Recursive Route to Mixed Poisson Distributions using Integration by Parts

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Abstract

Mixed Poisson distributions are very significant in modeling non-homogeneous populations; for instance in Actuarial applications for modeling total claims in insurance. However, the setback is in their use since the probability mass functions are difficult to evaluate, except for a few mixing distributions. One way of dealing with this problem is to express Mixed Poisson distributions in terms of recursive relations.

In this paper, recursive relations of some mixed Poisson distributions are obtained by use of integration by parts technique.

Keywords: mixtures; recursive relation; generating functions; moments

1. Introduction

The main difficulty with the use of mixed Poisson distributions is that, with the exception of a few mixing distributions, their probability mass functions are difficult to evaluate. Some of the ways of circumventing this problem is to express the mixed distributions in terms of expectation forms, special functions forms and recursive relations.

In this paper, we are confining ourselves to the recursive relations using integration by parts technique.

In the literature, we find that originally patterns of recursive relations in probabilities were devised; and the problem was to identify probability mass functions satisfying these patterns. This type of work was done by Katz(1965) and then later by Panjer(1981).

Sundt and Jewel(1981) showed that the only distributions satisfying Panjer's model were Poisson, Binomial and Negative Binomial distributions. These works motivated other researchers to come up with other patterns of recursive relations and identify corresponding probability distributions; [Schroter (1990), Sundt (1992), Hesselager (1994), Wang (1994)].

Further motivation led Willmot (1993) to obtain Mixed Poisson distributions in recursive forms. His approach is known as Willmot's Approach. It is interesting to note that the recursive relations obtained by Willmot's Approach can simply be obtained using integration by parts technique; which does not require any condition as that of Willmot's Approach. This is the main motivation of this paper.

Moments and probability generating functions can be obtained using the recursive formulae.

This paper is organized as follows: two-parameter, three-parameter and four-parameter Beta mixing distributions have been considered in sections 2, 3 and 4 respectively. Gamma mixing distributions are in section 5; Pareto in section 6 and Inverse Gaussian distributions in section 7. Other mixing distributions are Confluent Hypergeometric and Half-Normal which are in section 8 followed by conclusion in section 9.

2. Mixing with two-parameter Beta Distributions

2.1 Beta I distribution

$$g(\lambda) = \frac{\lambda^{\alpha-1}(1-\lambda)^{\beta-1}}{B(\alpha,\beta)};\ \ 0<\lambda<1 \tag{1}$$

The Mixed Poisson distribution is thus given by;

$$f(x) = \frac{t^x}{x!\,B(\alpha,\beta)}\int_0^1 e^{-\lambda t}\lambda^{x+\alpha-1}(1-\lambda)^{\beta-1}d\lambda$$

Using integration by parts, let

$$u = e^{-\lambda t}\lambda^{x+\alpha-1} \text{ and } dv = (1-\lambda)^{\beta-1}d\lambda$$

Therefore the recursive relation is given by:

$$(x+1)xf(x+1) = (\beta+t+x+\alpha-1)xf(x) - t(x+\alpha-1)f(x-1);\ \ x = 0,1,2,\ldots \quad (2)$$

with f(-1)=0.

2.2 Rectangular distribution

$$g(\lambda) = \frac{1}{b-a}\ ; a < \lambda < b \quad (3)$$

Therefore,

$$f(x) = \frac{t^x}{x!\,(b-a)}\left\{\int_0^b e^{-\lambda t}\lambda^x d\lambda - \int_0^a e^{-\lambda t}\lambda^x d\lambda\right\}$$

Let $y = \lambda t \Rightarrow \lambda = \frac{y}{t}$ and $d\lambda = \frac{dy}{t}$, then

$$f(x) = \frac{1}{t(b-a)x!}\left\{\int_0^{bt} e^{-y}y^x dy - \int_0^{at} e^{-y}y^x dy\right\}$$

$$= \frac{1}{t(b-a)x!}\{\gamma(x+1,bt) - \gamma(x+1,at)\}$$

where

$$\gamma(x,c) = \int_0^c y^{x-1}e^{-y}dy$$

is incomplete gamma.

