Some Types of Ideals on KS-Semigroups

Sajda mohammed1* Sundus Jaber2
1. Faculty of Education For Girls, Kufa university, Iraq
2. Faculty of Education For Girls, Kufa university, Iraq

*E-mail of the corresponding author Sajidak.mohammed@uokufa.edu.iq

Abstract:
In this paper we introduce a new types of ideals in KS-Semigroups in ordinary and fuzzy sense,we called it KS-H-ideal and fuzzy KS-H-ideal and study its properties.

1. Introduction

Keywords: Semigroup, BCK algebra, H-ideal, P-ideal, ideal, KS–semigroup,

2. Preliminary
This section contains some basic concepts we needed it in this paper.

Definition (2.1)[9]: An algebraic system (X, *, 0) is called a BCK algebra if it satisfies the following conditions:
1. ((x * y) * (x * z)) * (z * y) = 0,
2. (x * (x * y)) * y = 0,
3. x * x = 0,
4. 0 * x = 0
5. If x * y = 0 and y * x = 0 then x = y, for all x, y, z ∈ X.

Remarks (2.2)[6]: Let X be a BCK algebra then:

 a) A partial ordering ” ≤” on X can be defined by x ≤ y if and only if x * y = 0.

 b) A BCK-algebra X has the following properties:
1. x * 0 = x.
2. If x*y=0 and y*z=0 imply x*z=0.
3. If x*y=0 implies (x*z)(y*z)=0 and (z*y)(z*x)=0.
4. If (x*y)*z=(x*z)*y.

Definition (2.3)[9]
A KS-semigroup is a non-empty set X with two binary operation ” * “ and ” . “ , and a constant 0 satisfies the following axioms:
1. (X, *) is a BCK-algebra
2. (X, .) is a semigroup,
3. \( x(y \ast z) = (x \ast y) \ast (x \ast z) \) and \( (x \ast y)z = (x \ast z) \ast (y \ast z) \), for all \( x, y, z \in X \).

**Definition (2.4) [9]** A non-empty subset \( S \) of \( X \) with binary operation \( \ast \) and \( \cdot \), is called **sub KS-semigroup** of \( X \) if it satisfies the following conditions:

1. \( x \ast y \in S \quad \forall \ x, y \in S \).
2. \( x, y \in S \quad \forall \ x, y \in S \).

**Definition (2.5) [7]** A strong **KS-semigroup** is a KS-semigroup \( X \) satisfying: \( x \ast y = x \ast x, y \) for all \( x, y \in X \).

**Lemma (2.6) [7]**: Let \( X \) be a strong KS-semigroup then:

1. \( x, y \in X \)
2. \( x \ast y = 0 \quad \forall \ x, y \in X \).

**Definition (2.7) [11]** A non-empty subset \( I \) of a BCK-algebra \( X \) is called a **H-ideal** of \( X \) if the following conditions hold:

1. \( 0 \in I \).
2. \( x \ast (y \ast z) \in I \quad \text{and} \quad y \in I \implies x \ast z \in I \), for all \( x, y, z \in X \).

**Definition (2.8) [7]** Let \( X \) and \( Y \) be KS-semigroups, a mapping \( f : X \rightarrow Y \) is called a **KS-Semigroup homomorphism** (briefly homomorphism) if \( f(x \ast y) = f(x) \ast f(y) \) and \( f(xy) = f(x) \ast f(y) \) for all \( x, y \in X \).

Let \( f : X \rightarrow Y \) be a KS-semigroup homomorphism. Then the set \( \{x \in X / f(x) = 0 \} \) is called the **kernel of** \( f \), and denote by \( \ker f \). Moreover the set \( \{f(x) \in Y / x \in X \} \) is called the **image of** \( f \) and denote by \( \text{Im}f \).

**Definition (2.9) [9]** A non-empty subset \( A \) of a semigroup \( X \) is said to be **left (resp. right) stable** if \( xa \in A \) (resp. \( ax \in A \)) whenever \( x \in X \) and \( a \in A \).

Both left and right stable is called **two-sided stable** or simply **stable**.

**Definition (2.10) [9]** A non-empty subset \( A \) of a KS-semigroup \( X \) is said to be **left (resp. right) ideal** of \( X \) if:

1. \( A \) is left (resp. right) stable subset of \( (X, \ast) \) and
2. \( x \ast y \in A \) and \( y \in A \) imply that \( x \in A \), for all \( x, y \in X \).

If \( A \) is both left and right ideal then \( A \) is called two-sided ideal or simply an ideal.

**Remarks (2.11)**

- Let \( A \) be a KS-ideal then \( 0 \in A \) for all \( x \in X \) since \( A \neq \emptyset \) then \( \exists a \in A \) such that \( xa, ax \in A \), put \( x = 0 \) we get \( 0 \in A \).
- Let \( f : X \rightarrow Y \) be a KS-semigroup homomorphism then \( f(0) = 0 \) and if \( x \leq y \), then \( f(x) \leq f(y) \).
- \( \ker f \) is a KS-ideal.[7]

**Definition (2.12) [9]** A non-empty subset \( A \) of a KS-Semigroup \( X \) is said to be **left (resp. right) p-ideal** of \( X \) if:

1. \( A \) is a left (resp. right) stable subset of \( (X, \ast) \) and
2. \( (x \ast y) \ast z \in A \) and \( y \ast z \in A \) imply that \( x \ast z \in A \), for all \( x, y, z \in X \).

If \( A \) is both left and right p-ideal then \( A \) is called two sided ideal or simply p-ideal.

**Theorem (2.13) [7]** Every p-ideal of a **KS-Semigroup** \( X \) is an ideal but converses is not true.

**Definition (2.14) [10]** The element \( e \) is called a unity in a KS-semigroup \( X \) if \( e, x = xe = x \quad \forall x \in X \).

