiz criterion the equilibrium point E_6 is locally asymptotically stable.

Case 2: if the first part of the condition (7c) holds then we have $C_i > 0, i = 1,3,4$ and $\Delta_2 > 0$ if and only if in addition to the conditions (15c)-(15f) hold the following condition should be satisfy.

$$\Gamma_2 < \min\{H_1, H_2, H_3, H_4\} \tag{15g}$$

here $H_1 = \dfrac{j_{11}\Gamma_1 - j_{12}\Gamma_3 + j_{13}(j_{31}j_{44} - j_{21}j_{32})}{-j_{23}}$, $H_2 = \dfrac{-j_{12}\Gamma_4}{(j_{13}j_{21} - j_{11}j_{23})}$,

$H_3 = \dfrac{-j_{12}j_{21}j_{44}}{j_{23}}$, $H_4 = \dfrac{j_{11}j_{21}(j_{44}j_{12}\Gamma_5 - 2j_{34}\Gamma_8)}{j_{23}j_{34}j_{43}}$

So, according to Routh-Hawirtiz criterion the equilibrium point E_6 is locally asymptotically stable.

5. Global stability analysis of system (2):

In this section the global stability for the equilibrium points of system (2) is investigated by using the Lyapunov method as shown in the following theorems.

***Theorem* (2)**: Assume that the equilibrium point E_1 of system (2) is locally asymptotically stable in the R_+^4, and the following conditions are satisfied:

$$b_2 < \frac{n_1 d_1 + n_2}{n_1 a_2} < d_2 \tag{16a}$$

$$b_1 > 1 + a_1 \tag{16b}$$

Then E_1 is globally asymptotically stable in the R_+^4.

Proof: Consider the following function:

$$V_1 = c_1(x - 1 - \ln x) + c_2 y + c_3 z + c_4 w$$

Where $c_i;\ i = 1,2,3,4$ are positive constants to be determined. Clearly $V_1 : R_+^4 \to R$ is C^1 positive definite function. Now, by calculating the derivative of V_1 and choosing the positive constants as below:

$$c_1 = c_2 = \frac{n_1 d_1 + n_2}{a_2};\quad c_3 = n_1;\quad c_4 = 1$$

We obtain that:

$$\dot{V}_1 \le -\frac{n_1 d_1 + n_2}{a_2}(x-1)^2 + \frac{n_1 d_1 + n_2}{a_2}(1 + a_1 - b_1)y$$
$$+ \frac{n_1 d_1 + n_2 - n_1 a_2 d_2}{a_2} z - \frac{n_1 d_1 + n_2 - n_1 a_2 b_2}{a_2} xz$$

According to conditions (16a)-(16b) we get $\dot{V}_1 < 0$. Therefore E_1 is globally asymptotically stable in the R_+^4. And hence the proof is complete. ■

***Theorem* (3)**: Assume that the equilibrium point E_2 of system (2) is locally asymptotically stable in the R_+^4, and the following conditions are satisfied:

$$\frac{n_1 d_2 a_2}{n_1 d_1 + n_2} > a_2 y_2 + \frac{a_1 x_2}{1 + a_1} \tag{17a}$$

$$n_1 b_2 < 1 \tag{17b}$$

Then E_2 is globally asymptotically stable in the R_+^4.

Proof: Consider the following function:

$$V_2 = e_1\left(x - x_2 - x_2 \ln \frac{x}{x_2}\right) + e_2\left(y - y_2 - y_2 \ln \frac{y}{y_2}\right) + e_3 z + e_4 w$$

Where $e_i;\ i = 1,2,3,4$ are positive constants to be determined. Clearly $V_2 : R_+^4 \to R$ is C^1 positive definite function. Now, by calculating the derivative of V_2 and choosing the positive constants as below:

$$e_1 = \frac{a_1(n_1 d_1 + n_2)}{a_2(1 + a_1)};\quad e_2 = \frac{n_1 d_1 + n_2}{a_2};\quad e_3 = n_1;\quad e_4 = 1$$

We obtain that:

$$\dot{V}_2 \le -\frac{a_1(n_1 d_1 + n_2)}{a_2(1 + a_1)}(x - x_2)^2 + \left(\frac{n_1 d_1 + n_2}{a_2}\left(\frac{a_1 x_2}{1 + a_1} + a_2 y_2\right) - n_1 d_2\right) z$$
$$+ (n_1 b_2 - 1) xz$$

According to conditions (17a)-(17b) we get $\dot{V}_2 < 0$. Therefore E_2 is globally asymptotically stable in the R_+^4. And hence the proof is complete. ■

***Theorem* (4)**: Assume that the equilibrium point E_3 of system (2) is locally asymptotically stable in the R_+^4, and the following conditions are satisfied:

$$n_1 z_3 < d_2 + l \tag{18a}$$

$$\frac{a_1(n_1 d_1 + n_2)}{n_1 a_2} < b_2(1 + a_1) < \frac{d_1 z_3}{x_3} \tag{18b}$$

Then E_3 is globally asymptotically stable in the R_+^4.

Proof: Consider the following function:

$$V_3 = h_1\left(x - x_3 - x_3 \ln\frac{x}{x_3}\right) + h_2 y + h_3\left(z - z_3 - z_3 \ln\frac{z}{z_3}\right) + h_4 w$$

Where $h_i;\ i = 1,2,3,4$ are positive constants to be determined. Clearly $V_3 : R_+^4 \to R$ is C^1 positive definite function. Now, by calculating the derivative of V_3 and choosing the positive constants as below:

$$h_1 = n_1 b_2;\quad h_2 = \frac{n_1 d_1 + n_2}{a_2};\quad h_3 = n_1;\quad h_4 = 1$$

We obtain that:

$$\dot{V}_3 = -n_1 b_2 (x - x_3)^2 + \left(n_1\left(b_2 x_3 (1 + a_1) - d_1 z_3\right) - \frac{n_1 d_1 + n_2}{a_2} b_1\right) y$$
$$+ \left(n_1 z_3 - (d_2 + l)\right) w + \left(a_1 \frac{n_1 d_1 + n_2}{a_2} - n_1 b_2 (1 + a_1)\right) xy$$

According to conditions (18a)-(18b) we get $\dot{V}_3 < 0$. Therefore E_3 is globally asymptotically stable in the R_+^4. And hence the proof is complete. ■

***Theorem* (5)**: Assume that the equilibrium point E_4 of system (2) is locally asymptotically stable in the R_+^4, and the following condition is satisfied:

$$\frac{a_1(n_1 d_1 + n_2)}{n_1 a_2} < b_2(1 + a_1) < \frac{d_1 z_4}{x_4} \tag{19}$$

Then E_4 is globally asymptotically stable in the R_+^4.

Proof: Consider the following function:

$$V_4 = k_1\left(x - x_4 - x_4 \ln\frac{x}{x_4}\right) + k_2 y + k_3\left(z - z_4 - z_4 \ln\frac{z}{z_4}\right)$$
$$+ k_4\left(w - w_4 - w_4 \ln\frac{w}{w_4}\right)$$

Where $k_i;\ i = 1,2,3,4$ are positive constants to be determined. Clearly $V_4 : R_+^4 \to R$ is C^1 positive definite function. Now, by calculating the derivative of V_4 and choosing the positive constants as below:

$$k_1 = n_1 b_2;\quad k_2 = \frac{n_1 d_1 + n_2}{a_2};\quad k_3 = n_1;\quad k_4 = 1$$

We obtain that:

$$\dot{V}_4 \le -n_1 b_2 (x - x_4)^2 + \left(n_1\left(b_2 x_4 (1 + a_1) - d_1 z_4\right) - \frac{n_1 d_1 + n_2}{a_2} b_1\right) y$$
$$+ \left(a_1 \frac{n_1 d_1 + n_2}{a_2} - n_1 b_2 (1 + a_1)\right) xy$$

According to condition (19) we get $\dot{V}_4 < 0$. Therefore E_4 is globally asymptotically stable in the R_+^4. And hence the proof is complete. ■

***Theorem* (6)**: Assume that the equilibrium point E_5 of system (2) is locally asymptotically stable in the R_+^4, and the following conditions are satisfied:

$$a_1 d_1 - a_2 b_2 (1 + a_1) = 0 \tag{20a}$$

$$n_1 z_5 < d_2 + l \tag{20b}$$

Then E_5 is globally asymptotically stable in the R_+^4.

Proof: Consider the following function:

$$V_5 = m_1\left(x - x_5 - x_5 \ln\frac{x}{x_5}\right) + m_2\left(y - y_5 - y_5 \ln\frac{y}{y_5}\right)$$
$$+ m_3\left(z - z_5 - z_5 \ln\frac{z}{z_5}\right) + m_4 w$$

Where $m_i;\ i = 1,2,3,4$ are positive constants to be determined. Clearly $V_5 : R_+^4 \to R$ is C^1 positive definite function. Now, by calculating the derivative of V_5 and choosing the positive constants as below:

$$m_1 = \frac{a_2 b_2}{d_1}; \quad m_2 = 1; \quad m_3 = \frac{a_2}{d_1}; \quad m_4 = \frac{a_2}{n_1 d_1}$$

We obtain that:

$$\dot{V}_5 = -\frac{a_2 b_2}{d_1}(x - x_5)^2 + \left(\frac{a_1 d_1 - a_2 b_2 (1 + a_1)}{d_1}\right)(x - x_5)(y - y_5)$$
$$+ \frac{a_2}{d_1}\left(z_5 - \frac{d_2 + l}{n_1}\right) w$$

According to conditions (20a)-(20b) we get $\dot{V}_5 < 0$. Therefore E_5 is globally asymptotically stable in the R_+^4. And hence the proof is complete. ■

***Theorem* (7)**: Assume that the equilibrium point E_6 of system (2) is locally asymptotically stable in the R_+^4, and the following conditions are satisfied:

$$a_1 d_1 - a_2 b_2 (1 + a_1) = 0 \tag{21a}$$

$$(x - x_6)^2 > \frac{n_2}{n_1 b_2}(yz + y_6 z_6) \tag{21b}$$

Then E_6 is globally asymptotically stable in the sub region of R_+^4 that satisfy the above conditions.