Consider

$$\gamma(x+1,bt) = \int_0^{bt} e^{-y}y^x dy$$

Let $u = y^x$ and $dv = e^{-y}dy$; therefore the recursive relation is given by

$$f(x+1) = f(x) + \left\{\frac{e^{-at}(at)^{x+1} - e^{-bt}(bt)^{x+1}}{t(b-a)(x+1)!}\right\}; x = 0,1,2,\ldots \quad (4)$$

with

$$f(0) = \frac{e^{-at} - e^{-bt}}{t(b-a)}$$

2.3 Beta II distribution

$$g(\lambda) = \frac{\lambda^{\alpha-1}}{B(\alpha,\beta)(1+\lambda)^{\alpha+\beta}};\ \lambda > 0;\ \alpha,\beta > 0 \quad (5)$$

Then

$$f(x) = \frac{t^x}{x!\,B(\alpha,\beta)}\int_0^\infty \lambda^{x+\alpha-1}(1+\lambda)^{-\alpha-\beta}e^{-\lambda t}d\lambda$$

Let $u = \lambda^{x+\alpha-1}e^{-\lambda t}$ and $dv = (1+\lambda)^{-(\alpha+\beta)}d\lambda$. Therefore the recursive relation of the mixed Poisson distribution is given by:

$$(x+1)xf(x+1) = (x-\beta-t)xf(x) + t(x+\alpha-1)f(x-1); x = 0,1,2,\ldots \quad (6)$$

with $f(-1) = 0$.

3. Mixing with three-parameter Beta distributions

3.1 Scaled Beta distribution

$$g(\lambda) = \frac{\lambda^{\alpha-1}(\mu-\lambda)^{\beta-1}}{B(\alpha,\beta)\mu^{\alpha+\beta-1}}; 0 \le \lambda \le \mu \quad (7)$$

Therefore,

$$f(x) = \frac{t^x}{x!\,B(\alpha,\beta)\mu^{\alpha+\beta-1}}\int_0^{\mu}\lambda^{x+\alpha-1}(\mu-\lambda)^{\beta-1}e^{-\lambda t}d\lambda$$

Let $\lambda = \mu z \Rightarrow d\lambda = \mu dz$ and $z = \frac{\lambda}{\mu}$, therefore

$$f(x) = \frac{(\mu t)^x}{x!\,B(\alpha,\beta)}\int_0^1 z^{x+\alpha-1}(1-z)^{\beta-1}e^{-\mu t z}dz$$

Putting $u = e^{-\mu t z}z^{x+\alpha-1}$ and $dv = (1-z)^{\beta-1}dz$, the recursive relation is therefore given by

$$x(x+1)f(x+1) = (\beta+\mu t+x+\alpha-1)xf(x) - (x+\alpha-1)(\mu t)f(x-1); x = 0,1,2,\ldots \quad (8)$$

with $f(-1) = 0$.

3.2 Full Beta Model

The probability density function (pdf) of a Full Beta model is given by:

$$g(\lambda) = \frac{b^p\lambda^{p-1}}{B(p,q)(1+b\lambda)^{p+q}};\ \lambda > 0; b,p,q > 0 \quad (9)$$

Therefore,

$$f(x) = \frac{t^x b^p}{x!\,B(p,q)}\int_0^{\infty}\lambda^{x+p-1}(1+b\lambda)^{-(p+q)}e^{-\lambda t}d\lambda$$

Let $z = b\lambda; \Rightarrow dz = bd\lambda$ and $\lambda = \frac{z}{b}$, then

$$f(x) = \left(\frac{t}{b}\right)^x\frac{1}{x!\,B(p,q)}\int_0^{\infty}z^{x+p-1}(1+z)^{-(p+q)}e^{-\frac{t}{b}z}dz$$

Put $u = z^{x+p-1}e^{-\frac{t}{b}z}$ and $v = (1+z)^{-(p+q)}dz$; the recursive relation is therefore given by:

$$b^2x(x+1)f(x+1) = [b(x-q)-t]bxf(x) + bt(x+p-1)f(x-1); x = 0,1,2,\ldots \quad (10)$$

with $f(-1) = 0$.