**Definition (2.15) [11]** Let \( X \) be a non-empty set, a **fuzzy subset** of \( X \) is a function \( \mu : X \rightarrow [0, 1] \).

**Remarks (2.16) [11]**

Let \( X \) be a non-empty set then:

1. Each fuzzy subset \( \lambda \) and \( \mu \) of \( X \), if \( \lambda \subseteq \mu \) mean that \( \lambda(x) \leq \mu(x) \) for all \( a \in X \).
2. If \( x \leq y \) implies that \( \mu(x) \geq \mu(y) \) for all \( x, y \in X \).
3. If \( \mu, \nu \) be two fuzzy set of \( X \) and \( a \leq b \) such that \( a, b \in [0, 1] \), then \( \mu_b \leq \mu_a \).

58
**Definition (2.17) [9]** Let X be a non-empty set and let \( \mu \) be the fuzzy subset of \( X \) for a fixed \( 0 \leq t \leq 1 \), the set \( \mu_t = \{ x \in X \mid \mu(x) \geq t \} \) is called an upper level set of \( \mu \).

**Definition (2.18) [9]** Let \( f : X \rightarrow Y \) be a mapping of KS-Semigroup and \( \mu \) be a fuzzy subset of \( Y \). The map \( \mu^f \) is the pre-image of \( \mu \) under \( f \) if \( \mu^f = \{ f(x) \mid \forall x \in X \} \).

**Definition (2.19) [5]** Let \( X \) be a BCK-algebra. A fuzzy subset \( \mu \) of \( X \) is called a fuzzy subalgebra of \( X \) if it satisfies the following condition: \( \mu(x * y) \geq \min\{\mu(x), \mu(y)\} \quad \forall x, y \in X \).

**Definition (2.20) [11]** A fuzzy set \( \mu \) of BCK-algebra \( X \) is called a fuzzy H-ideal if it satisfies:

1. \( \mu(0) \geq \mu(x) \quad \forall x \in X \),
2. \( \mu(x * z) \geq \min\{\mu(x * y), \mu(y)\} \quad \forall x, y, z \in X \).

**Definition (2.21) [9]** A fuzzy set \( \mu \) defined on \( X \) is called a fuzzy subKS-seminigroup of \( X \) if it satisfies the following conditions:

1. \( \mu(x_1 * x_2) \geq \min\{\mu(x_1), \mu(x_2)\} \),
2. \( \mu(x_1 x_2) \geq \min\{\mu(x_1), \mu(x_2)\} \quad \forall x_1, x_2 \in X \).

**Definition (2.22) [9]** A fuzzy subset \( \mu \) of \( X \) is called a left fuzzy KS-ideal if:

- KSI1: \( \mu(0) \geq \mu(x) \)
- KSI2: \( \mu(x) \geq \min\{\mu(x * y), \mu(y)\} \)
- KSI3: \( \mu(xa) \geq \min\{\mu(x), \mu(a)\} \quad \text{for all } x, y, a \in X \).

A fuzzy subset \( \mu \) of \( X \) is called a right fuzzy KS-ideal if it satisfies KSI1, KSI2 and

- KSI4: \( \mu(ax) \geq \min\{\mu(x), \mu(a)\} \quad \text{for all } x, y, a \in X \).

A fuzzy subset \( \mu \) of \( X \) is called a fuzzy KS-ideal if it is both left and right fuzzy KS-ideal of \( X \).

**Definition (2.23) [9]** A fuzzy subset \( \mu \) of \( X \) is called a left fuzzy p-ideal if:

- KSP1: \( \mu(0) \geq \mu(x) \)
- KSP2: \( \mu(x * z) \geq \min\{\mu(x * y), \mu(y * z)\} \)
- KSP3: \( \mu(xa) \geq \min\{\mu(x), \mu(a)\} \quad \text{for all } x, y, z, a \in X \).

A fuzzy subset \( \mu \) of \( X \) is called a right fuzzy p-ideal if it satisfies KSP1, KSP2 and

- KSP4: \( \mu(ax) \geq \min\{\mu(x), \mu(a)\} \quad \text{for all } x, y, a \in X \).

A fuzzy subset \( \mu \) of \( X \) is called a fuzzy p-ideal if it is both left and right fuzzy p-ideal of \( X \).

**Theorem (2.24) [9]** Every left (resp. right) fuzzy p-ideal of \( X \) is a left (resp. right) fuzzy KS-ideal of \( X \).

**Definition (2.25) [9]** Let \( \lambda \) and \( \mu \) be the fuzzy subsets in a set \( X \). The cartesian product \( \lambda \times \mu : X \times X \rightarrow [0, 1] \) is defined by \( (\lambda \times \mu)(x, y) = \min\{\lambda(x), \mu(y)\} \) for all \( x, y \in X \).

**Definition (2.26) [9]** Let \( V \) be a fuzzy subset in \( X \). The strong fuzzy relation on \( X \) that is a fuzzy relation on \( V \) is \( \rho_V \) given by \( \rho_V(x, y) = \min\{\rho(x), \rho(y)\} \).

3. **KS-H-Ideal**

**Definition (3.1)**

A non-empty subset \( I \) of a KS-semigroup \( X \) is said to be left KS-H-ideal of \( X \) if it satisfies:

1. If \( x * (y * z) \in I \) and \( y \in I \) then \( x * z \in I \)
2. \( xa \in I \) (resp. \( ax \in I \)) whenever \( x \in X \) and \( a \in I \).

A non-empty subset \( I \) is said to be right KS-H-ideal of \( X \) if it satisfies (1) and (3):

\[ xa \in I \text{ whenever } x \in X \text{ and } a \in I . \]

If \( I \) is both left and right KS-H-ideal then \( I \) is called two-sided KS-H-ideal or simply KS-H-ideal.

**Example (3.2)**
Let $X = \{0, 1, 2, 3\}$ be defined by the following tables:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Then by usual calculations we can prove that $X$ is a **KS-semigroup**. If $A = \{0, 1\}$ then $A$ is a **KS-H-ideal** of a KS-semigroup $X$.