Proof: Consider the following function:

$$V_6 = r_1\left(x - x_6 - x_6 \ln\frac{x}{x_6}\right) + r_2\left(y - y_6 - y_6 \ln\frac{y}{y_6}\right)$$
$$+ r_3\left(z - z_6 - z_6 \ln\frac{z}{z_6}\right) + r_4\left(w - w_6 - w_6 \ln\frac{w}{w_6}\right)$$

Where $r_i;\ i = 1,2,3,4$ are positive constants to be determined. Clearly $V_6 : R_+^4 \to R$ is C^1 positive definite function. Now , by calculating the derivative of V_6 and choosing the positive constants as below:

$$r_1 = 1; \quad r_2 = \frac{d_1}{a_2 b_2}; \quad r_3 = \frac{1}{b_2}; \quad r_4 = \frac{1}{n_1 b_2}$$

We obtain that:

$$\dot{V}_6 \le -(x-x_6)^2 + \left(\frac{a_1 d_1 - a_2 b_2 (1+a_1)}{a_2 b_2}\right)(x-x_6)(y-y_6)$$
$$+\frac{n_2}{n_1 b_2}(yz + y_6 z_6)$$

According to conditions (21a)-(21b) we get $\dot{V}_6 < 0$. Therefore E_6 is globally asymptotically stable in the sub region of R_+^4 that satisfy the above conditions. And hence the proof is complete. ■

6. The local bifurcation analysis of system (2):

In this section, the effect of varying the parameter values on the dynamical behavior of the system (2) around each equilibrium points is studied. Recall that the existence of nonhyperbolic equilibrium point of system (2) is the necessary but not sufficient condition for bifurcation to occur. Therefore, in the following theorems an application to the Sotomayor's theorem [16] for local bifurcation is adapted.

Now, according to Jacobian matrix of system (2) given in equation (8), it is clear to verify that for any non-zero vector $V = (v_1, v_2, v_3, v_4)^T$ we have:

$$D^2F(V,V) = \begin{pmatrix} -2v_1(v_1 + (1+a_1)v_2 + v_3) \\ 2v_2(a_1 v_1 - a_2 v_3) \\ 2v_3(b_2 v_1 + d_1 v_2 - v_4) \\ 2v_3(n_2 v_2 + n_1 v_4) \end{pmatrix} \quad (22) \text{ and}$$

$D^3F(V,V,V) = (0,0,0,0)^T$

So, according to Sotomayor's theorem the pitchfork bifurcation does not occur at each point $E_i, i = 0,1,2,3,4,5,6$.

Note that, according to the Jacobian matrix J_0 given by Eq.(9) the system (2) at the equilibrium point E_0 has four non-zero eigenvalues. So, the Jacobian matrix J_0 has no non hyperbolic equilibrium point. Thus, the system has not bifurcation at E_0.

***Theorem* (8):** Suppose that the condition (10b) is satisfied. Then for the parameter value $b_2^* = d_2$ system (2) at the equilibrium point E_1 has

1. No saddle-node bifurcation.
2. Transcritical bifurcation.

Proof: According to the Jacobian matrix J_1 given by Eq.(10a) the system (2) at the equilibrium point E_1 has zero eigenvalue (say $\lambda_{1z} = 0$) at $b_2 = b_2^*$, and the Jacobian matrix J_1 with $b_2 = b_2^*$ becomes:

$$J_1^* = J(\lambda_{1z} = 0) = \begin{pmatrix} -1 & -(1+a_1) & -1 & 0 \\ 0 & a_1 - b_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(d_2 + l) \end{pmatrix}$$

Now, let $V^{[1]} = (v_1^{[1]}, v_2^{[1]}, v_3^{[1]}, v_4^{[1]})^T$ be the eigenvector corresponding to the eigenvalue $\lambda_{1z} = 0$. Thus $(J_1^* - \lambda_{1z} I)V^{[1]} = 0$, which gives:
$v_1^{[1]} = -v_3^{[1]}$, $v_2^{[1]} = 0$, $v_4^{[1]} = 0$ and $v_3^{[1]}$ any nonzero real number.

Let $\Psi^{[1]}=\left(\psi_1^{[1]},\psi_2^{[1]},\psi_3^{[1]},\psi_4^{[1]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{1z}=0$ of the matrix J_1^{*T}. Then we have $\left(J_1^{*T}-\lambda_{1z}I\right)\Psi^{[1]}=0$. By solving this equation for $\Psi^{[1]}$ we obtain $\Psi^{[1]}=\left(0,0,\psi_3^{[1]},0\right)^T$, where $\psi_3^{[1]}$ any nonzero real number. Now, consider:

$$\frac{\partial f}{\partial b_2}=f_{b_2}\left(X,b_2\right)=\left(\frac{\partial f_1}{\partial b_2},\frac{\partial f_2}{\partial b_2},\frac{\partial f_3}{\partial b_2},\frac{\partial f_4}{\partial b_2}\right)^T=(0,0,xz,0)^T$$

So, $f_{b_2}\left(E_1,b_2^*\right)=(0,0,0,0)^T$ and hence $\left(\Psi^{[1]}\right)^T f_{b_2}\left(E_1,b_2^*\right)=0$

So, according to Sotomayor's theorem the saddle-nod bifurcation can not occur. While the first condition of transcritical bifurcation is satisfied. Now, since

$$Df_{b_2}\left(X,b_2\right)=\begin{pmatrix}0&0&0&0\\0&0&0&0\\z&0&x&0\\0&0&0&0\end{pmatrix}$$

Where $Df_{b_2}\left(X,b_2\right)$ represents the derivative of $f_{b_2}\left(X,b_2\right)$ with respect to $X=(x,y,z,w)^T$. Further, it is observed

$$Df_{b_2}\left(E_1,b_2^*\right)V^{[1]}=\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}-v_3^{[1]}\\0\\v_3^{[1]}\\0\end{pmatrix}=\begin{pmatrix}0\\0\\v_3^{[1]}\\0\end{pmatrix}$$

$$\left(\Psi^{[1]}\right)^T\left[Df_{b_2}\left(E_1,b_2^*\right)V^{[1]}\right]=\left(0,0,\psi_3^{[1]},0\right)\left(0,0,v_3^{[1]},0\right)^T=\psi_3^{[1]}v_3^{[1]}\neq 0$$

Now, by substituting $V^{[1]}$ in (22) we get

$$D^2 f\left(E_1,b_2^*\right)\left(V^{[1]},V^{[1]}\right)=\left(0,0,-2b_2^*\left(v_3^{[1]}\right)^2,0\right)^T$$

Hence, it is obtain that:

$$\left(\Psi^{[1]}\right)^T\left[D^2 f\left(E_1,b_2^*\right)\left(V^{[1]},V^{[1]}\right)\right]=$$

$$\left(0,0,\psi_3^{[1]},0\right)\left(0,0,-2b_2^*\left(v_3^{[1]}\right)^2,0\right)^T=-2\psi_3^{[1]}b_2^*\left(v_3^{[1]}\right)^2\neq 0$$

Thus, according to Sotomayor's theorem system (2) has transcritical bifurcation at E_1 with the parameter $b_2=b_2^*$. ■

***Theorem* (9):** Suppose that the condition

$$a_2b_2(1+a_1)\left(d_2^*+l\right)\neq d_1\left(a_1+a_2\right)\left(d_2^*+l\right)+n_2a_1\left(1-x_2\right) \quad (23)$$

is satisfied. Then for the parameter value $d_2^*=b_2x_2+\dfrac{d_1\left(1-x_2\right)}{1+a_1}$ system (2) at the equilibrium point E_2 has

1. No saddle-node bifurcation.
2. Transcritical bifurcation.

Proof: According to the Jacobian matrix J_2 given by Eq.(11a) the system (2) at the equilibrium point E_2 has zero eigenvalue (say $\lambda_{2z}=0$) at $d_2=d_2^*$, and the Jacobian matrix J_2 with $d_2=d_2^*$ becomes:

$$J_2^* = J_2(\lambda_{2z}=0) = \begin{pmatrix} -x_2 & -x_2(1+a_1) & -x_2 & 0 \\ \frac{a_1(1-x_2)}{1+a_1} & 0 & \frac{-a_2(1-x_2)}{1+a_1} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{n_2(1-x_2)}{1+a_1} & -(d_2^*+l) \end{pmatrix}$$

Now, let $V^{[2]} = (v_1^{[2]}, v_2^{[2]}, v_3^{[2]}, v_4^{[2]})^T$ be the eigenvector corresponding to the eigenvalue $(\lambda_{2z}=0)$. Thus $(J_2^* - \lambda_{2z} I)V^{[2]} = 0$, which gives:

$v_1^{[2]} = \frac{a_2}{a_1} v_3^{[2]}$, $v_2^{[2]} = \frac{-(a_1+a_2)}{a_1(1+a_1)} v_3^{[2]}$, $v_4^{[2]} = \frac{n_2(1-x_2)}{(d_2^*+l)(1+a_1)} v_3^{[2]}$ and $v_3^{[2]}$ any nonzero real number.

Let $\Psi^{[2]} = (\psi_1^{[2]}, \psi_2^{[2]}, \psi_3^{[2]}, \psi_4^{[2]})^T$ be the eigenvector associated with the eigenvalue $\lambda_{2z} = 0$ of the matrix J_2^{*T}. Then we have $(J_2^{*T} - \lambda_{2_z} I)\Psi^{[2]} = 0$.