4. Mixing with a four-parameter Beta distribution

The pdf of a Transformed Beta distribution is given by:

$$g(\lambda) = \frac{c\mu^{\alpha}\lambda^{c\beta-1}}{B(\alpha,\beta)(\mu+\lambda^c)^{\alpha+\beta}};\ \lambda > 0 \quad (11)$$

Then,

$$f(x) = \frac{c\mu^{\alpha}t^x}{B(\alpha,\beta)x!}\int_0^{\infty}\lambda^{x+c\beta-1}(\mu+\lambda^c)^{-\alpha-\beta}e^{-\lambda t}d\lambda$$

Let $u = \lambda^{x-c+c\beta}e^{-\lambda t}$ and $v = c\lambda^{c-1}(\mu+\lambda^c)^{-\alpha-\beta}d\lambda$; therefore the recursive relation is given by:

$$\left\{\prod_{i=1}^{c+1}(x-c+i)\right\}f(x+1) = (x-c\alpha)\left\{\prod_{i=1}^{c}(x-c+i)\right\}f(x) + t^c(x-c+c\beta)\mu f(x-c)$$

$$-\mu t^c (x-c+1) f(x-c+1) \tag{12}$$

for $x = 0,1,2,\ldots$

5. Mixing with Gamma distributions

5.1 Inverse Gamma distribution

The pdf of an Inverse Gamma distribution is given by:

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\frac{\beta}{\lambda}} \lambda^{-\alpha-1};\ \lambda > 0;\ \alpha, \beta > 0 \tag{13}$$

Then,

$$f(x) = \frac{\beta^\alpha t^x}{\Gamma(\alpha) x!} \int_0^\infty \lambda^{x-\alpha-1} e^{-\left(\lambda t + \frac{\beta}{\lambda}\right)} d\lambda$$

Put $u = e^{-\left(\lambda t + \frac{\beta}{\lambda}\right)}$ and $dv = \lambda^{x-\alpha-1} d\lambda$; therefore the recursive relation is given by:

$$x(x+1) f(x+1) = (x-\alpha) x f(x) + \beta t f(x-1); x = 0,1,2,\ldots \tag{14}$$

with $f(-1) = 0$.

5.2 Shifted Gamma distribution

The pdf of a Shifted Gamma distribution is given by:

$$g(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1};\ \lambda > \mu > 0;\ \alpha, \beta > 0 \tag{15}$$

Then;

$$f(x) = \frac{t^x \beta^\alpha}{x!\, \Gamma(\alpha)} \int_\mu^\infty e^{-\lambda t} \lambda^x e^{-\beta(\lambda-\mu)} (\lambda-\mu)^{\alpha-1} d\lambda$$

Put $z = \lambda - \mu;\ \Rightarrow dz = d\lambda$ and $\lambda = \mu + z;$ therefore

$$f(x) = \frac{t^x \beta^\alpha}{x!\, \Gamma(\alpha)} \int_0^\infty e^{-(\mu+z)t} (\mu+z)^x e^{-\beta z} z^{\alpha-1} dz$$

Put $z = \mu y;\ \Rightarrow dz = \mu dy$; therefore,

$$f(x) = \frac{(\mu t)^x (\mu\beta)^\alpha}{x!\, \Gamma(\alpha)} e^{-\mu t} \int_0^\infty y^{\alpha-1} (1+y)^x e^{-(t+\beta)\mu y} dy$$

Using integration by parts, let $u = (1+y)^x e^{-(t+\beta)\mu y}$ and $dv = y^{\alpha-1} dy$; therefore the recursive relation is given by:

$$(t+\beta)(x+1) f(x+1) = [x + \alpha + (t+\beta)\mu] t f(x) - \mu t^2 f(x-1); x = 0,1,2,\ldots \tag{16}$$

with $f(-1) = 0$.