**Proposition (3.3)**

Let $X$ be a KS-semigroup and let $A$ be left (resp. right) KS-H-ideal of $X$ then $A$ is a left (resp. right) KS-ideal of $X$.

**Proof:**

Let $A$ be a left KS-H-ideal of $X$ then $A$ is a stable. Now, let $x, y \in X$ such that $x^* y \in A$ and $y \in A$ then $x^* y = x^*(y^* 0) \in A$ and $y \in A$ then $x \in A$ and since $A$ is a left KS-H-ideal then it is clear $A$ is a left KS-H-ideal.

**Proposition (3.4)**

Let $I$ and $J$ are left (resp. right) KS-H-ideal of KS-Semigroups $X$ then $I \cap J$ is a left (resp. right) KS-H-ideal of $X$.

**Proof:** it is clear

**Proposition (3.5)**

Let $I$ and $J$ are left (resp. right) KS-H-ideal of KS-Semigroups $X$ then $I \cup J$ is a left (resp. right) KS-H-ideal if $I \subseteq J$ or $J \subseteq I$.

**Proof:** it is clear

**Proposition (3.6)**

Let $I$ and $J$ are left (resp. right) KS-H-ideal of KS-Semigroups $X$ then $I \times J$ is a left (resp. right) KS-H-ideal of $X \times X$.

**Proof:**

Let $I$ and $J$ are left KS-H-ideal of KS-Semigroups $X$

For any $x_1, x_2, a_1, a_2 \in X$ and $(x_1, x_2) \in X \times X, (a_1, a_2) \in I \times J$ then

$((x_1, x_2)(a_1, a_2)) = (x_1 a_1, x_2 a_2)$, since $I, J$ are left $KS-H$-ideal so $x_1 a_1 \in I$ and $x_2 a_2 \in J$ then $(x_1 a_1, x_2 a_2) \in I \times J$ therefore $(x_1, x_2)(a_1, a_2) \in I \times J$

If $(x_1, x_2)(y_1, y_2) \in I \times J$ and $(y_1, y_2) \in I \times J$ then $(x_1, x_2)(y_1, y_2) \in I \times J$

$(x_1, x_2)(y_1, y_2) \in I \times J$ then $(x_1, x_2)(y_1, y_2)(z_1, z_2) \in I \times J$

$(x_1, x_2)(y_1, y_2)(z_1, z_2) \in I \times J$ then $(x_1, x_2)(y_1, y_2)(z_1, z_2) \in I \times J$

$x_1, y_1 \in I$ and $y_2 \in J$ then $x_1 z_1 \in I$ and $x_2 z_2 \in J$ (since $I, J$ are left $KS-H$-ideal) so $(x_1, x_2)(z_1, z_2) \in I \times J$

$x_1, x_2 \in I \times J$ then $x_1^* z_1 \in I$ and $x_2^* z_2 \in J$ (since $I, J$ are left $KS-H$-ideal) so $(x_1, x_2)(z_1, z_2) \in I \times J$

$x^* z \in I \times J$

then $I \times J$ is a left KS-H-ideal.
Let \( f : X \to Y \) be a KS-semigroup epimorphism if \( A \) is a left (resp. right) KS-H-ideal in \( X \) then \( f(A) \) is a left (resp. right) KS-H-ideal in \( Y \).

**Proof:**
Let \( A \) be a left KS-H-ideal of \( X \). let \( a^- = f(a) \in f(A) \) and \( y \in Y \) where \( a \in A \) since \( f \) onto then there exists \( x \in X \) such that \( f(x) = y \) since \( xa \in A \quad \forall x \in X \) and \( a \in A \) so \( f(xa) \in f(A) \) but \( f(xa) = f(x)f(a) = y a^- \) [since \( f \) is epimorphism]

therefore \( f(A) \) is stable. Now, Suppose that \( f(x), f(y), f(z) \in f(A) \) for some \( x, y, z \in A \) such that \( f(x)[f(y)*f(z)] \in f(A) \) and \( f(y) \in f(A) \), since \( f \) is a homomorphism then \( f(x)[f(y)*f(z)] = f(x*[y*z]) \in f(A) \) and since \( f(y) \in f(A) \), thus \( x*[y*z] \in A \), \( y, A \to x*z \in A \) [since \( A \) is KS-H-ideal] therefore \( f(x*z) \in f(A) \) but \( f(x)*f(z) = f(x*z) \in f(A) \) so hence \( f(A) \) is a left KS-H-ideal.

**Proposition (3.8)**
Let \( f : X \to Y \) be a KS-semigroup homomorphism then ker \( f \) is a KS-H-ideal of \( X \).

**Proof:**
Let \( f : X \to Y \) be a KS-semigroup homomorphism, since ker \( f \) is an ideal of \( X \) [3] it follows that ker \( f \) is a stable, now, let \( x, y, z \in X \) such that \( x*[y*z] \in \ker f \) and \( y \in \ker f \), so \( f(x*[y*z]) = 0 \) and \( f(y) = 0 \) so \( f(x)[f(y)*f(z)] = 0 \) and \( f(y) = 0 \) so \( f(x*[0*f(z)] = 0 \) so \( f(x) = 0 \) so \( x \in \ker f \), now, \( f(x*z) = f(x)f(z) = 0 \) therefore \( x*z \in \ker f \) hence ker \( f \) is a KS-H-ideal.

**Proposition (2.1.9)**
Let \( I \) be a KS-ideal of KS-semigroup \( X \) such that \( x*y = y*x \) for all \( x \neq 0 \) and \( y \neq 0 \) and \( x*y = 0 \) just when \( x = 0 \). Then \( I \) is a KS-H-ideal of \( X \).