By solving this equation for $\Psi^{[2]}$ we obtain $\Psi^{[2]} = (0, 0, \psi_3^{[2]}, 0)^T$, where $\psi_3^{[2]}$ any nonzero real number. Now, by using the same steps in the previous theorem we will get

$$(\Psi^{[2]})^T f_{d_2}(E_2, d_2^*) = 0$$

So, according to Sotomayor's theorem the saddle-nod bifurcation can not occur, while the first condition of transcritical bifurcation is satisfied. Further, it is observed

$$(\Psi^{[2]})^T [Df_{d_2}(E_2, d_2^*)V^{[2]}] = (0, 0, \psi_3^{[2]}, 0)(0, 0, -v_3^{[2]}, -v_4^{[2]})^T = -\psi_3^{[2]} v_3^{[2]} \neq 0$$

Now, by substituting $V^{[2]}$ in (22) we get

$$(\Psi^{[2]})^T [D^2 f(E_2, d_2^*)(V^{[2]}, V^{[2]})] =$$
$$2\psi_3^{[2]}(v_3^{[2]})^2 \left[\frac{(a_2 b_2(1+a_1) - d_1(a_1+a_2))(d_2^*+l) - n_2 a_1(1-x_2)}{a_1(1+a_1)(d_2^*+l)} \right]$$

According to condition (23) we obtain that:

$$(\Psi^{[2]})^T [D^2 f(E_2, d_2^*)(V^{[2]}, V^{[2]})] \neq 0$$

Thus, according to Sotomayor's theorem system (2) has transcritical bifurcation at E_2 with the parameter $d_2 = d_2^*$. ■

***Theorem* (10):** Suppose that the following conditions

$$x_3(a_1+a_2) > a_2 \tag{24}$$
$$n_1(1-x_3) \neq d_2 + l \tag{25}$$
$$d_1 + \frac{n_2(1-x_3)}{n_1(1-x_3) - d_2 - l} < b_2(1+a_1) \tag{26}$$

are satisfied. Then for the parameter value $b_1^* = x_3(a_1+a_2) - a_2$ system (2) at the equilibrium point E_3 has

1. No saddle-node bifurcation.
2. Transcritical bifurcation.

Proof: According to the Jacobian matrix J_3 given by Eq.(12a) the system (2) at the equilibrium point E_3 has zero eigenvalue (say $\lambda_{3y}=0$) at $b_1=b_1^*$, it is clearly that $b_1^*>0$ provided that condition (24) holds, and the Jacobian matrix J_3 with $b_1=b_1^*$ becomes:

$$J_3^*=J_3(\lambda_{3y}=0)=\begin{pmatrix} -x_3 & -x_3(1+a_1) & -x_3 & 0 \\ 0 & 0 & 0 & 0 \\ b_2(1-x_3) & d_1(1-x_3) & b_2x_3-d_2 & x_3-1 \\ 0 & n_2(1-x_3) & 0 & n_1(1-x_3)-(d_2+l) \end{pmatrix}$$

Now, let $V^{[3]}=\left(v_1^{[3]},v_2^{[3]},v_3^{[3]},v_4^{[3]}\right)^T$ be the eigenvector corresponding to the eigenvalue $(\lambda_{3y}=0)$. Thus $\left(J_3^*-\lambda_{3y}I\right)V^{[3]}=0$, which gives:

$$v_1^{[3]}=\left[\frac{(x_3-1)}{d_2-b_2}\left(d_1-b_2(1+a_1)+\frac{n_2(1-x_3)}{n_1(1-x_3)-d_2-l}\right)-(1+a_1)\right]v_2^{[3]}$$

$$v_3^{[3]}=\frac{(1-x_3)}{d_2-b_2}\left(d_1-b_2(1+a_1)+\frac{n_2(1-x_3)}{n_1(1-x_3)-d_2-l}\right)v_2^{[3]}$$

$$v_4^{[3]}=\frac{n_2(x_3-1)}{n_1(1-x_3)-d_2-l}v_2^{[3]}$$

and $v_2^{[3]}$ any nonzero real number. Here we have $v_1^{[3]},v_3^{[3]}$ and $v_4^{[3]}$ are defined under the existence condition (4b) and (25). While $v_1^{[3]}$ and $v_3^{[3]}$ not equal zero under the condition (26).

Let $\Psi^{[3]}=\left(\psi_1^{[3]},\psi_2^{[3]},\psi_3^{[3]},\psi_4^{[3]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{3y}=0$, of the matrix J_3^{*T}. Then we have $\left(J_3^{*T}-\lambda_{3y}I\right)\Psi^{[3]}=0$. By solving this equation for $\Psi^{[3]}$ we obtain $\Psi^{[3]}=\left(0,\psi_2^{[3]},0,0\right)^T$, where $\psi_2^{[3]}$ any nonzero real number. Now, by using the same steps in the previous theorems we will get $\left(\Psi^{[3]}\right)^T f_{b_1}\left(E_3,b_1^*\right)=0$. So, according to Sotomayor's theorem the saddle-nod bifurcation can not occur. While the first condition of transcritical bifurcation is satisfied. Further, it is observed

$$\left(\Psi^{[3]}\right)^T\left[Df_{b_1}\left(E_3,b_1^*\right)V^{[3]}\right]=\left(0,\psi_2^{[3]},0,0\right)\left(0,-v_2^{[3]},0,0\right)^T=-\psi_2^{[3]}v_2^{[3]}\neq 0$$

Now, by substituting $V^{[3]}$ in (22) we get

$$\left(\Psi^{[3]}\right)^T\left[D^2f\left(E_3,b_1^*\right)\left(V^{[3]},V^{[3]}\right)\right]=-2\psi_2^{[3]}\left(v_2^{[3]}\right)^2R$$

where:

$$R=a_1(1+a_1)$$
$$+(a_1+a_2)\frac{1-x_3}{d_2-b_2}\left(d_1-b_2(1+a_1)+\frac{n_2(1-x_3)}{n_1(1-x_3)-d_2-l}\right)$$

According to condition (26) with existence condition (4b) we obtain that:

$$\left(\Psi^{[3]}\right)^T\left[D^2f\left(E_3,b_1^*\right)\left(V^{[3]},V^{[3]}\right)\right]\neq 0$$

Thus, according to Sotomayor's theorem system (2) has transcritical bifurcation at E_3 with the parameter $b_1=b_1^*$. ■

***Theorem* (11):** Suppose that the following conditions

$$a_1 > z_4(a_1 + a_2) \tag{27}$$

$$d_1 > b_2(1 + a_1) \tag{28}$$

$$n_2 z_4 \neq n_1 w_4 \tag{29}$$

$$n_2 z_4(a_1 + a_2) \neq n_1 w_4(1 + a_1) \tag{30}$$

are satisfied. Then for the parameter value $\hat{b}_1 = a_1 - z_4(a_1 + a_2)$ system (2) at the equilibrium point E_4 has
1. No saddle-node bifurcation.
2. Transcritical bifurcation.

Proof: According to the Jacobian matrix J_4 given by Eq.(13a) the system (2) at the equilibrium point E_4 has zero eigenvalue (say $\lambda_{4y} = 0$) at $b_1 = \hat{b}_1$, it is clearly that $\hat{b}_1 > 0$ provided that condition (27) holds, and the Jacobian matrix J_4 with $b_1 = \hat{b}_1$ becomes:

$$J_4^* = J_4(\lambda_{4y} = 0) = \begin{pmatrix} -x_4 & -x_4(1+a_1) & -x_4 & 0 \\ 0 & 0 & 0 & 0 \\ b_2 z_4 & d_1 z_4 & 0 & -z_4 \\ 0 & n_2 z_4 & n_1 w_4 & 0 \end{pmatrix}$$

Now, let $V^{[4]} = \left(v_1^{[4]}, v_2^{[4]}, v_3^{[4]}, v_4^{[4]}\right)^T$ be the eigenvector corresponding to the eigenvalue $(\lambda_{4y} = 0)$. Thus $\left(J_4^* - \lambda_{4y} I\right)V^{[4]} = 0$, which gives:

$$v_1^{[4]} = \frac{n_2 z_4 - n_1 w_4}{n_1 w_4} v_2^{[4]}, \quad v_3^{[4]} = \frac{-n_2 z_4}{n_1 w_4} v_2^{[4]}, v_4^{[4]} = \frac{n_1 w_4\left(d_1 - b_2(1 + a_1)\right) + b_2 n_2 z_4}{n_1 w_4} v_2^{[4]}$$

and $v_2^{[4]}$ any nonzero real number. Here we have $v_1^{[4]}$ and $v_4^{[4]}$ not equal zero under the condition (28) and (29). Let $\Psi^{[4]} = \left(\psi_1^{[4]}, \psi_2^{[4]}, \psi_3^{[4]}, \psi_4^{[4]}\right)$ be the eigenvector associated with the eigenvalue $\lambda_{4y} = 0$ of the matrix J_4^{*T}. Then we have $\left(J_4^{*T} - \lambda_{4y} I\right)\Psi^{[4]} = 0$. By solving this equation for $\Psi^{[4]}$ we obtain $\Psi^{[4]} = \left(0, \psi_2^{[4]}, 0, 0\right)^T$, where $\psi_2^{[4]}$ any nonzero real number. Now, by using the same steps in the previous theorems we will get $\left(\Psi^{[4]}\right)^T f_{b_1}\left(E_4, \hat{b}_1\right) = 0$. So, according to Sotomayor's theorem the saddle-nod bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Further, it is observed

$$\left(\Psi^{[4]}\right)^T \left[Df_{b_1}\left(E_4, \hat{b}_1\right)V^{[4]}\right] = \left(0, \psi_2^{[4]}, 0, 0\right)\left(0, -v_2^{[4]}, 0, 0\right) = -\psi_2^{[4]} v_2^{[4]} \neq 0$$

Now, by substituting $V^{[4]}$ in (22) we get

$$\left(\Psi^{[4]}\right)^T \left[D^2 f\left(E_4, \hat{b}_1\right)\left(V^{[4]}, V^{[4]}\right)\right] = 2\psi_2^{[4]}\left(v_2^{[4]}\right)^2 \left[\frac{n_2 z_4(a_1 + a_2) - n_1 w_4(1 + a_1)}{n_1 w_4}\right]$$