5.3 Gamma Truncated from below

The pdf of a Gamma distribution truncated from below is given by:

$$g(\lambda) = \frac{\beta^\alpha e^{-\beta\lambda} \lambda^{\alpha-1}}{\{1 - \gamma(\alpha, \beta\lambda_0)\}};\lambda > \lambda_0 > 0;\ \alpha, \beta > 0 \tag{17}$$

where

$$\gamma(\alpha, \beta\lambda_0) = \int_0^{\beta\lambda_0} e^{-y} y^{\alpha-1} dy$$

Now,

$$f(x)=\frac{\beta^{\alpha}t^{x}}{x!\{1-\gamma(\alpha,\beta\lambda_0)\}}\int_{\lambda_0}^{\infty}\lambda^{x+\alpha-1}e^{-(t+\beta)\lambda}d\lambda$$

Using integration by parts, let $u=e^{-(t+\beta)\lambda}$ and $dv=\lambda^{x+\alpha-1}d\lambda$; therefore the recursive relation is given by:

$$(t+\beta)(x+1)f(x+1)=t(x+\alpha)f(x)+t^{x+1}e^{-t\lambda_0}g(\lambda_0)\frac{\lambda_0^{x+1}}{x!} \quad (18)$$

5.4 Generalized Gamma distribution

If the Generalized Gamma mixing distribution is given by:

$$g(\lambda)=\frac{\alpha^{m-\delta}e^{-\alpha\lambda}\lambda^{m-1}}{\Gamma_\delta(m,\alpha n)(\lambda+n)^{\delta}};\ \lambda\geq 0; m,\alpha,n>0;\delta\geq 0 \quad (19)$$

where

$$\Gamma_\delta(m,\alpha n)=\int_0^{\infty}\frac{y^{m-1}e^{-y}}{(y+\alpha n)^{\delta}}dy$$

then the integrand of the mixed distribution becomes

$$f(x)=\frac{t^{x}\alpha^{m-\delta}}{x!\,\Gamma_\delta(m,\alpha n)}\int_0^{\infty}e^{-(\alpha+t)\lambda}\lambda^{x+m-1}(n+\lambda)^{-\delta}d\lambda$$

Therefore

$$\frac{x!\,f(x)\Gamma_\delta(m,\alpha n)}{t^{x}\alpha^{m-\delta}}=\int_0^{\infty}e^{-(\alpha+t)\lambda}\lambda^{x+m-1}(\lambda+n)^{-\delta}d\lambda$$

Put $\lambda=nz\Rightarrow d\lambda=ndz$; therefore

$$RHS=\int_0^{\infty}e^{-(\alpha+t)nz}(nz)^{x+m-1}n^{-\delta}(1+z)^{-\delta}ndz$$

$$=n^{x+m-\delta}\int_0^{\infty}e^{-(\alpha+t)nz}z^{x+m-1}(1+z)^{-\delta}dz$$

Put $u=z^{x+m-1}e^{-(\alpha+t)nz}$ and $dv=(1+z)^{-\delta}dz$; therefore the recursive relation is given by:

$$(\alpha+t)x(x+1)f(x+1)=[x+m-\delta-n(\alpha+t)]x(nt)f(x)$$
$$+(x+m-1)(nt)^2f(x-1) \quad (20)$$

as obtained by Ong (1995).

5.5 Transformed (Generalized) Gamma distribution

If the probability density function of a Transformed Gamma mixing distribution is given by;

$$g(\lambda)=\frac{c\beta^{\alpha}}{\Gamma(\alpha)}\lambda^{c\alpha-1}e^{-\beta\lambda^{c}};\ \lambda>0;\ \alpha,\beta,c>0 \quad (21)$$

then

$$f(x)=\frac{c\beta^{\alpha}t^{x}}{\Gamma(\alpha)x!}\int_0^{\infty}\lambda^{x+c\alpha-1}e^{-\lambda t-\beta\lambda^{c}}d\lambda$$

Let $u=e^{-\lambda t-\beta\lambda^{c}}$ and $dv=\lambda^{x+c\alpha-1}d\lambda$; therefore the recursive relation is given by:

$$(x+c\alpha)f(x)=(x+1)f(x+1)+\frac{c\beta}{t^{c}}\left\{\prod_{i=1}^{c}(x+i)\right\}f(x+c) \quad (22)$$

