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Error Estimate for the Finite Element Method of the Fractional Perona-Malik

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Abstract

In this paper, we present the finite element method for the time-space fractional Perona-Malik equation on a finite domain $\Omega=[0,\ T]\times[0,\ X]$. Here the fractional derivative indicates the Caputo derivative for the first-order time and space derivatives of orders $(0<\alpha<1)$ and $(0<\beta<1)$, respectively. The fully discrete scheme is considered by using a finite element method and error estimate in L^2- norm is proved of $O\left((\Delta t)^{2-\alpha}+(\Delta x)^{3-\beta}\right)$.

Key words: Perona-Malik equation, finite element methods, fractional time-space derivatives, the left Caputo fractional derivatives

1. Introduction

Perona-Malik equation is a technique aiming at reducing the noise in the corrupted images. In the process of imaging and transmission, images are often been polluted by a lot of noise, which not only influences the vision effect heavily, but also takes some difficulties into image analyzing and understanding. Therefore, image smoothing plays an important role in image preprocessing (Feng *et al.* 2014). The existence and uniqueness are proved of solutions of Perona-Malik equation for H^1 initial data in (Greer & Bertozzi 2004) . In (Handlovicova & Kriva 2005) are derived and proved the error estimates in the L^2-norm for the explicit fully discrete numerical finite volume scheme for Perona-Malik equation. Since, we see in (Zhao *et al.* (2013) there exist two kinds of the fractional derivatives, left(right) Caputo derivative and left(right) Riemann-Liouville derivative for both the time and space derivatives with order $(n-1<\mu<n)$, for any positive integer n . In this paper, we paid attention to study of Perona-Malik equation by the concept of the Caputo derivative for the first-order time and space derivatives of orders $(0<\alpha<1)$ and $(0<\beta<1)$, respectively on a finite domain. Therefore we named the left Caputo time-space fractional Perona-Malik equation on a finite domain. The new equation can be analyzed by using the finite element method and we show that the order of the error estimate in L^2- norm . This paper is organized as follows. In section **2** we present the fractional Perona-Malik equation with assumptions, properties of this equation and the important lemma of the Gauss function G_σ . The discretization of time-space fractional meshes in a finite domain is shown in section **3**. In section **4** we present the weak form of a new equation. The linear finite element approximation scheme of a new equation is shown in section **5**. In section **6** we present the error estimate. The conclusions are shown in section **7**.

2. The Fractional Perona-Malik Equation

First, we shall present the Perona-Malik equation has the following form as In (Handlovicova & Kriva 2005)**:**

$$\frac{\partial}{\partial t}u(x,\ t)-\nabla\cdot\left(g\left(\left|\nabla G_{\sigma}*u(x,\ t)\right|\right)\nabla u(x,\ t)\right)=0,\qquad \text{in } \Omega, \tag{1}$$

$$u(t,\ 0)=u(t,\ X)=0,\qquad \text{for } t\in[0,\ T], \tag{2}$$

$$u(0,\ x)=u^{0},\qquad \text{for } x\in(0,\ X), \tag{3}$$

Where $\Omega=[0,\ T]\times[0,\ X]$ is a finite domain. The functions g and G_{σ} are a Lipchitz continuous decreasing functions and $G_{\sigma}\in C^{\infty}(\Omega)$ is a smoothing kernel (e.g., **Gauss function**). In this paper, we paid attention to study of Perona-Malik equation by the concept of the Caputo derivative for the first-order time and space derivatives of orders $(0<\alpha<1)$ and $(0<\beta<1)$, respectively on a finite domain. Therefore we will define the left Caputo time-space fractional Perona-Malik equation on a finite domain with the following new form :

$$ {}_{0}^{C}D_{t}^{\alpha}u^{n}(x,\ t)-\nabla\cdot\left(g\left(\left|\nabla G_{\sigma}*u^{n}(x,\ t)\right|\right){}_{0}^{C}D_{x}^{\beta}u^{n}(x,\ t)\right)=0,\quad \text{in } \Omega,\ n=1,\dots,j, \tag{4}$$

$$u(t_{n},\ 0)=u(t_{n},\ X)=0,\qquad \text{for } t_{n}\in[0,\ T], \tag{5}$$

$$u(0,\ x_{n})=u^{0},\qquad \text{for } x_{n}\in(0,\ X). \tag{6}$$

Hence, Gauss function $G_{\sigma}\in C^{\infty}(\Omega)$ can be presented as in (Handlovicova *et al.* 2002):

$$\nabla G_{\sigma}*u^{n}(x,\ t)=\int_{x_{n-1}}^{x_{n}}\nabla G_{\sigma}(x_{n}-\xi)u^{n}(\xi,\ t)\,d\xi\qquad \text{for } n=1,\dots,j, \tag{7}$$