According to condition (30) we obtain that:

$$\left(\Psi^{[4]}\right)^T \left[D^2 f\left(E_4, \hat{b}_1\right)\left(V^{[4]}, V^{[4]}\right)\right] \neq 0$$

Thus, according to Sotomayor's theorem system (2) has transcritical bifurcation at E_4 with the parameter $b_1 = \hat{b}_1$. ■

***Theorem* (12):** For the parameter value $n_1^* = \dfrac{d_2 + l}{z_5}$ system (2) at the equilibrium point E_5 has

1. No saddle-node bifurcation.

2. Transcritical bifurcation.

Proof: According to the Jacobian matrix J_5 given by Eq.(14a) the system (2) at the equilibrium point E_5 has zero eigenvalue (say $\lambda_{5w}=0$) at $n_1=n_1^*$, and the Jacobian matrix J_5 with $n_1=n_1^*$ becomes:

$$J_5^*=J_5(\lambda_{5w}=0)=\begin{pmatrix} -x_4 & -x_4(1+a_1) & -x_4 & 0 \\ a_1y_5 & 0 & -a_2y_5 & 0 \\ b_2z_5 & d_1z_5 & 0 & -z_5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Let $V^{[5]}=\left(v_1^{[5]},v_2^{[5]},v_3^{[5]},v_4^{[5]}\right)^T$ be the eigenvector corresponding to the eigenvalue $(\lambda_{5w}=0)$. Thus $\left(J_5^*-\lambda_{5w}I\right)V^{[5]}=0$, which gives:

$$v_1^{[5]}=\frac{a_2}{a_1}v_3^{[5]},\quad v_2^{[5]}=\frac{-(a_1+a_2)}{a_1(1+a_1)}v_3^{[5]},v_4^{[5]}=\frac{b_2a_2(1+a_1)-d_1(a_1+a_2)}{a_1(1+a_1)}v_3^{[5]}$$

and $v_3^{[5]}$ any nonzero real number. It is clear that v_4^5 not equals zero under the existence condition (6b) or (6c). Let $\Psi^{[5]}=\left(\psi_1^{[5]},\psi_2^{[5]},\psi_3^{[5]},\psi_4^{[5]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_{5w}=0$ of the matrix J_5^{*T}. Then we have $\left(J_5^{*T}-\lambda_{5_w}I\right)\Psi^{[5]}=0$. By solving this equation for $\Psi^{[5]}$ we obtain $\Psi^{[5]}=\left(0,0,0,\psi_4^{[5]}\right)^T$, where $\psi_4^{[5]}$ any nonzero real number. Now, by using the same steps in the previous theorems we will get $\left(\Psi^{[5]}\right)^T f_{n_1}\left(E_5,n_1^*\right)=0$. So, according to Sotomayor's theorem the saddle-nod bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Further, it is observed

$$\left(\Psi^{[5]}\right)^T\left[Df_{n_1}\left(E_5,n_1^*\right)V^{[5]}\right]=\left(0,0,0,\psi_4^{[5]}\right)\left(0,0,0,z_5v_4^{[5]}\right)^T=z_5\psi_4^{[5]}v_4^{[5]}\neq 0$$

Now, by substituting $V^{[5]}$ in (22) we get

$$\left(\Psi^{[5]}\right)^T\left[D^2f\left(E_5,n_1^*\right)\left(V^{[5]},V^{[5]}\right)\right]=2n_1^*\psi_4^{[5]}\left(v_3^{[5]}\right)^2\left[\frac{b_2a_2(1+a_1)-(a_1+a_2)d_1}{a_1(1+a_1)}\right]$$

According to existence condition (6b) or (6c) we obtain that:

$$\left(\Psi^{[5]}\right)^T\left[D^2f\left(E_5,n_1^*\right)\left(V^{[5]},V^{[5]}\right)\right]\neq 0$$

Thus, according to Sotomayor's theorem system (2) has transcritical bifurcation at E_5 with the parameter $n_1=n_1^*$. ■

***Theorem* (13):** Suppose that the condition (15c) with the following conditions are satisfied

$$\varsigma_{42}S_1\neq\varsigma_{12}\varsigma_{21}\varsigma_{43} \tag{31}$$

$$\frac{v_1^{[6]}\Gamma_2\left(S_2-S_1(1+a_1)\right)}{\varsigma_{12}}\neq\frac{v_3^{[6]}\varsigma_{34}\left(n_1\left(\varsigma_{42}S_1-\varsigma_{12}\varsigma_{21}\varsigma_{43}\right)-n_2\varsigma_{44}S_1\right)}{\varsigma_{44}} \tag{32}$$

$$\frac{\Gamma_2S_1}{j_{12}y_6z_6}<a_1j_{43} \tag{33}$$

Or the conditions (15c) and (31)-(32) with the following conditions are satisfied

$$\varsigma_{11}\Gamma_2\neq\varsigma_{12}\varsigma_{31}\varsigma_{44} \tag{34}$$

then for the parameter value $\hat{b}_2=\frac{1}{a_2j_{44}}\left[\frac{\Gamma_2S_1}{j_{12}y_6z_6}-a_1j_{43}\right]$ system (2) at the equilibrium point E_6 has

1. No saddle-node bifurcation.
2. Transcritical bifurcation.

with, $S_1 = j_{13}j_{21} - j_{11}j_{23}$, $S_2 = \varsigma_{12}\varsigma_{21} - \varsigma_{23}\varsigma_{12}$

Proof: The characteristic equation given by Eq.(15b) having zero eigenvalue (say $\lambda_6 = 0$) if and only if $C_4 = 0$ and then E_6 becomes a nonhyperbolic equilibrium point. Clearly the Jacobian matrix of system (2) at the equilibrium point E_6 with parameter $b_2 = \hat{b}_2$ becomes

$$J_6^* = J_6(\lambda_6 = 0) = \left[\varsigma_{ij}\right]_{4\times 4}$$

where $\varsigma_{ij} = j_{ij}$ for all $i, j = 1,2,3,4$ except ς_{31} which is given by: $\varsigma_{31} = \hat{b}_2 z_6$. Not that, if the second part of existence condition (7c) holds then we have $\hat{b}_2 > 0$ if the condition (15c) holds. While, if the first part of the condition (7c) holds then we have $\hat{b}_2 > 0$ if and only if in addition to the condition (15c) holds and the condition (33) should be satisfy. Let $V^{[6]} = \left(v_1^{[6]}, v_2^{[6]}, v_3^{[6]}, v_4^{[6]}\right)^T$ be the eigenvector corresponding to the eigenvalue $\lambda_6 = 0$. Thus $\left(J_6^* - \lambda_6 I\right)V^{[6]} = 0$, which gives:

$$v_1^{[6]} = -\frac{\varsigma_{23}}{\varsigma_{21}}v_3^{[6]},\ v_2^{[6]} = \frac{-S_1}{\varsigma_{12}\varsigma_{21}}v_3^{[6]},\ v_4^{[6]} = \frac{\varsigma_{42}S_1 - \varsigma_{12}\varsigma_{21}\varsigma_{43}}{\varsigma_{12}\varsigma_{21}\varsigma_{44}}v_3^{[6]}$$

and $v_3^{[6]}$ any nonzero real number. Clearly, according to conditions (15c) and (31), we have $v_2^{[6]}$ and $v_4^{[6]}$ not equal zero. Let $\Psi^{[6]} = \left(\psi_1^{[6]}, \psi_2^{[6]}, \psi_3^{[6]}, \psi_4^{[6]}\right)^T$ be the eigenvector associated with the eigenvalue $\lambda_6 = 0$ of the matrix J_6^{*T}. Then we have $\left(J_6^{*T} - \lambda_6 I\right)\Psi^{[6]} = 0$. Which gives :

$$\psi_1^{[6]} = \frac{-\Gamma_2}{\varsigma_{12}\varsigma_{44}}\psi_3^{[6]},\ \psi_2^{[6]} = \frac{\varsigma_{11}\Gamma_2 - \varsigma_{12}\varsigma_{31}\varsigma_{44}}{\varsigma_{12}\varsigma_{21}\varsigma_{44}}\psi_3^{[6]},\ \psi_4^{[6]} = -\frac{\varsigma_{34}}{\varsigma_{44}}\psi_3^{[6]}$$

and $\psi_3^{[6]}$ any nonzero real number. Not that, if the first part of existence condition (7c) holds then we have $\psi_2^{[6]} \neq 0$ if the condition (15c) holds. While, if the second part of the condition (7c) holds then we have $\psi_2^{[6]} \neq 0$ if and only if in addition to the condition (15c) holds and the condition (34) should be satisfy. Now, by using the same steps in the previous theorems we will get

$$\left(\Psi^{[6]}\right)^T f_{b_2}\left(E_6, \hat{b}_2\right) = x_6 z_6 \psi_3^{[6]} \neq 0$$

So, according to Sotomayor's theorem the saddle-nod bifurcation cannot occur. While the first condition of transcritical bifurcation is satisfied. Now, by substituting $V^{[6]}$ in (22) we get

$$\left(\Psi^{[6]}\right)^T\left[D^2 f\left(E_6, \hat{b}_2\right)\left(V^{[6]}, V^{[6]}\right)\right] =$$
$$\frac{2v_3^{[6]}\psi_3^{[6]}}{\varsigma_{21}\varsigma_{12}\varsigma_{44}}\left(\frac{v_1^{[6]}\Gamma_2\left(S_2 - S_1(1+a_1)\right)}{\varsigma_{12}} - \frac{v_3^{[6]}\varsigma_{34}\left(n_1\left(\varsigma_{42}S_1 - \varsigma_{12}\varsigma_{21}\varsigma_{43}\right) - n_2\varsigma_{44}S_1\right)}{\varsigma_{44}}\right)$$

So, according to condition (32) we obtain that:

$$\left(\Psi^{[6]}\right)^T\left[D^2 f\left(E_6, \hat{b}_2\right)\left(V^{[6]}, V^{[6]}\right)\right] \neq 0$$

Thus, according to Sotomayor's theorem system (2) has transcritical bifurcation at E_6 with the parameter $b_2 = \hat{b}_2$. ■

7. The Hopf bifurcation analysis of system (2):

In this section, the existence of periodic dynamic in system (2) due to changing the value of one parameter is studied.