**Proof:**
First since \( I \) is a KS-ideal so \( xa \in I \quad \forall x \in X \) and \( a \in I \), Now let \( x, y, z \in X \) and \( x*(y*z), y \in I \) to prove \( x*z \in I \). There are several cases:

1) If \( x, y, z \neq 0 \) and \( x \neq y \neq z \) so
\[
x*(y*z) = x*(z*y) \quad \text{[since \( x*y = y*x \ \forall x, y \neq 0 \) and \( x*y = 0 \)]}
\[
= (x*y)*x \quad \text{[since \( x*y = y*x \ \forall x, y \neq 0 \) and \( x*y = 0 \)]}
\[
= (x*y)*y \quad \text{[since \( x*(y*z) = (x*y)*y \) in BCK]}
\[
= (x*z)*y \in I \quad \text{[since \( x*y = y*x \ \forall x, y \neq 0 \) and \( x*y = 0 \)]}
\]
and \( y \in I \) then \( x*z \in I \) [since \( I \) is a KS-ideal].

2) If \( x = 0 \) and \( y, z \neq 0 \) so
\[
0*(y*z) \in I \quad \text{and} \quad y \in I \quad I \text{is a KS-ideal so}
\[
x*z = 0*z \in I.
\]

3) If \( x = y = 0 \) and \( z \neq 0 \) then
\[
0*(0*z) \in I \quad \text{and} \quad z \in I \quad I \text{is a KS-ideal so}
\[
x*z = 0*z \in I.
\]

4) If \( x = 0, z = 0 \) then \( x*z \in I \) [by the same way of (3)].
5) If \( x \neq 0 \), and \( y = 0 \) then
\[ x^* (0^* z) \in I \text{ , } 0 \in I \]
then \( x \in I, 0 \in I \).

so if \( z = 0 \) then \( x^* z = x \in I \) and
If \( z \neq 0 \) then \( (x^* y)^* z = 0 \) and \( x \in I \) so \( x^* z \in I \) \([\text{I is a KS-ideal}].\)

6) If \( x = 0, y = 0, z = 0 \) then \( 0^* (0^* 0) = 0 \in I \) so \( x^* z \in I \).

7) If \( x \neq 0, y \neq 0, z = 0 \)
\[ x^* (y^* 0) \in I \text{ and } y \in I \]
then \( x^* y \in I \) and \( y \in I \) \([\text{since I is a KS-ideal}].\)
so \( x^* z = x^* 0 \in I \)

8) If \( x = 0, y \neq 0, z = 0 \) then \( 0^* (y^* 0) \in I \) so \( x^* z \in I \).

9) If \( x = 0, z \neq 0, y \neq 0 \) then \( 0^* (y^* z) \in I \) so \( x^* z \in I \).

10) If \( z = 0, x \neq 0, y = 0 \)
\[ x^* (y^* z) = x^* 0 \in I \text{ and } z = 0 \in I \rightarrow x^* z = x^* 0 \in I \]
\[ \text{[since I is a KS-ideal].} \]

Then \( I \) is a KS-H-ideal.

4. Fuzzy KS-H-Ideals

In this section, we define the notion of the fuzzy KS-H-ideal of KS-semigroup \( X \) and and prove some results and examples.

**Definition (4.1)**

A fuzzy subset \( \mu \) of KS-semigroup \( X \) is called a **left fuzzy KS-H-ideal** if the following conditions hold:

**KSH1** \( \mu(0) \geq \mu(x), \)

**KSH2** \( \mu(x^* z) \geq \min\{\mu(x^* (y^* z)), \mu(y)\}, \)

**KSH3** \( \mu(xa) \geq \min\{\mu(x), \mu(a)\}, \text{ for all } x, y, z, a \in X. \)

A fuzzy subset \( \mu \) is called a **right fuzzy KS-H-ideal** if it satisfies **KSH1**, **KSH2** and **KSH4**: \( \mu(ax) \geq \min\{\mu(x), \mu(a)\}, \text{ for all } x, y, a \in X. \)

A fuzzy subset \( \mu \) is called a **fuzzy KS-H-ideal** if it is both left and right fuzzy KS-H-ideal of \( X \).

**Example (4.2)**

Let \( X = \{0, 1, 2, 3\} \) be a KS-semigroup defined by the following tables:

<table>
<thead>
<tr>
<th>( \ast )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( \cdot )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Define a fuzzy subset \( \mu : X \rightarrow [0, 1] \) by \( \mu(0) = 0.4 \), \( \mu(x) = 0.2 \ \forall x \neq 0 \in X \). By usual calculations, we can prove that \( \mu \) is a left fuzzy KS-H-ideal of \( X \).

**Remark (4.3)**

Every fuzzy KS-H-ideal is a fuzzy KS-ideal.

**Proof:**

Let \( \mu \) be a fuzzy KS-H-ideal of \( X \) since \( x^* 0 = x \ \forall x \in X \) \([\text{by Remark (2.2)}].\)
\( \mu(x) = \mu(x^*0) \geq \min\{\mu(x^*(y^*0)), \mu(y)\} \)
\[
= \min\{\mu(x^*y), \mu(y)\}
\]

thus \( \mu(x) \geq \min\{\mu(x^*y), \mu(y)\}\)

and since \( \mu(0) \geq \mu(x) \forall x \in X \) and \( \mu(ax) \geq \min\{\mu(x), \mu(a)\}, \mu(ax) \geq \min\{\mu(a), \mu(x)\} \forall x, a \in X \)

Hence \( \mu \) is a fuzzy KS-ideal.

**Proposition (4.4)**

Let \( \mu \) and \( \lambda \) are left (resp. right) fuzzy KS-H-ideal of KS-semigroup \( X \) then \( \mu \cap \lambda \) is a left (resp. right) fuzzy KS-H-ideal.