6. Mixing with Pareto Distributions

6.1 Pareto I distribution

The pdf of Pareto I distribution is given by:

$$g(\lambda) = \frac{\alpha\beta^{\alpha}}{\lambda^{\alpha+1}}; \lambda > \beta; \alpha > 0 \qquad (23)$$

Now,

$$f(x) = \frac{t^x \alpha\beta^{\alpha}}{x!}\int_{\beta}^{\infty} e^{-\lambda t}\lambda^{x-\alpha-1}d\lambda$$

Let $u = e^{-\lambda t}$ and $dv = \lambda^{x-\alpha-1}d\lambda$; therefore the recursive relation is given by:

$$(x+1)f(x+1) = (x-\alpha)f(x) + \frac{\alpha t^x e^{-\beta t}\beta^{\alpha}}{x!} \qquad (24)$$

6.2 Pareto II (Lomax) distribution

The pdf of Lomax distribution is given by:

$$g(\lambda) = \frac{\alpha\beta^{\alpha}}{(\lambda+\beta)^{\alpha+1}}; \lambda > 0; \beta, \alpha > 0 \qquad (25)$$

Therefore,

$$f(x) = \alpha\beta^{\alpha}\frac{t^x}{x!}\int_0^{\infty} e^{-\lambda t}\lambda^x(\lambda+\beta)^{-\alpha-1}d\lambda$$

Putting $u = e^{-\lambda t}\lambda^x$ and $dv = (\lambda+\beta)^{-\alpha-1}d\lambda$; the recursive relation becomes:

$$(x+1)f(x+1) = (x+\beta t-\alpha)f(x) + t\beta f(x-1); x = 0,1,2,\ldots \qquad (26)$$

with $f(-1) = 0$.

6.3 Generalized Pareto distribution

The pdf of a Generalized Pareto distribution is given by:

$$g(\lambda) = \frac{\mu^{\alpha}\lambda^{\beta-1}}{B(\alpha,\beta)(\mu+\lambda)^{\alpha+\beta}}; \lambda > 0; \alpha, \beta, \mu > 0 \qquad (27)$$

Therefore,

$$f(x) = \frac{\mu^{\alpha}t^x}{x!\,B(\alpha,\beta)}\int_0^{\infty} e^{-\lambda t}\lambda^{x+\beta-1}(\lambda+\mu)^{-\alpha-\beta}d\lambda$$

Put $u = e^{-\lambda t}\lambda^{x+\beta-1}$ and $dv = (\lambda+\mu)^{-\alpha-\beta}d\lambda$; therefore the recursive relation is given by:

$$x(x+1)f(x+1) = (x-\alpha-\mu t)xf(x) + t\mu(x+\beta-1)f(x-1); x = 0,1,2,\ldots \qquad (28)$$

with $f(-1) = 0$.

6.3 Another Generalized Pareto distribution

$$g(\lambda) = \frac{1}{k}\left(1-\frac{c}{k}\lambda\right)^{\frac{1}{c}-1}; \lambda > 0 \qquad (29)$$

Case (i): When *c<0*

Let $c = -d$ where $d > 0$, therefore;

$$g(\lambda) = \frac{1}{k}\left(1+\frac{d}{k}\lambda\right)^{-\frac{1}{d}-1}; \lambda > 0$$

$$f(x) = \frac{t^x}{kx!}\int_0^\infty e^{-\lambda t}\lambda^x\left(1+\frac{d}{k}\lambda\right)^{-\frac{1}{d}-1} d\lambda$$

Put $u = e^{-\lambda t}\lambda^x$ and $dv = \left(1+\frac{d}{k}\lambda\right)^{-\frac{1}{d}-1} d\lambda$; therefore the recursive relation is given by:

$$c(x+1)f(x+1) = (cx+kt+1)f(x) - tkf(x-1); c<0; x = 0,1,2,\ldots \quad (30)$$

with $f(-1) = 0$.