The Hopf bifurcation analysis near E_0 and E_1:

According to the Jacobian matrix of system (2) at E_0 and E_1, it is clear that J_0 given by (9) and J_1 given by (10a) has always four real eigenvalues. So, the necessary and sufficient conditions for a Hopf bifurcation to occur are not satisfied.

The Hopf bifurcation analysis near E_2 and E_3:

From the characteristic equations of J_2 and J_3, it is observed that the eigenvalues are given respectively as in Eq.(11b) and (12b). Clearly $\mathrm{Re}(\lambda_{2x,y}) \neq 0$ and $\mathrm{Re}(\lambda_{3z,w}) \neq 0$. So, there is no possibility for Hopf bifurcation to occur.

The Hopf bifurcation analysis near E_4 :

From the characteristic equations of J_4, which given by Eq.(13b) we get:

$$\Delta = L_1 L_2 - L_3 = a_{11}a_{13}a_{31} = b_2 z_4 x_4^2 = 0 \text{ if and only if } b_2 = 0.$$

But in our assumptions $b_2 > 0$. So, there is no possibility for having complex eigenvalues, then the Hopf bifurcation can not occur.

The Hopf bifurcation analysis near E_5 and E_6 : The possibility of Hopf bifurcation to occur is discussed in the following theorems.

***Theorem* (14):** Suppose that the condition (14c) with the following conditions are satisfied:

$$a_1 d_1 y_5 z_5 > x_5\left(a_1(1+a_1)y_5 + b_2 z_5\right) \tag{35}$$

$$b_2(2+a_1) \neq d_1 \tag{36}$$

Then for the parameter value $a_2^* = \dfrac{a_1 d_1 y_5 z_5 - x_5\left(a_1(1+a_1)y_5 + b_2 z_5\right)}{y_5 z_5 b_2 (1+a_1)}$ the system (2) has a Hopf bifurcation near the point E_5.

Proof: Consider the characteristic equation of the system (2) at E_5 which is given by Eq.(14b). Now, to verify the necessary and sufficient conditions for a Hopf bifurcation to occur we need to find a parameter satisfy $\Delta = 0$. Therefore it is observed that $\Delta = 0$ gives :

$$a_2^* = \frac{a_1 d_1 y_5 z_5 - x_5\left(a_1(1+a_1)y_5 + b_2 z_5\right)}{y_5 z_5 b_2 (1+a_1)}$$

Clearly, $a_2^* > 0$ provided that the condition (35) holds. The coefficients of the characteristic equation given by Eq.(14b) can be rewritten as

$$D_1(a_2^*) = x_5 > 0$$
$$D_2(a_2^*) = -\left(-a_2^* y_5 b_{32} + b_{12}b_{21} + b_{13}b_{31}\right) > 0$$
$$D_3(a_2^*) = -a_2^* y_5 (b_{11}b_{32} - b_{12}b_{31}) - b_{21}b_{32}b_1$$

Hence, for $a_2 = a_2^*$ the characteristic equation given by Eq.(14b) becomes:

$$(b_{44} - \lambda_{5w})(\lambda_5 + D_1)(\lambda_5^2 + D_2) = 0$$

which has four roots $\lambda_{5w} = b_{44}$, $\lambda_{51} = -D_1$, $\lambda_{52} = i\sqrt{D_2}$ and $\lambda_{53} = -i\sqrt{D_2}$

Clearly, at $a_2 = a_2^*$ there are two pure imaginary eigenvalues (λ_{52} and λ_{53}) and two real and negative eigenvalues provided that the condition (14c) holds. Now for all values of a_2 in the neighborhood of a_2^*, the roots in general of the following form:

$$\lambda_{5w}(a_2) = b_{44},\ \lambda_{51}(a_2) = -D_1(a_2),\ \lambda_{52} = \omega_1(a_2) + i\omega_2(a_2),\ \lambda_{53}(a_2) = \omega_1(a_2) - i\omega_2(a_2)$$

Clearly, $\operatorname{Re}(\lambda_{5j}(a_2))\big|_{a_2=a_2^*} = \omega_1(a_2^*) = 0,\ \ j = 2,3$ that means the first condition of the necessary and sufficient conditions for Hopf bifurcation is satisfied at $a_2 = a_2^*$. Now by substituting $\lambda_{52} = \omega_1(a_2) + i\omega_2(a_2)$ and $\lambda_{53}(a_2) = \omega_1(a_2) - i\omega_2(a_2)$ in the equation $(\lambda_5 + D_1)(\lambda_5^2 + D_2) = 0$ and calculating these derivative with respect to the parameter a_2, and then comparing the two sides of this equation by equating their real and imaginary parts, it is obtain that:

$$\begin{aligned}
&\Psi(a_2)\omega_1'(a_2) - \Phi(a_2)\omega_2'(a_2) + \Theta(a_2) = 0\\
&\Phi(a_2)\omega_1'(a_2) + \Psi(a_2)\omega_2'(a_2) + \Gamma(a_2) = 0
\end{aligned} \tag{37}$$

here

$$\begin{aligned}
\Psi(a_2) &= 3(\omega_1(a_2))^2 + 2D_1(a_2)\omega_1(a_2) + D_2(a_2) - 3(\omega_2(a_2))^2\\
\Phi(a_2) &= 6\omega_1(a_2)\omega_2(a_2) + 2D_1(a_2)\omega_2(a_2)\\
\Theta(a_2) &= (\omega_1(a_2))^2 D_1'(a_2) + D_2'(a_2)\omega_1(a_2) + D_3'(a_2) - D_1'(a_2)(\omega_2(a_2))^2\\
\Gamma(a_2) &= 2\omega_1(a_2)\omega_2(a_2)D_1'(a_2) + D_2'(a_2)\omega_2(a_2)
\end{aligned} \tag{38}$$

Solving the linear system (37) by using Cramer's rule for the unknowns $\omega_1'(a_2)$ and $\omega_2'(a_2)$, gives that

$$\omega_1'(a_2) = -\frac{\Theta(a_2)\Psi(a_2) + \Gamma(a_2)\Phi(a_2)}{(\Psi(a_2))^2 + (\Phi(a_2))^2};\ \omega_2'(a_2) = \frac{-\Gamma(a_2)\Psi(a_2) + \Theta(a_2)\Phi(a_2)}{(\Psi(a_2))^2 + (\Phi(a_2))^2}$$

Therefore the second necessary and sufficient condition of Hopf bifurcation

$$\frac{d}{da_2}\left(\operatorname{Re}(\lambda_{5j})\right)\Big|_{a_2=a_2^*} = \omega_1'(a_2)\big|_{a_2=a_2^*} \neq 0,\quad j = 2,3$$

will be satisfied if and only if

$$\Theta(a_2^*)\Psi(a_2^*) + \Gamma(a_2^*)\Phi(a_2^*) \neq 0 \tag{39}$$

Note that for $a_2 = a_2^*$ we have $\omega_1 = 0$ and $\omega_2 = \sqrt{D_2(a_2^*)}$, substitution into (38) gives the following simplifications:

$$\begin{aligned}
&\Psi(a_2^*) = -2D_2(a_2^*),\ \Phi(a_2^*) = 2D_1(a_2^*)\sqrt{D_2(a_2^*)}\\
&\Theta(a_2^*) = D_3'(a_2^*) - D_1'(a_2^*)D_2(a_2^*),\ \Gamma(a_2^*) = D_2'(a_2^*)\sqrt{D_2(a_2^*)}
\end{aligned}$$

where:

$$D_1' = \frac{dD_1}{da_2}\Big|_{a_2=a_2^*} = 0,\ D_2' = \frac{dD_2}{da_2}\Big|_{a_2=a_2^*} = b_2 y_5 z_5,\ D_3' = \frac{dD_3}{da_2}\Big|_{a_2=a_2^*} = x_5 y_5 z_5(d_1 - b_2(1 + a_1))$$

Consequently,

$$\Theta(a_2^*)\Psi(a_2^*) + \Gamma(a_2^*)\Phi(a_2^*) = 2D_2(a_2^*)x_5 y_5 z_5(b_2(2 + a_1) - d_1)$$

So, according to condition (36) we have:

$$\Theta(a_2^*)\Psi(a_2^*) + \Gamma(a_2^*)\Phi(a_2^*) \neq 0$$

Yields, the Hopf bifurcation occurs around the equilibrium point E_5 at the parameter $a_2 = a_2^*$ and the proof is complete. ■

Now the conditions of Hopf bifurcation for $n=4$ are constructed according to the Haque and Venturino methods [17]. Consider the characteristic equation given by:

$$P_4(\tau)=\tau^4+C_1\tau^3+C_2\gamma^2+C_3\gamma+C_4=0$$

here $C_1=-tr\left(J\left(x^*\right)\right)$, $C_2=M_1\left(J\left(x^*\right)\right)$, $C_3=-M_2\left(J\left(x^*\right)\right)$ and $C_4=\det\left(J\left(x^*\right)\right)$ with $M_1\left(J\left(x^*\right)\right)$ and $M_2\left(J\left(x^*\right)\right)$ represent the sum of the principal minors of order two and three of $J\left(x^*\right)$ respectively. Clearly, the first condition of Hopf bifurcation holds if and only if