**Proof:** Let \( \mu \) and \( \lambda \) are left fuzzy KS-H-ideal of \( X \) then
\[
(\mu \cap \lambda)(0) = \min\{\mu(0), \lambda(0)\} \geq \min\{\mu(x), \lambda(x)\} = (\mu \cap \lambda)(x) \forall x \in X \quad \text{[since \( \mu, \lambda \) are left fuzzy KS \(-H\)-ideal]}
\]

now, \( (\mu \cap \lambda)(ax) = \min\{\mu(ax), \lambda(ax)\} \geq \min\{\mu(x), \lambda(x)\} \}
\[
= \min\{\mu(x^*y), \lambda(x^*y)\} \quad \forall x, a \in X
\]

so \( (\mu \cap \lambda)(x^*z) = \min\{\mu(x^*(y^*z)), \lambda(x^*(y^*z))\} \quad \forall x, y, z \in X \)

hence \( \mu \cap \lambda \) is a left fuzzy KS-H-ideal.

**Proposition (4.5)**

Let \( \mu \) and \( \nu \) be two fuzzy KS-H-ideal of KS-semigroup \( X \) if \( \mu \subseteq \nu \) or \( \nu \subseteq \mu \) then \( \mu \cup \nu \) is a fuzzy KS-H-ideal.

**Proof:**

Let \( \mu \) and \( \nu \) are fuzzy KS-H-ideal of \( X \), without loss of generality we may assume that let \( \mu \subseteq \nu \)

since \( \mu \) and \( \nu \) are fuzzy KS-H-ideal and \( x, y, a \in X \) so \( \mu(0) \geq \mu(x) \quad \text{and} \quad \nu(0) \geq \nu(x) \forall x \in X \) therefore,
\[
(\mu \cup \nu)(0) = \max\{\mu(0), \nu(0)\} \geq \max\{\mu(x), \nu(x)\} = (\mu \cup \nu)(x) \quad \text{since} \quad \mu \quad \text{and} \quad \nu \quad \text{are fuzzy KS-H-ideal so}
\]
\[
(\mu \cup \nu)(ax) = \min\{\max\{\mu(x), \mu(a)\}, \max\{\nu(x), \nu(a)\}\} \quad \forall x, a \in X
\]

so \( (\mu \cup \nu)(x^*y) = \min\{\max\{\mu(x^*y), \mu(y)\}, \max\{\nu(x^*y), \nu(y)\}\} \quad \forall x, y \in X \)

hence \( \mu \cup \nu \) is a fuzzy KS-H-ideal.

**Proposition (4.6)**

Let \( I \) and \( J \) are left (resp. right) fuzzy KS-H-ideal of KS-semigroup \( X \) then \( I \times J \) is a left (resp. right) fuzzy KS-H-ideal of \( X \times X \).

**Proof:**

Let \( I \) and \( J \) are left fuzzy KS-H-ideal of \( X \) then
\[
(I \times J)(0,0) = \min\{I(0), J(0)\} \geq \min\{I(x), J(y)\} = (I \times J)(x, y) \quad \forall x, y \in X \times X \quad \text{[since \( I, J \) are left fuzzy KS-H-ideal of \( X \), let \( (x, x) \in X \times X \quad \text{and} \quad (a_1, a_2) \in I \times J \quad \text{so}
\]
\[
(I \times J)(x, x)(a_1, a_2) = (I \times J)(xa_1, xa_2) \geq \min\{I(xa_1), J(xa_2)\} \quad \geq \min\{\min\{I(x), I(a_1)\}, \min\{J(x), J(a_2)\}\}
\]
\[
= \min\{I(x), J(x)\}, \min\{I(a_1), J(a_2)\}\}
\]

\[
= \min\{I \times J\}(x, x), \min\{I \times J\}(a_1, a_2) \}
\]

\[
\text{now, let} \quad (x_1, x_2) \quad \text{and} \quad (z_1, z_2) \in X \times X,
\]

\[
(I \times J)(x_1, x_2)(z_1, z_2) = (I \times J)(x_1^*z_1, x_2^*z_2) = \min\{I(x_1^*z_1), J(x_2^*z_2)\}
\]
\[
\geq \min\{\min\{I(x_1^*y_1^*z_1), I(y_1)\}, \min\{J(x_2^*y_2^*z_2), J(y_2)\}\}
\]
\[
= \min\{I \times J\}(x_1, x_2), \min\{I \times J\}(z_1, z_2) \}
\]

\[
= \min\{I \times J\}(y_1, y_2)
\]

\[
= \min\{I \times J\}(x_1, x_2)(y_1, y_2) = (I \times J)(y_1, y_2)
\]

hence \( I \times J \) is a fuzzy KS-H-ideal.
hence $I \times J$ is a left fuzzy KS-H-ideal.

**Proposition (4.7)**

If $A$ be a left (resp. right ) KS-H-ideal of KS-semigroup $X$ then $\forall$ $0 < t \leq 1$ their exist a left (resp. right ) fuzzy KS-H-ideal $\mu$ such that $A = \mu_I$.

**Proof:**

Let $A$ be a left KS-H-ideal and $\mu$ be defined by

$$\mu(x) = \begin{cases} t & \text{if } x \in A \\ 0 & \text{if } x \notin A \\ \text{where } o < t \leq 1 \end{cases}$$

Let $x \in A$ then $\mu(x) = t$, then $x \in \mu_I$, so $A \subseteq \mu_I$, and if $x \notin A$ then $\mu(x) \geq t$, then $x \in A$ so $A = \mu_I$

Since $A$ is a left KS-H-ideal so $0 \in A$ then $\mu(0) = t \geq \mu(x)$ \quad $\forall x \in X$, now let $x, a \in X$ there are several cases:

1. If $x, a \in X$ so $xa \in A$ since $A$ is a left $H$-ideal $\mu(xa) = t \geq \min\{\mu(x), \mu(a)\}$.

2. If $x \notin A$ and $a \notin A$ then $\mu(xa) \geq \min\{\mu(x), \mu(a)\} = 0$

3. If at most one of $x,a$ belong to $A$, then at most one of $\mu(x)$ and $\mu(a)$

   is equal to 0. therefore $\mu(xa) \geq \min\{\mu(x), \mu(a)\} = 0$

   $\mu(xa) \geq \min\{\mu(x), \mu(a)\}$ \quad $\forall x, a \in X$ let $x^*(y^* z), y \in X$ there are several cases:

   1. If $x^*(y^* z), y \in A$ then $x^* z \in A$ since $A$ is a left $KS-H$-ideal so $(x^* z) = t \geq \min\{\mu(x^*(y^* z)), \mu(y)\}$

   2. If $x^*(y^* z), y \notin A$ then $\mu(x^*(y^* z)) = \mu(y) = 0$ so $\mu(x^* z) \geq \min\{\mu(x^*(y^* z)), \mu(y)\}$

3. If at most one of $x^*(y^* z), y$ belong to $A$, then at most one of $\mu(x^*(y^* z))$ and $\mu(y)$ is equal to 0, therefore $x^* z \notin A \mu(x^* z) \geq \min\{\mu(x^*(y^* z)), \mu(y)\} = 0$ so $\mu(x^* z) \geq \min\{\mu(x^*(y^* z)), \mu(y)\}$ for all $x, y, z \in X$. hence $\mu$ is a left fuzzy KS-H-ideal.

**Proposition (4.8)**

Let $\mu$ be a left (resp. right) fuzzy KS-H-ideal in $X$-semigroup $X$ then a fuzzy set $\mu^+$ defined by $\mu^+ = \mu(x) + 1 - \mu(0)$ is a left (resp. right) fuzzy KS-H-ideal such that $\mu \subseteq \mu^+$.

**Proof:**

Let $\mu$ be a left fuzzy $KS-H$-ideal and $\mu^+$ is a fuzzy set then

$$\mu^+(0) = \mu(0) + 1 - \mu(0) = 1 \geq \mu^+(x) \quad \forall x \in X$$. now, let $x, a \in X$ so

$$\mu^+(xa) = \mu(xa) + 1 - \mu(0)$$

$$\geq \min\{\mu(x), \mu(a)\} + 1 - \mu(0) \quad [\text{since } \mu \text{ is a left fuzzy } KS-H \text{-ideal}]$$

$$= \min\{\mu(x) + 1 - \mu(0), \mu(a) + 1 - \mu(0)\} = \min\{\mu^+(x), \mu^+(a)\}$$.

Let $x, y, z \in X$ so

$$\mu^+(x^*(y^* z)) = \mu(x^*(y^* z)) + 1 - \mu(0)$$

$$\geq \min\{\mu(x^*(y^* z)), \mu(y)\} + 1 - \mu(0) \quad [\text{since } \mu \text{ is a left fuzzy } KS-H \text{-ideal}]$$

$$= \min\{\mu(x^*(y^* z)) + 1 - \mu(0), \mu(y) + 1 - \mu(0)\}$$

$$= \min\{\mu^+(x^*(y^* z)), \mu^+(y)\}$$

hence $\mu^+$ is a left fuzzy KS-H-ideal.

**Proposition (4.9)**
Let \( f : X \rightarrow Y \) be a homomorphism if \( \mu \) is a left (resp. right) fuzzy KS-H-ideal of Y then \( \mu^f \) is a left (resp. right) fuzzy KS-H-ideal of X.

**Proof:**

Let \( \mu \) be a left fuzzy KS-H-ideal of Y then \( \mu^f(0) = \mu(f(0)) \geq \mu(f(x)) = \mu^f(x) \quad \forall x \in X \).

Now, let \( x, a \in X \) so

\[
\mu^f(xa) = \mu(f(xa)) = \mu(f(x)f(a)) \quad \geq \min\{\mu(f(x)), \mu(f(a))\} \quad [f \text{ is a homomorphism}]
\]

\[
\geq \min\{\mu(f(x)), \mu(f(a))\} \quad [\text{since } \mu \text{ is a left } H-\text{ideal}]
\]

\[
= \min\{\mu^f(x), \mu^f(a)\}
\]

Let \( x, y, z \in X \), \( \mu^f(x^*z) = \mu(f(x^*z)) = \mu(f(x)^*f(z)) \)

\[
\geq \min\{\mu(f(x)^*f(y))\}. \mu(f(y)) \quad [\text{since } \mu \text{ is a left } H-\text{ideal}]
\]

\[
= \min\{\mu(f(x^*(y^*z)), \mu(f(y))\}
\]

\[
= \min\{\mu^f(x^*(y^*z)), \mu^f(y)\}
\]

Hence \( \mu^f \) is a left fuzzy KS-H-ideal of X.

**Proposition (4.10)**

Let \( f : X \rightarrow Y \) be an epimorphism if \( \mu^f \) is a left (resp. right) fuzzy KS-H-ideal of X then \( \mu \) is a left (resp. right) fuzzy KS-H-ideal of Y.

**Proof:**

Let \( \mu^f \) is a fuzzy KS-H-ideal of X and

let \( y \in Y \) then \( \exists x \in X \) such that \( f(x) = y \)

\[
\mu(y) = \mu(f(x)) = \mu^f(x) \leq \mu^f(0) \quad [\text{since } \mu^f \text{ is a left fuzzy } KS - H-\text{ideal}]
\]

\[
= \mu(f(0)) = \mu(0) \quad [\text{by remark } 2.13]
\]

Now, let \( x, a \in Y \) then \( \exists t, m \in X \) such that

\[
f(t) = x \quad \text{and} \quad f(m) = a \quad \text{then} \quad \mu(xa) = \mu(f(t)f(m)) = \mu(f(tm))
\]

\[
= \mu^f(tm) \geq \min\{\mu^f(t), \mu^f(m)\} = \min\{\mu(f(t)), \mu(f(m))\} = \min\{\mu(x), \mu(a)\}
\]

So, let \( x, y, z \in Y \) then \( \exists a, b, c \in X \) such that \( f(a) = x \), \( f(b) = y \), \( f(c) = z \) then

\[
\mu(x^*c) = \mu(f(a)^*f(c)) = \mu(f(a^*c)) = \mu^f(a^*c)
\]

\[
\geq \min\{\mu^f(a^*(b^*c)), \mu^f(b)\} = \min\{\mu(f(a^*(b^*c)), \mu(f(b))\} = \min\{\mu(f(a)^*[f(b)^*f(c)]), \mu(f(b))\}
\]

\[
= \min\{\mu(x^*(y^*z)), \mu(y)\}
\]

Hence \( \mu \) is a fuzzy KS-H-ideal of Y.