Case (ii): When *c=0*

$$g(\lambda) = lim_{c\to\infty}\frac{1}{k}\left(1-\frac{c}{k}\lambda\right)^{\frac{1}{c}-1}$$
$$= \frac{1}{k}e^{-\frac{\lambda}{k}};\ \lambda > 0$$

which is an exponential distribution with mean *k*.
The recursive relation is given by:

$$f(x+1) = \frac{tk}{kt+1}f(x); x = 0,1,2,\ldots$$

with

$$f(0) = \int_0^\infty e^{-\lambda t}\frac{1}{k}\left(1+\frac{d}{k}\lambda\right)^{-\frac{1}{d}-1} d\lambda$$

Case (iii): When *c>0*

$$g(\lambda) = \frac{1}{k}\left(1-\frac{c}{k}\lambda\right)^{\frac{1}{c}-1}; 0<\lambda<\frac{k}{c}$$

Therefore,

$$f(x) = \frac{t^x}{kx!}\int_0^{\frac{k}{c}} e^{-\lambda t}\lambda^x\left(1-\frac{c}{k}\lambda\right)^{\frac{1}{c}-1} d\lambda$$

Let $u = e^{-\lambda t}\lambda^x$ and $dv = \left(1-\frac{c}{k}\lambda\right)^{\frac{1}{c}-1} d\lambda$. Therefore the recursive relation is given by:

$$c(x+1)f(x+1) = (1+tk+cx)f(x) - tkf(x-1); x = 0,1,2,\ldots \quad (31)$$

with $f(-1) = 0$.

7. Mixing with Inverse Gaussian Distributions

7.1 Inverse Gaussian distribution

$$g(\lambda) = \left(\frac{\phi}{2\pi\lambda^3}\right)^{\frac{1}{2}} e^{-\frac{\phi(\lambda-\mu)^2}{2\lambda\mu^2}}; \lambda > 0 \quad (32)$$

Therefore,

$$f(x) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}}\frac{t^x}{x!}e^{\frac{\phi}{\mu}}\int_0^\infty \lambda^{x-\frac{3}{2}}e^{-\left(t+\frac{\phi}{2\mu^2}\right)\lambda-\frac{\phi}{2\lambda}}d\lambda$$

Put $u = e^{-\left(t+\frac{\phi}{2\mu^2}\right)\lambda-\frac{\phi}{2\lambda}}$ and $dv = \lambda^{x-\frac{3}{2}}d\lambda$. Therefore the recursive formula is given by:

$$(2\mu^2 t+\phi)x(x+1)f(x+1) = \mu^2(2x-1)xtf(x) + \mu^2\phi t^2 f(x-1); x = 0,1,2,\ldots \quad (33)$$

with $f(-1) = 0$.

7.2 Reciprocal Inverse Gaussian distribution

$$g(\lambda) = \left(\frac{\phi}{2\pi\lambda}\right)^{\frac{1}{2}} exp\left\{-\frac{\phi(1-\mu\lambda)^2}{2\mu^2\lambda}\right\}; \lambda > 0 \tag{34}$$

Therefore,

$$f(x) = \left(\frac{\phi}{2\pi}\right)^{\frac{1}{2}} \frac{t^x}{x!} \int_0^\infty \lambda^{x-\frac{1}{2}} exp\left\{-\lambda t - \frac{\phi(1-\mu\lambda)^2}{2\mu^2\lambda}\right\} d\lambda$$

Put $u = exp\left\{-\frac{\phi}{2\mu^2\lambda} - \frac{(2t+\phi)\lambda}{2}\right\}$ and $dv = \lambda^{x-\frac{1}{2}} d\lambda$; therefore the recursive relation becomes:

$$\mu^2(\phi + 2t)x(x+1)f(x+1) = t\mu^2(2x+1)xf(x) + \phi t^2 f(x-1); x = 0,1,2, \ldots \tag{35}$$

with $f(-1) = 0$ and $f(0) = \left(\frac{\phi}{\phi+2t}\right)^{\frac{1}{2}} exp\left[\frac{\phi}{\mu}\left(1 - \sqrt{1 + \frac{2t}{\phi}}\right)\right]$.