$C_i>0;\ i=1,3;\ \Delta_1=C_1C_2-C_3>0;\ C_1^3-4\Delta_1>0$ and $\Delta_2=C_3(C_1C_2-C_3)-C_1^2C_4=0$

consequently, $C_4=\dfrac{C_3(C_1C_2-C_3)}{C_1^2}$. So, the characteristic equation becomes:

$$P_4(\tau)=\left(\tau^2+\frac{C_3}{C_1}\right)\left(\tau^2+C_1\tau+\frac{\Delta_1}{C_1}\right)=0 \tag{40}$$

Clearly the roots of Eq.(40) are $\tau_{1,2}=\dfrac{1}{2}\left(-C_1\pm\sqrt{C_1^2-4\dfrac{\Delta_1}{C_1}}\right)$, $\tau_{3,4}=\pm i\sqrt{\dfrac{C_3}{C_1}}$

Now, to verify the transversality condition of Hopf bifurcation, we substitute $\tau(q)=\alpha_1(q)\mp i\alpha_2(q)$ into Eq.(40), and then calculating its derivative with respect to the bifurcation parameter q, $P_4'(\tau(q))=0$, comparing the two sides of this equation and then equating their real and imaginary parts, we have

$$\begin{aligned}&\overline{\Psi}(q)\alpha_1'(q)-\overline{\Phi}(q)\alpha_2'(q)+\overline{\Theta}(q)=0\\&\overline{\Phi}(q)\alpha_1'(q)+\overline{\Psi}(q)\alpha_2'(q)+\overline{\Gamma}(q)=0\end{aligned} \tag{41}$$

Where

$$\begin{aligned}\overline{\Psi}(q)&=4(\alpha_1(q))^3+3C_1(q)(\alpha_1(q))^2+C_3(q)+2C_2(q)\alpha_1(q)\\&\quad-12\alpha_1(q)\alpha_2^2(q)-3C_1(q)(\alpha_2(q))^2\\\overline{\Phi}(q)&=12(\alpha_1(q))^2\alpha_2(q)+6C_1(q)\alpha_1(q)\alpha_2(q)+2C_2(q)\alpha_2(q)\\&\quad-4(\alpha_2(q))^3\\\overline{\Theta}(q)&=(\alpha_1(q))^3C_1'(q)+C_3'(q)\alpha_1(q)+C_2'(q)(\alpha_1(q))^2+C_4'(q)\\&\quad-3C_1'(q)\alpha_1(q)(\alpha_2(q))^2-C_2'(q)(\alpha_2(q))^2\\\overline{\Gamma}(q)&=3(\alpha_1(q))^2\alpha_2(q)C_1'(q)+C_3'(q)\alpha_2(q)+2C_2'(q)\alpha_1(q)\alpha_2(q)\\&\quad-C_1'(q)(\alpha_2(q))^3\end{aligned} \tag{42}$$

Solving the linear system (41) by using Cramer's rule for the unknowns $\alpha_1'(q)$ and $\alpha_2'(q)$, gives that

$$\alpha_1'(q)=-\frac{\overline{\Theta}(q)\overline{\Psi}(q)+\overline{\Gamma}(q)\overline{\Phi}(q)}{(\overline{\Psi}(q))^2+(\overline{\Phi}(q))^2};\quad \alpha_2'(q)=\frac{-\overline{\Gamma}(q)\overline{\Psi}(q)+\overline{\Theta}(q)\overline{\Phi}(q)}{(\overline{\Psi}(q))^2+(\overline{\Phi}(q))^2}$$

Hence the transversality condition not being zero if and only if

$$\overline{\Theta}(q)\overline{\Psi}(q)+\overline{\Gamma}(q)\overline{\Phi}(q)\neq 0 \tag{43}$$

***Theorem* (15):** Suppose that the conditions (15c), (15e) with the following conditions are satisfied:

$$\frac{b_2(1+a_1)x_6}{2x_6+(1+a_1)y_6+z_6-1}>d_1 \tag{44}$$

$$j_{13}j_{21}j_{32} > \max\left\{-j_{11}j_{13}j_{31}, \frac{B_6}{\Gamma_5 j_{44}^2}\right\} \tag{45}$$

$$\frac{\hat{a}_2 b_2 z_6}{a_1} < d_2 + l - n_1 z_6 \tag{46}$$

$$d_1 < \frac{\hat{a}_2 b_2 (1 + a_1)}{a_1} \tag{47}$$

$$C_1^3 - 4\Delta_1 > 0 \tag{48}$$

$$\Delta_1 > C_3 \tag{49}$$

$$\Gamma_2 < \frac{j_{11}\Gamma_1 - j_{12}\Gamma_3 + j_{13}(j_{31}j_{44} - j_{21}j_{32})}{-j_{23}} \tag{50}$$

$$C_1^2 j_{11} < C_3 - \Delta_1 \tag{51}$$

Or the conditions (15c),(15e) and (3.16)-(3.21) with the following conditions are satisfied

$$\Gamma_7 > -\Gamma_2 \tag{52}$$

$$C_1^2 j_{11} > C_3 - \Delta_1 \tag{53}$$

Then at the parameter value $\hat{a}_2 = \frac{1}{2N_1 y_6}\left(-N_2 - \sqrt{N_2^2 - 4N_1N_3}\right)$, the system (2) has a Hopf bifurcation near the point E_6, where

$$N_1 = (j_{34}j_{42} + j_{31}j_{12})(\Gamma_2 + \Gamma_7)$$

$$\begin{aligned} N_2 = {} & j_{13}j_{21}j_{32}(j_{34}j_{42} + j_{31}j_{12}) + \Gamma_5(j_{34}j_{42}j_{11}^2 + j_{31}j_{12}j_{44}^2) \\ & - \Gamma_2 j_{34}j_{43}j_{44} - \Gamma_7[j_{11}(j_{12}j_{21} + j_{13}j_{31}) + j_{34}j_{43}j_{44} + j_{13}j_{21}j_{32}] \\ & - (j_{34}j_{42} + j_{31}j_{12})(j_{13}j_{31}j_{44} + j_{11}j_{34}j_{43}) \\ & - \Gamma_2[j_{11}(j_{12}j_{21} + j_{13}j_{31}) + j_{13}j_{21}j_{32}] - j_{12}j_{21}j_{44}(j_{34}j_{42} + j_{31}j_{12}) \end{aligned}$$

$$\begin{aligned} N_3 = {} & [j_{11}(j_{13}j_{31} - j_{44}\Gamma_5) + j_{34}j_{43}j_{44} + j_{13}j_{21}j_{32}][j_{13}j_{31}j_{44} + j_{11}j_{34}j_{43}] \\ & - j_{13}j_{21}j_{32}[j_{11}(j_{12}j_{21} + j_{13}j_{31}) + j_{34}j_{43}j_{44} + j_{13}j_{21}j_{32}] \\ & + j_{11}j_{21}j_{44}(2j_{34}\Gamma_8 - j_{44}j_{12}\Gamma_5) - \Gamma_5 j_{13}j_{21}j_{32}j_{44}^2 + B_6 \\ & + j_{12}j_{21}j_{44}[j_{11}(j_{12}j_{21} + j_{13}j_{31}) + j_{13}j_{21}j_{32}] \end{aligned}$$

Proof: Consider the characteristic equation of the system (2) at E_6 which is given by Eq.(15b). Now, to verify the necessary and sufficient conditions for a Hopf bifurcation to occur we need to find a parameter satisfy $\Delta_2 = 0$.Therefore it is observed that $\Delta_2 = 0$ gives :

$$a_2^2 y_6^2 N_1 + a_2 y_6 N_2 + N_3 = 0 \tag{54}$$

Now, we have two cases:

Case1: if the first part of existence conditions (7c) hold, then by using Descartes Rule Eq.(54) has a unique positive root $\hat{a}_2 = \frac{1}{2N_1 y_6}\left(-N_2 - \sqrt{N_2^2 - 4N_1N_3}\right)$ provided that the conditions (15c), (15e), (44) and (45) hold.

Case2: if the second part of existence conditions (7c) hold, then by using Descartes Rule Eq.(54) has a unique positive root $\hat{a}_2 = \frac{1}{2N_1 y_6}\left(-N_2 - \sqrt{N_2^2 - 4N_1N_3}\right)$ provided that the conditions (15c), (15e), (44), (45) and (52) hold.

Now, at $a_2 = \hat{a}_2$ the characteristic equation given by Eq.(15b) can be written as :

$$\left(\lambda_6^2 + \frac{C_3}{C_1}\right)\left(\lambda_6^2 + C_1\lambda_6 + \frac{\Delta_1}{C_1}\right) = 0$$

Which has four roots $\lambda_{61,2} = \pm i\sqrt{\frac{C_3}{C_1}}$ and $\lambda_{63,4} = \frac{1}{2}\left(-C_1 \pm \sqrt{C_1^2 - 4\frac{\Delta_1}{C_1}}\right)$. Clearly, at $a_2 = \hat{a}_2$ there are two pure imaginary eigenvalues (λ_{61} and λ_{62}) and two eigenvalues which are real and negative provided the conditions (15c) and (46)-(48), and (50) hold when the first part of existence conditions (7c) hold or the conditions (15c) and (46)-(48) hold when the second part of existence conditions (7c) hold.