**Proposition (4.13)**

Let \( I \) be a non-empty subset of a strong KS-semigroup X then \( I \) is a left (resp. right) KS-H-ideal of X if and only if \( \chi_I \) is a left (resp. right) fuzzy KS-H-ideal where \( \chi_I : X \rightarrow [0, 1] \) define as follows:

\[
\chi_I = \begin{cases} 
1 & \text{if } x \in I \\
0 & \text{if } x \notin I
\end{cases}
\]

**Proof:**

It is clear that \( \chi_I \) is a fuzzy set.

Suppose that \( I \) is a left KS-H-ideal of X and \( x, y, a \in X \)

since \( 0 \in X \) so \( 0a = 0 \in I \) \( \forall a \in I \) then \( \chi_I(0) = 1 \geq \chi_I(x) \quad \forall \ x \in X \).

There are several cases: let \( x, a \in X \)
1– If \( x \in I, a \in I \) \( \text{so } xa \in I \) [since \( I \) is a left fuzzy KS-\( H \)-ideal] 
\[
\chi_I(x) = 1, \chi_I(a) = 1 \text{ and } \chi_I(xa) = 1 \text{ then } \chi_I(xa) \geq \min(\chi_I(x), \chi_I(a))
\]

2– If \( x \in I, a \notin I \) \( \text{so } xa \notin I \) \( \text{thus } \chi_I(x) = 1, \chi_I(a) = 0 \) \( \text{and } \chi_I(xa) = 0 \) 
\[
\text{then } \chi_I(xa) \geq \min(\chi_I(x), \chi_I(a))
\]

3– If \( x \notin I, a \in I \) \( \text{so } xa \in I \) \( \text{thus } \chi_I(x) = 0, \chi_I(a) = 1 \) \( \text{and } \chi_I(xa) = 1 \) 
\[
\text{then } \chi_I(xa) \geq \min(\chi_I(x), \chi_I(a))
\]

4– If \( x \notin I, a \notin I \) \( \text{so } xa \notin I \) \( \text{thus } \chi_I(x) = 0, \chi_I(a) = 0 \) \( \text{and } \chi_I(xa) = 0 \) 
\[
\text{then } \chi_I(xa) \geq \min(\chi_I(x), \chi_I(a))
\]
so \( \chi_I(xa) \geq \min(\chi_I(x), \chi_I(a)) \)

In similar way we can prove that \( \chi_I(x^* z) \geq \min(\chi_I(x^*(y^* z)), \chi_I(y)) \) \( \forall x, y, z \in X \).

Hence \( \chi_I \) is a left KS-H-ideal.

Conversely, assume that \( X \) is a strong KS-semigroup and let \( \chi_I \) is a fuzzy KS-H-ideal of \( X \)
let \( x \in X \) and \( a \in I \) since \( X \) is a strong KS-semigroup so \( xa^* a = 0 \) and since \( 0 \in I \) so \( xa^* a \in I \) and \( a \in I \) then \( \chi_I(xa) \geq \min(\chi_I([xa^* a]), \chi_I(a)) = \min(\chi_I(0), \chi_I(a)) = \chi_I(a) = 1 \)
so \( xa \in I \)
now, let \( x^*(y^* z) \in I \) and \( y \in I \) so \( \chi_I(x^*(y^* z)) = \chi_I(\chi_I(y) = 1 \) since \( \chi_I \) is fuzzy KS-\( H \)-ideal we have \( \chi_I(x^*(y^* z)) \geq \min(\chi_I(x^*(y^* z)), \chi_I(y)) = 1 \) so \( x^* z \in I \)
therefore \( I \) is a left KS-H-ideal

**Proposition (4.14)**

If \( \mu \) be a right fuzzy KS-H-ideal of KS-semigroup \( X \) with left identity \( e \) and satisfying the condition \( (xy)z = (xz)y \) \( \forall x, y, z \in X \) then \( \mu \) is a left fuzzy H-ideal of \( X \).

**Proof:**

Let \( \mu \) be a right fuzzy KS-H-ideal of KS-semigroup \( X \) with left identity 
and let \( x, a \in X \) 
\[
\mu(xa) = \mu((ex)a) = \mu((ea)x) = \mu(ax) \geq \min(\mu(a), \mu(x)) \ [	ext{by hypothesis}]
\]
\[
\mu(xa) \geq \min(\mu(x), \mu(a))
\]
since \( \mu(0) \geq \mu(x) \) \( \forall x \in X \) and 
\[
\mu(x^* z) \geq \min(\mu(x^*(y^* z)), \mu(y)) \ [	ext{if } \mu \text{ is a right fuzzy KS-} H \text{-ideal}]
\]
therefore \( \mu \) is a left fuzzy KS-H-ideal.

**Corollary (4.15)**

Every right fuzzy KS-H-ideal of KS-semigroup \( X \) with left identity \( e \) satisfying the condition is a fuzzy KS-H-ideal of \( X \).

**Proof:**

Let \( \mu \) be a right fuzzy KS-H-ideal with left identity then \( \mu \) is a left fuzzy KS-H-ideal therefore \( \mu \) is a fuzzy KS-H-ideal.