Where ${}_{0}^{C}D_{t}^{\alpha}u^{n}(x,\ t)$ and ${}_{0}^{C}D_{x}^{\beta}u^{n}(x,\ t)$ refer to the left Caputo fractional derivatives, For any positive integer m and real numbers $(m-1<\alpha<m)$ and $(m-1<\beta<m)$. We consider the definition of the left Caputo fractional derivatives in general case as in (Zhao *et al.* (2013),

$$ {}_{0}^{C}D_{t}^{\alpha}y(x,\ t)=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}\frac{1}{(t_{m}-s)^{\alpha-m+1}}\frac{d^{m}}{ds^{m}}y(x,\ s)ds, \qquad \text{and}$$

$$ {}_{0}^{C}D_{x}^{\beta}y(x,\ t)=\frac{1}{\Gamma(m-\beta)}\int_{0}^{x}\frac{1}{(x_{m}-s)^{\beta-m+1}}\frac{d^{m}}{ds^{m}}y(s,\ t)ds,$$

Where $\alpha,\ \beta > 0$ are the orders of Caputo fractional derivatives for the time and the space, respectively. $\Gamma(\cdot)$ is the gamma function. But in this paper we take $m = 1$, then the orders of Caputo fractional derivatives for the time and the space it's become $(0 < \alpha < 1)$ and $(0 < \beta < 1)$, respectively.

2.1 Assumptions

(a) By ε we denote a generic constant independent of $\Delta t,\ \Delta x,\ n, \ldots$ which attains in general different values in different places.

(b) Assume $u(x_n,\ t_n) \in L^\infty(\Omega)$. $u_x(x_n,\ t_n),\ u_{xx}(x_n,\ t_n) \in L^\infty\left(L^2(\Omega)\right)$ and $u_t(x_n,\ t_n)$, $u_{tt}(x_n,\ t_n) \in L^\infty\left(L^2(\Omega)\right)$.

(c) For simplicity we shall write $u(x_n,\ t_n) = u^n$.

2.1 Basic Properties

(1) Since the functions $g,\ G_\sigma$ are **locally Lipschitz –continuous** with respect $y,\ z$ for any constants $L_g,\ L_{G_\sigma}$ respectively such that

$$|g(y) - g(z)| \le L_g |y - z| \quad \text{and} \quad |G_\sigma(y) - G_\sigma(z)| \le L_{G_\sigma} |y - z|.$$

(2) We shall denote by $(\cdot,\cdot)$ is the scalar product in $L^2(\Omega)$ as in (Debnath & Mikusński 1990) i.e.

$$(u,\ v) = \int_\Omega uv\ dx, \qquad u, v \in L^2(\Omega),$$

The norm $\|u\|_{L^2(\Omega)} = (u,\ u)^{1/2}$, $u \in L^2(\Omega)$ and the seminorm

$$|u|_{H_0^1(\Omega)} = \left(\int_\Omega (\nabla u)^2 dx \right)^{1/2}.$$

Next, in the following lemma we sketch how bounds of the functions g and G_σ on a finite domain. *Lemma 2.1***.** Since the functions $g,\ G_\sigma \in C^\infty(\Omega)$ are satisfy the property (1) and (7) on a finite domain, for $n = 1, \ldots, j$ and $r = 1, \ldots, n$, gives

(1) $$\left| g\left(\left| \nabla G_\sigma * u^n \right| \right) \right| \le C L_g L_{G_\sigma} \Delta x_n,$$

(2) $$\sum_{r=1}^{n} \left| g\left(\left| \nabla G_\sigma * u_h^r \right| \right) \right| \le C L_g L_{G_\sigma} \sum_{r=1}^{n} \Delta x_r.$$

Proof. First, from (5), this observation immediately yields

$$g\left(\left| \nabla G_\sigma * u(0,\ t_n) \right| \right) = 0. \tag{8}$$

By using (8) and the property (1) then

(1) $$\left|g\left(\left|\nabla G_\sigma * u(x_n,\ t_n)\right|\right)-g\left(\left|\nabla G_\sigma * u(0,\ t_n)\right|\right)\right| \le L_g \left\|\nabla G_\sigma * u(x_n,\ t_n)\right|-\left|\nabla G_\sigma * u(0,\ t_n)\right\|$$

$$= L_g \left|\nabla G_\sigma * u(x_n,\ t_n)\right|$$

$$\le L_g \int_{x_{n-1}}^{x_n} \nabla G_\sigma(x_n-\xi) u(\xi,\ t_n)\, d\xi ,$$

From (7), assumption (a) and property (1), then

$$\le L_g \left| u(x_n,\ t_n)\right|_{L^\infty(\Omega)} \left| -G_\sigma(x_n-\xi)\Big|_{x_{n-1}}^{x_n}\right|$$

$$\le \varepsilon\, L_g \left|-G_\sigma(x_n-x_n)+G_\sigma(x_n-x_{n-1})\right|$$

$$\le \varepsilon\, L_g L_{G_\sigma} \left|x_n-x_{n-1}-x_n+x_n\right|$$

$$\le \varepsilon\, L_g\, L_{G_\sigma} \Delta\, x_r.$$

(2) Similar for the proof of part **(1)**. ■

3. Discretization of Time-Space Fractional Meshes in a Finite Domain

Define the time and space meshes, respectively as follows:

(A) Let $[0,\ T] = [t_{n-1},\ t_n] = \bigcup_{k=1}^{n} [t_{k-1},\ t_k],$ for $n = 1, ..., j$,

$\Delta t_n = t_n - t_{n-1},\quad n = 1, ..., j\quad ,\ \Delta t_k = t_k - t_{k-1},\quad k = 1, ..., n$ and $\max_{1\le n\le j} \Delta t_n = \Delta t$.

(B) Let $[0,\ X] = [x_{n-1},\ x_n] = \bigcup_{r=1}^{n} [t_{r-1},\ t_r],$ for $n = 1, ..., j$,

$\Delta x_n = x_n - x_{n-1},\quad n = 1, ..., j\quad ,\ \Delta x_r = x_r - x_{r-1},\quad r = 1, ..., n$ and $\max_{1\le n\le j} \Delta x_n = \Delta x$.

4. The Weak Form

we shall introduce a weak form of the linear finite element approximation for the left Caputo time-space fractional Perona-Malik equation on a finite domain. By multiplying the equations (4)-(6) in both sides by an arbitrary $v^n \in H_0^1(\Omega)$ and using the integral by part to find regular exact solution $u^n \in H_0^1(\Omega)$ such that:

$$\left({}_0^C D_t^\alpha u^n,\ v^n\right) + \varepsilon\, L_g L_{G_\sigma} \Delta x_n \left({}_0^C D_x^\beta u^n,\ \nabla v^n\right) = 0,\quad \text{for } n = 1, ..., j,\quad \forall v^n \in H_0^1(\Omega) \tag{9}$$

$$\left(u^0,\ v^n\right) = \left(\varphi(x_n),\ v^n\right) \tag{10}$$

5. The Linear Finite Element Approximation Scheme

We say that $u_h^n \in V_h^n$, $n=1,\dots,j$ is the piecewise linear finite element approximate solution of the left Caputo time-space fractional Perona-Malik equation on a finite domain (4)-(6) such that

$$\left(\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n} b_k^n \left(u_h^k - u_h^{k-1}\right),\ v_h^n\right) + \varepsilon\, L_g L_{G_\sigma}\left(\frac{1}{\Gamma(1-\beta)}\sum_{r=1}^{n} \Delta x_r\, b_r^n \left(u_h^r - u_h^{r-1}\right),\ \nabla v_h^n\right) = 0,\ \forall v_h^n \in V_h^n \quad (11)$$

$$\left(u_h^0,\ v_h^n\right) = \left(\varphi(x_n),\ v_h^n\right)\ , \quad (12)$$

Where

$$V_h^n = \left\{\ v_h^n \in C(\Omega) : v\big|_{[x_{r-1},x_r]} \in P_r,\ r=1,\dots,n\ \right\},\ n=1,\dots,j\ ,$$

P_r denote the set of piecewise polynomials of degree not exceeding r and

$$b_k^n = \frac{1}{\Delta t_k}\int_{t_{k-1}}^{t_k} \frac{ds}{(t_n - s)^{\alpha}},\ k=1,\dots,n \quad \text{and} \quad b_r^n = \frac{1}{\Delta x_r}\int_{x_{r-1}}^{x_r} \frac{ds}{(x_n - s)^{\beta}},\ r=1,\dots,n\ .$$

Now, we present the error estimation of the finite element method.

6. The Error Estimate

First, we introduce the two definitions which will be used frequently in the following theorem. *Definition 6.1* (Debnath & Mikusński 1990). In a Hilbert space V^n, $n=1,\dots,j$, then **Cauchy-Schwartz inequality** is holds

$$\left\|\left(u^n, v^n\right)\right\| \le \|u^n\|\|v^n\| \qquad \text{for each } u^n,\ v^n,\ n=1,\dots,j. \quad (13)$$

Definition 6.2 (Quarteroni & Valli 1997). The two norms $\|\cdot\|$ and $|\cdot|$ on v^n are equivalent if there exist two positive constants ε_1 and ε_2 such that

$$\varepsilon_1\|v^n\| \le |v^n| \le \varepsilon_2\|v^n\| \qquad \text{for each } u^n,\ v^n,\ n=1,\dots,j\ . \quad (14)$$