Now for all values of a_2 in the neighborhood of $\hat{a}_2$, the roots in general of the following form:

$$\lambda_{61} = \alpha_1 + i\alpha_2\,;\quad \lambda_{62} = \alpha_1 - i\alpha_2\,;\quad \lambda_{63,4} = \frac{1}{2}\left(-C_1 \pm \sqrt{C_1^2 - 4\frac{\Delta_1}{C_1}}\right)$$

Clearly, $\mathrm{Re}\left(\lambda_{6k}(a_2)\right)\big|_{a_2=\hat{a}_2} = \alpha_1(\hat{a}_2) = 0,\ k = 1,2$ that means the first condition of the necessary and sufficient conditions for Hopf bifurcation is satisfied at $a_2 = \hat{a}_2$. Now to verify the transversality condition we must prove that $\overline{\Theta}(\hat{a}_2)\overline{\Psi}(\hat{a}_2) + \overline{\Gamma}(\hat{a}_2)\overline{\Phi}(\hat{a}_2) \neq 0$, where the form of $\overline{\Theta}$, $\overline{\Psi}$, $\overline{\Gamma}$ and $\overline{\Phi}$ are given in Eq.(42). Note that for $a_2 = \hat{a}_2$ we have $\alpha_1 = 0$ and $\alpha_2 = \sqrt{\frac{C_3}{C_1}}$, substitution into (42) gives the following simplifications:

$$\overline{\Psi}(\hat{a}_2) = -2C_3(\hat{a}_2);\quad \overline{\Phi}(\hat{a}_2) = 2\frac{\alpha_2(\hat{a}_2)}{C_1}(C_1C_2 - 2C_3)$$

$$\overline{\Theta}(\hat{a}_2) = C_4'(\hat{a}_2) - \frac{C_3}{C_1}C_2'(\hat{a}_2);\quad \overline{\Gamma}(\hat{a}_2) = \alpha_2(\hat{a}_2)\left(C_3'(\hat{a}_2) - \frac{C_3}{C_1}C_1'(\hat{a}_2)\right)$$

Where:

$$C_1' = \left.\frac{dC_1}{da_2}\right|_{a_2=\hat{a}_2} = 0;\quad C_2' = \left.\frac{dC_2}{da_2}\right|_{a_2=\hat{a}_2} = y_5 j_{32}$$

$$C_3' = \left.\frac{dC_3}{da_2}\right|_{a_2=\hat{a}_2} = -y_5(\Gamma_2 + j_{11}j_{32} + j_{12}j_{31});\quad C_4' = \left.\frac{dC_4}{da_2}\right|_{a_2=\hat{a}_2} = j_{11}y_5(\Gamma_2 - j_{12}j_{31}j_{44})$$

Then by using Eq.(43) we get that:

$$\overline{\Theta}(\hat{a}_2)\overline{\Psi}(\hat{a}_2) + \overline{\Gamma}(\hat{a}_2)\overline{\Phi}(\hat{a}_2) =$$

$$-2C_3y_6\left[\Gamma_2\left(j_{11} + \frac{\Delta_1 - C_3}{C_1^2}\right) + \left(\frac{\Delta_1 - C_3}{C_1^2}\right)(j_{11}j_{32} + j_{12}j_{31}) - j_{12}j_{31}j_{44} - j_{32}\alpha_2^2(\hat{a}_2)\right]$$

Now, we have two cases:

Case1: if the first part of existence conditions (7c) hold, then

$$\overline{\Theta}(\hat{a}_2)\overline{\Psi}(\hat{a}_2) + \overline{\Gamma}(\hat{a}_2)\overline{\Phi}(\hat{a}_2) \neq 0$$

provided that the conditions (15c), (49) and (51) hold.

Case2: if the second part of existence conditions (7c) hold, then

$$\overline{\Theta}(\hat{a}_2)\overline{\Psi}(\hat{a}_2) + \overline{\Gamma}(\hat{a}_2)\overline{\Phi}(\hat{a}_2) \neq 0$$

provided that the conditions (15c), (49) and (53) hold.

So, we obtain that the Hopf bifurcation occurs around the equilibrium point E_6 at the parameter $a_2 = \hat{a}_2$ and the proof is complete. ■

8. Numerical analysis of system (2)

In this section the dynamical behavior of system (2) is studied numerically for different sets of parameters and different sets of initial points. The objectives of this study are: first investigate the effect of varying the value of each parameter on the dynamical behavior of system (2) and second confirm our obtained analytical results. It is observed that, for the following set of hypothetical parameters:

$$\begin{array}{lllll} a_1 = 0.2 & , a_2 = 0.1 & , b_1 = 0.1 & , b_2 = 0.2 & , d_1 = 0.2 \\ d_2 = 0.1 & , n_1 = 0.2 & , n_2 = 0.4 & , l = 0.1 & \end{array} \tag{55}$$

The trajectory of the system (2) is drawn in the figure (1) for different initial points.

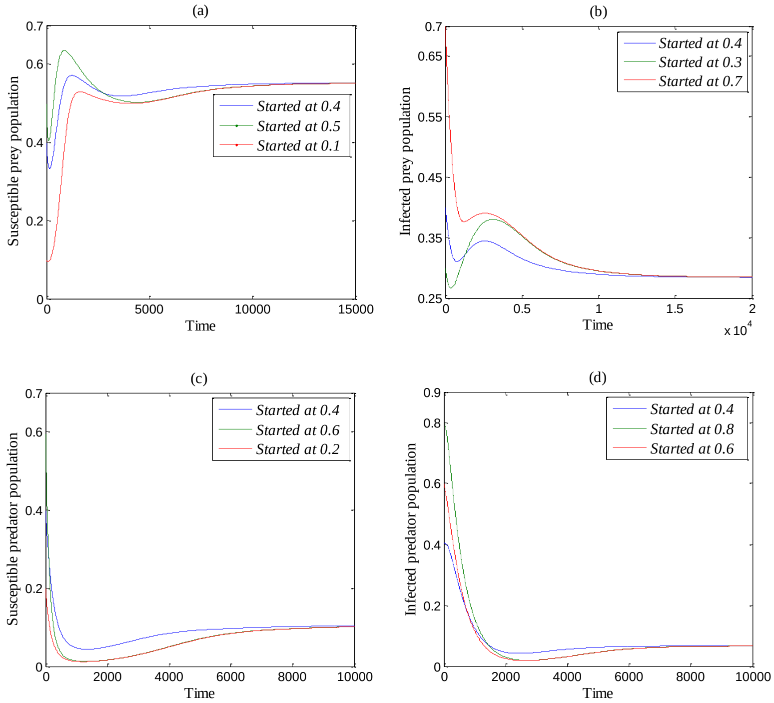

Figure (1): Time series of the solution of system (2) (a) trajectories of x as a function of time, (b) trajectories of y as a function of time, (c) trajectories of z as a function of time, (d) trajectories of w as a function of time.

Clearly, figure (1) shows that the solution of system (2) approaches asymptotically to the positive equilibrium point $E_6 = (0.5, 0.2, 0.1, 0.06)$ starting from three different initial points and this is confirming our obtained analytical results regarding to global stability of the positive equilibrium point.

Now in order to discuss the effect of the varying the parameters values of system (2) on the dynamical behavior of the system, the system is solved numerically for the data given in Eq.(55) with varying one parameter each time. It is observed that for the data given in Eq.(55) with varying the parameter value of n_1 there is no effect on the dynamical behavior of system (2) and the system still approaches to positive equilibrium point. It is observed that for the data as given in Eq.(55) with $a_1 < 0.18$, the solution of system (2) approaches asymptotically to $E_3 = (x_3, 0, z_3, 0)$ in the interior of positive quadrant of $xz-$plane, however for $a_1 \geq 0.18$ the system approaches to the positive equilibrium point, as shown in figure (2).

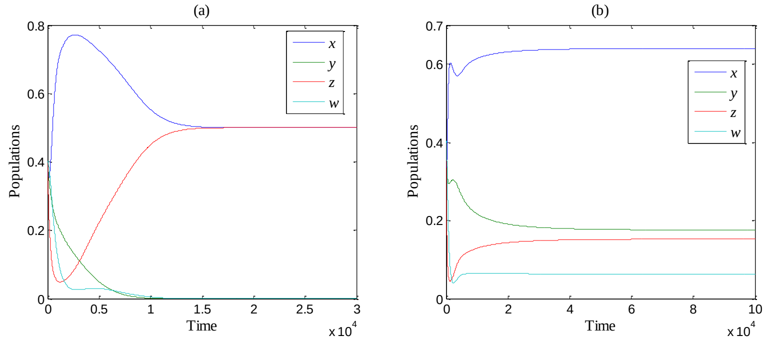

Figure (2): Time series of the solution of system (2) for the data given by Eq.(55) with (a) $a_1 = 0.1$, which approaches to (0.5, 0, 0.5, 0) in the interior of positive quadrant of $xz-$plane, (b) $a_1 = 0.18$, which approaches to (0.64, 0.17, 0.15, 0.06) in the interior of R_+^4.

Similarly varying the parameter a_2 keeping the rest of parameter values as in Eq.(55), It is observed that for $a_2 \geq 0.19$, the solution of system (2) approaches asymptotically to $E_3 = (x_3, 0, z_3, 0)$ in the interior of the positive quadrant of $xz-$plane, however for $a_2 < 0.19$ the system approaches to the positive equilibrium point.

Also varying the parameter b_1 keeping the rest of parameter values as in Eq.(55), it is observed that for $b_1 \geq 0.12$, the solution of system (2) approaches asymptotically to $E_3 = (x_3, 0, z_3, 0)$ in the interior of positive quadrant of $xz-$plane, while for $b_1 < 0.12$ the system approaches to the positive equilibrium point.

For the parameter values given in Eq.(55) with varying b_2 in the range $b_2 \geq 0.26$ system (2) approaches asymptotically to $E_3 = (x_3, 0, z_3, 0)$ in the interior of positive quadrant of $xz-$plane, however for $0.04 < b_2 < 0.26$ the system approaches to the positive equilibrium point, while for $b_2 \leq 0.04$ the solution of system (2) approaches asymptotically to $E_2 = (x_2, y_2, 0, 0)$ in the interior of positive quadrant of $xy-$plane, as shown in figure (3).