**Proposition (4.16)**

Let \( \mu \) be a fuzzy set of strong KS-semigroup \( X \) if \( \mu \) is a left fuzzy KS-H-ideal then \( \mu_t \) left KS-H-ideal where \( t \in [0, \mu(0)] \).

**Proof:**

Let \( \mu_t \) be a left fuzzy KS-H-ideal of \( X \), and \( t \in [0, \mu(0)] \). let \( x \in \mu_t \)
since $\mu(0) \geq t$ then $0 \in \mu_I$ then $\mu_I \neq \emptyset$.

now, let $x \in X$ and $a \in \mu_I$ so $\mu(a) \geq t$ since $X$ is a strong so $xa^*a = 0$ and since $I$ is a left fuzzy KS-H-ideal so

$$\mu(xa) \geq \min\{\mu(xa^*(a^*0)), \mu(0)\} = \min\{\mu(xa), \mu(a)\} = \min\{\mu(0), \mu(a)\} = \mu(a) \geq t$$

$$\Rightarrow xa \in \mu_I$$

let $x^*(y^*z) \in \mu_I$ and $y \in \mu_I$ then $\mu(x^*(y^*z)) \geq t$ and $\mu(y) \geq t$, since $I$ is a left fuzzy KS-H-ideal so $\mu(x^*(y^*z)) \geq t$ so $x^*y \in \mu_I$. Therefore $\mu_I$ is a left KS-H-ideal.

**Theorem (4.17)**

Let $X$ be a KS-semigroup and $\mu, \lambda$ be two fuzzy sets in $X$ such that $\mu \times \nu$ is a fuzzy KS-H-ideal of $X$ then:

1. either $\mu(x) \leq \mu(0)$ or $\lambda(x) \leq \lambda(0)$ for all $x \in X$.
2. If $\mu(x) \leq \mu(0)$ for all $x \in X$ then either $\mu(x) \leq \lambda(0)$ or $\lambda(x) \leq \lambda(0)$.
3. If $\lambda(x) \leq \lambda(0)$ for all $x \in X$ then either $\mu(x) \leq \mu(0)$ or $\lambda(x) \leq \mu(0)$.
4. either $\mu$ or $\lambda$ is a fuzzy KS-H-ideal of $X$.

**Proof:**

since $\mu \times \nu$ is a fuzzy KS-H-ideal of $X$ then it is fuzzy sub KS semigroup by [ ], so (1),(2) and (3) satisfied by [12]. Now, to prove 4. Since by (1) either $\mu(x) \leq \mu(0)$ or $\lambda(x) \leq \lambda(0)$ for all $x \in X$ without loss of generality we may assume that $\lambda(x) \leq \lambda(0)$ it follows from (3) that either $\mu(x) \leq \mu(0)$ or $\lambda(x) \leq \mu(0)$ if $\lambda(x) \leq \mu(0)$ $\forall x \in X$ then

$$\lambda(x,a) = \min\{\mu(0), \lambda(x,a)\} = (\mu \times \lambda)(0, x, a) = (\mu \times \lambda)((0, 0, x)) \geq \min\{\mu \times \lambda(0, x), (\mu \times \lambda)(0, a)\}$$

$$= \min\{\mu(0), \lambda(x), \lambda(a)\} = \lambda(\lambda(x), \lambda(a)) = \lambda(x, a)$$

Now,

$$\lambda(x^*z) = \min\{\mu(0), \lambda(x^*z)\} = (\mu \times \lambda)(0, x^*z) = (\mu \times \lambda)((0, 0, x^*z)) = \min\{\mu \times \lambda(0, x), \lambda(0, y)\}$$

$$= \min\{\lambda(x^*(y^*z)), \lambda(y)\}$$

so $\lambda$ is a fuzzy KS-H-ideal in $X$.

If $\lambda(y) \leq \mu(0)$ is not satisfied then $\lambda(y) > \mu(0)$ for some $y \in X$ and by our assumption, $\mu(x) \leq \mu(0)$ for all $x \in X$ we have $\lambda(0) \geq \lambda(y) > \mu(0) \geq \lambda(x)$ i.e $\lambda(0) \geq \lambda(x) \forall x \in X$.

therefore $(\mu \times \lambda)(x, 0) = \min\{\mu(x), \lambda(0)\} = \mu(x)$ and

$$\mu(x, a) = (\mu \times \lambda)(x, a, 0)$$

$$= \min\{\mu(x \times \lambda), \mu(a)\} = \min\{\mu(x), \mu(a)\}$$.

so,

$$\mu(x^*z) = (\mu \times \lambda)(x^*z, 0) = (\mu \times \lambda)((x^*z), (0, a)) = \min\{\mu \times \lambda(0, x^*z), \lambda(0)\} = \min\{\mu, \lambda(0)\} = \min\{\mu(x), \lambda(0)\}.$$


The IISTE is a pioneer in the Open-Access hosting service and academic event management. The aim of the firm is Accelerating Global Knowledge Sharing.

More information about the firm can be found on the homepage: http://www.iiste.org

CALL FOR JOURNAL PAPERS

There are more than 30 peer-reviewed academic journals hosted under the hosting platform.

Prospective authors of journals can find the submission instruction on the following page: http://www.iiste.org/journals/ All the journals articles are available online to the readers all over the world without financial, legal, or technical barriers other than those inseparable from gaining access to the internet itself. Paper version of the journals is also available upon request of readers and authors.

MORE RESOURCES

Book publication information: http://www.iiste.org/book/

Academic conference: http://www.iiste.org/conference/upcoming-conferences-call-for-paper/

IISTE Knowledge Sharing Partners

EBSCO, Index Copernicus, Ulrich's Periodicals Directory, JournalTOCS, PKP Open Archives Harvester, Bielefeld Academic Search Engine, Elektronische Zeitschriftenbibliothek EZB, Open J-Gate, OCLC WorldCat, Universe Digital Library, NewJour, Google Scholar