7.3 Generalized Inverse Gaussian distribution

The pdf of a Generalized Inverse Gaussian distribution is given by:

$$g(\lambda) = \frac{\left(\frac{\psi}{\phi}\right)^{\frac{v}{2}}}{2K_v(\sqrt{\psi\phi})} \lambda^{v-1} exp\left\{-\frac{1}{2}\left(\psi\lambda + \frac{\phi}{\lambda}\right)\right\}; \lambda > 0 \tag{36}$$

where

$$K_v(w) = \frac{1}{2}\int_0^\infty y^{v-1} e^{-\frac{w}{2}\left(y+\frac{1}{y}\right)} dy$$

which is the modified Bessel function of the third kind.

The parameters ϕ, ψ and v take values in one of the following ranges:

i. $\phi>0$, $\psi\geq 0$ if $v<0$
ii. $\phi>0$, $\psi>0$ if $v=0$
iii. $\phi\geq 0$, $\psi=0$ if $v>0$

Therefore,

$$f(x) = \frac{\left(\frac{\psi}{\phi}\right)^{v} t^x}{2K_v(\sqrt{\psi\phi})x!} \int_0^\infty e^{-\lambda t} \lambda^{x+v-1} exp\left\{-\frac{1}{2}\left(\psi\lambda + \frac{\phi}{\lambda}\right)\right\} d\lambda$$

Put $u = exp\left\{-\lambda t - \frac{1}{2}\left(\psi\lambda + \frac{\phi}{\lambda}\right)\right\}$ and $dv = \lambda^{x+v-1} d\lambda$. Therefore the recursive relation becomes:

$$(2t + \psi)x(x+1)f(x+1) = 2t(x+v)xf(x) + \phi t^2 f(x-1); x = 0,1,2, \ldots \tag{37}$$

with $f(-1) = 0$.

8. Other Distributions

8.1 Confluent Hypergeometric mixing distribution

The Confluent Hypergeometric distribution is given by:

$$g(\lambda) = \frac{\lambda^{a-1}(1+\lambda)^{c-a-1} e^{-k\lambda}}{\Gamma(a)\psi(a;c;k)}; \lambda > 0;\ -\infty < a < \infty; -\infty < c < \infty \tag{38}$$

where

$$\psi(a;c;k) = \frac{1}{\Gamma(a)}\int_0^\infty y^{a-1}(1+y)^{c-a-1}e^{-ky}dy$$

$$f(x) = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^x}{x!}\frac{\lambda^{a-1}(1+\lambda)^{c-a-1}e^{-k\lambda}}{\Gamma(a)\psi(a;c;k)}d\lambda$$

Put $u = \lambda^{x+a-1}e^{-(k+t)\lambda}$ and $dv = (1+\lambda)^{c-a-1}d\lambda$.. Therefore the recursive relation is given by:

$$(k+t)x(x+1)f(x+1) = (c+x-1-k-t)xtf(x) + (x+a-1)t^2 f(x-1) \quad (39)$$

8.2 Half-Normal mixing distribution

The pdf of Half Normal mixing distribution is given by:

$$g(\lambda) = \frac{2}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-\frac{(\lambda-\mu)^2}{2\sigma^2}};\ \lambda > 0; -\infty < \mu < \infty; \sigma^2 > 0 \quad (40)$$

$$f(x) = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^x}{x!}.\frac{2}{\sqrt{2\pi\sigma^2}}e^{-\frac{(\lambda-\mu)^2}{2\sigma^2}}d\lambda$$

Put $u = e^{-\lambda t-\frac{(\lambda-\mu)^2}{2\sigma^2}}$ and $dv = \lambda^x$; therefore the recursive relation is given by:

$$(x+2)f(x+2) = t^2\sigma^2 f(x) - (t\sigma^2 - \mu)tf(x+1); x = 0,1,2,\ldots \quad (41)$$

with $f(-1) = 0$.

9. Conclusion

The classical method of integration; Integration by Parts is used to obtain recursive relations in Mixed Poisson distributions which have been obtained by other methods. This technique is simple and straight forward provided the choice of *u* and *dv* in the integrand is done correctly to facilitate integration.

Recursive relation is one of the useful computation method once the initial conditions are known. It has been applied in random sums of independent and identically distributed (iid) random variables; which is useful in calculating total aggregate claims in insurance.

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