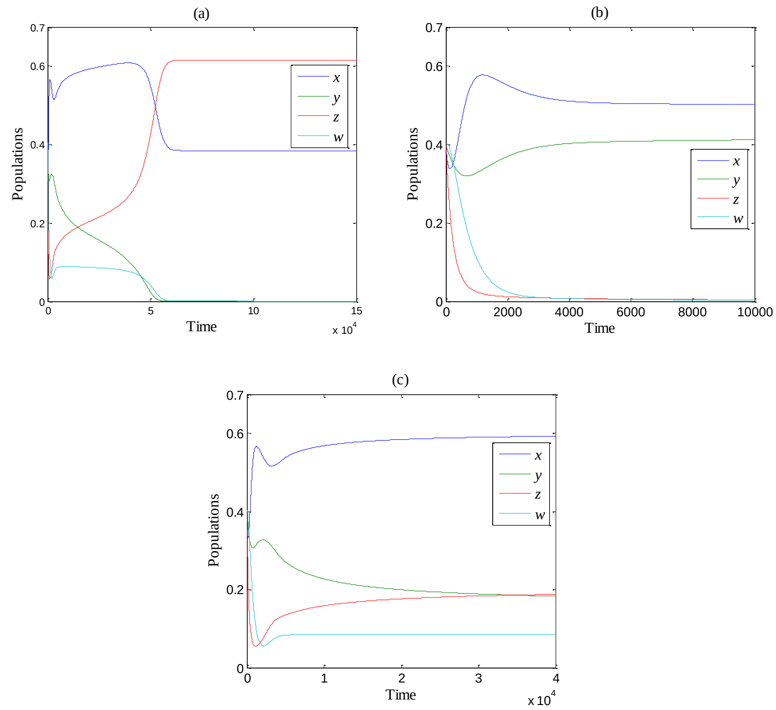

Figure (3): Time series of the solution of system (2) for the data given by Eq.(55) with (a) $b_2 = 0.26$, which approaches to (0.38, 0, 0.6, 0) in the interior of positive quadrant of $xz-$plane, (b) $b_2 = 0.02$,

which approaches to (0.5, 0.4, 0, 0) in the interior of positive quadrant of $xy-$plane, (c) $b_2 = 0.25$, which approaches to (0.59, 0.16, 0.19, 0.08) in the interior of R_+^4.

Again for the parameter values given in Eq.(55) with varying d_1 in the range $d_1 \geq 0.44$ system (2) approaches asymptotically to $E_3 = (x_3, 0, z_3, 0)$ in the interior of positive quadrant of $xz-$plane, however for $d_1 < 0.44$ the system approaches to the positive equilibrium point.

For the parameter values given in Eq.(55) with varying d_2 in the range $d_2 \geq 0.19$ system (2) approaches asymptotically to $E_2 = (x_2, y_2, 0, 0)$ in the interior of positive quadrant of $xy-$plane, however for $d_2 < 0.19$ the system approaches to the positive equilibrium point.

For the parameter values given in Eq.(55) with varying n_2 in the range $n_2 < 0.25$ system (2) approaches asymptotically to $E_3 = (x_3, 0, z_3, 0)$ in the interior of positive quadrant of $xz-$plane, however for $n_2 \geq 0.25$ the system approaches to the positive equilibrium point.

Varying the parameter l keeping the rest of parameter values as in Eq.(55), It is observed that for $l \geq 0.2$, the solution of system (2) approaches asymptotically to $E_3 = (x_3, 0, z_3, 0)$ in the interior of positive quadrant of $xz-$plane, while for $l < 0.2$ the system approaches to the positive equilibrium point.

Moreover, for the parameter values given in Eq.(55) with $b_1 = 0.3$ and $d_2 = 0.3$ the solution of system (2) approaches asymptotically to $E_1 = (1, 0, 0, 0)$ as shown in figure (4).

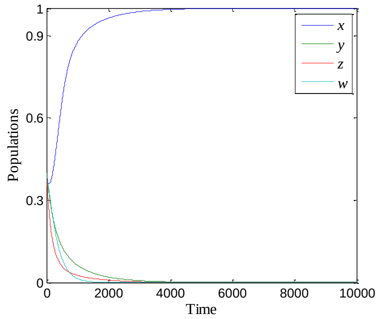

Figure (4): Time series of the solution of system (2) for the data given by Eq.(55) with $b_1 = 0.3$ and $d_2 = 0.3$, which approaches to (1, 0, 0, 0).

For the parameter values given in Eq.(55) with $b_1 = 0.09$, $d_1 = 0.12$, $d_2 = 0.01$ and $n_2 = 0.03$ the solution of system (2) approaches asymptotically to $E_4 = (x_4, 0, z_4, w_4)$ as shown in figure (5).

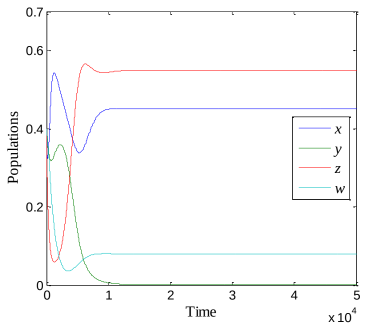

Figure (5): Time series of the solution of system (2) for the data given by Eq.(55) with $b_1 = 0.09$, $d_1 = 0.12$, $d_2 = 0.01$ and $n_2 = 0.03$, which approaches to (0.45, 0, 0.55, 0.08) in the interior of positive octant of $xzw-$plane

However, for the parameter values given in Eq.(55) with $b_1 = 0.09$, $d_1 = 0.12$, $d_2 = 0.13$ and $n_2 = 0$ the solution of system (2) approaches asymptotically to $E_5 = (x_5, y_5, z_5, 0)$ as shown in figure (6).

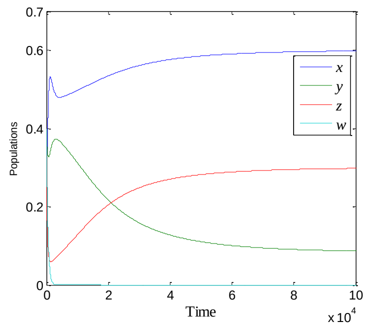

Figure (6): Time series of the solution of system (2) for the data given by Eq.(55) with $b_1 = 0.09$, $d_1 = 0.12$, $d_2 = 0.13$ and $n_2 = 0$, which approaches to (0.59, 0.08, 0.29, 0) in the interior of positive octant of $xyz-$plane.

9. Conclusions and Discussion

In this paper, we proposed and analyzed an eco-epidemiological model that described the dynamical behavior of prey-predator model with Lotka-Volterra type of functional response and linear incidence rate for the disease in prey and predator respectively. It is assumed that the disease is transmitted from a prey to predator during the predation process, also the disease transmitted within the same species by contact with an infected individual. The model included four non-linear autonomous differential equations that describe the dynamics of four different population namely susceptible prey x, infected prey y, susceptible predator z and infected predator w. The boundedness of the system (2) has been discussed. The dynamical behavior of system (2) has been investigated locally as well as globally. Further, it is observed that the vanishing equilibrium point (E_0) always exist, and it is unstable saddle point. The axial equilibrium point (E_1) always exist, and it is locally asymptotically stable point if and only if the conditions (10b)-(10c) hold as well as it is globally if the conditions (16a)-(16b) hold. The predator free equilibrium point (E_2) exists provided that the condition (3b) holds, and it is locally asymptotically stable point if and only if the condition (11c) holds as well as it is globally if the conditions (17a)-(17b) hold. The disease free equilibrium point (E_3) exists provided that the condition (4b) holds, and it is locally asymptotically stable point if and only if the condition (12c) holds, while it is globally if the conditions (18a)-(18b) hold. The infected prey free equilibrium point (E_4) exists provided that the condition (5b) holds, and it is locally asymptotically stable point if and only if the condition (13c) holds, while it is globally if the condition (19) holds. The infected predator free equilibrium point (E_5) exists provided that the condition (6b) or (6c) holds, and it is locally asymptotically stable point if and only if the conditions (14c)-(14d) hold, further it is globally if the conditions (20a)-(20b) hold. The positive equilibrium point of system (2) exists provided that the condition (7c) holds. It is locally asymptotically stable point if and only if conditions (15c)-(15g) or (15c)-(15f) hold, in addition it is globally if the conditions (22a)-(22b) hold. To understand the effect of varying each parameter on the global dynamics of system (2) and to confirm our above analytical results, system (2) has been solved numerically and the following results are obtained:

1. For the set of hypothetical parameters values given Eq.(55), system (2) approaches asymptotically to a globally asymptotically stable point $E_6 = (0.5, 0.2, 0.1, 0.06)$.

2. It is observed that the system (2) has no effect of the dynamical behavior for the data given in Eq.(55) with varying the parameter values n_1 and the system still approaches to positive equilibrium point.

3. As the infection rate of prey a_1 decreases keeping other parameters fixed as in Eq.(55) then the infected species of prey and predator will face extinction and the solution of system (2) approaches asymptotically to the equilibrium point $E_3 = (x_3, 0, z_3, 0)$. Otherwise the system still have a globally

asymptotically stable positive point in the $Int.R_+^4$. Further, it is observed that n_2 have the same effect as a_1.

4. As the attack rate a_2 increases keeping other parameters as in Eq.(4.1) then the infected species of prey and predator will face extinction and the solution of system (2.2) approaches asymptotically to the equilibrium point $E_3 = (x_3, 0, z_3, 0)$. Otherwise the system still have a globally asymptotically stable positive point in the $Int.R_+^4$. Moreover it is observed that the parameter d_1, mortality rate of infected prey b_1 and mortality rate of infected predator l have the same effect as a_2.

5. As the conversion rate b_2 increases keeping other parameters as in Eq.(55) then the infected species of prey and predator will face extinction and the solution of system (2) approaches asymptotically to the equilibrium point $E_3 = (x_3, 0, z_3, 0)$. However decreasing the parameter b_2 causes extinction of susceptible and infected predator and the solution of system (2) approaches asymptotically to the equilibrium point $E_2 = (x_2, y_2, 0, 0)$.

6. As the natural death rate of susceptible and infected predator d_2 increases keeping other parameters fixed as in Eq.(55) then the susceptible and infected predator face extinction and the solution of system (2) approaches asymptotically to the equilibrium point $E_2 = (x_2, y_2, 0, 0)$. Otherwise the system still have a globally asymptotically stable positive point in the $Int.R_+^4$.

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