Lemma 6.1 (Jun & Tang 2013). Suppose that positives δ_n $n=0,1,\dots,j$, satisfy

$$b_n^n \delta_n \le \sum_{k=2}^{n}(b_k^n - b_{k-1}^n)\delta_{k-1} + b_1^n \mu + \omega, \qquad n=1,\dots,j\,,$$

Where $\omega,\ \mu$ are positives. Then

$$\delta_n \le \mu + \omega / b_1^n\,, \qquad n=1,\dots,j\ .$$

Theorem 6.1 . Let u^n be the exact solution satisfy (9)-(10) and u_h^n be the piecewise linear finite element approximate satisfy (11)-(12). For $(0<\alpha<1)$, $(0<\beta<1)$ and by using assumption (a), the error estimation is given by,

$$\| u_h^n - u^n \|_{L^2(\Omega)} \le \varepsilon \left((\Delta t)^{2-\alpha} + (\Delta x)^{3-\beta} \right), \quad n = 1, \dots, j.$$

Proof. First, by subtracting Equation (9) from (11). setting $v^n = v_h^n \in V_h^n$, we get

$$\left(\frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} b_k^n \left(u_h^k - u_h^{k-1}\right) - {}_0^C D_t^{\alpha} u^n,\ v_h^n\right) +$$

$$\varepsilon\, L_g L_{G_\sigma} \left(\frac{1}{\Gamma(1-\beta)} \sum_{r=1}^{n} \Delta x_r\, b_r^n \left(u_h^r - u_h^{r-1}\right) - \Delta x_n\, {}_0^C D_x^{\beta} u^n,\ \nabla v_h^n\right) = 0. \tag{15}$$

By adding and subtracting the following terms

$$\left(\frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} b_k^n \left(u^k - u^{k-1}\right),\ v_h^n\right) \quad \text{and} \quad \varepsilon\, L_g L_{G_\sigma} \left(\frac{1}{\Gamma(1-\beta)} \sum_{r=1}^{n} \Delta x_r\, b_r^n \left(u^r - u^{r-1}\right),\ \nabla v_h^n\right)$$

We set $e^n = u_h^n - u^n$ then the Equation (15) become

$$\left(\frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} b_k^n \left(e^k - e^{k-1}\right),\ v_h^n\right) + \left(\frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} b_k^n \left(u^k - u^{k-1}\right) - {}_0^C D_t^{\alpha} u^n,\ v_h^n\right)$$

$$+ \varepsilon\, L_g L_{G_\sigma} \left(\frac{1}{\Gamma(1-\beta)} \sum_{r=1}^{n} \Delta x_r\, b_r^n \left(e^r - e^{r-1}\right),\ \nabla v_h^n\right)$$

$$+ \varepsilon\, L_g L_{G_\sigma} \left(\frac{1}{\Gamma(1-\beta)} \sum_{r=1}^{n} \Delta x_r\, b_r^n \left(u^r - u^{r-1}\right) - \Delta x_n\, {}_0^C D_x^{\beta} u^n,\ \nabla v_h^n\right) = 0. \tag{16}$$

Over $k = 1, \dots, n$ for $t_n \in [0,\ T]$, by adding and subtracting the term $\sum_{k=1}^{n} b_{k-1}^n e^{k-1}$ then we get

$$\sum_{k=1}^{n} b_k^n e^k - \sum_{k=1}^{n} b_{k-1}^n e^{k-1} + \sum_{k=1}^{n} b_{k-1}^n e^{k-1} - \sum_{k=1}^{n} b_k^n e^{k-1} = b_n^n e^n - b_0^n e^0 + \sum_{k=2}^{n} \left(b_{k-1}^n - b_k^n\right) e^{k-1}$$

$$= b_n^n e^n - b_1^n e^0 - \sum_{k=2}^{n} \left(b_k^n - b_{k-1}^n\right) e^{k-1} \tag{17}$$

Now, we substitute (17) in (16) and multiplying the both sides by the quantity $\Gamma(1-\alpha)$, then

$$b_n^n (e^n,\ v_h^n) + \varepsilon\, L_g L_{G_\sigma} \left(\frac{\Gamma(1-\alpha)}{\Gamma(1-\beta)} \sum_{r=1}^{n} \Delta x_r\, b_r^n \left(e^r - e^{r-1}\right),\ \nabla v_h^n\right) = b_1^n (e^0,\ v_h^n)$$

$$+ \left(\sum_{k=2}^{n} \left(b_k^n - b_{k-1}^n\right) e^{k-1},\ v_h^n\right) - \Gamma(1-\alpha) \left(\frac{1}{\Gamma(1-\alpha)} \sum_{k=1}^{n} b_k^n \left(u^k - u^{k-1}\right) - {}_0^C D_t^{\alpha} u^n,\ v_h^n\right)$$

$$-\varepsilon\, L_g L_{G_\sigma}\Gamma(1-\alpha)\Big(\frac{1}{\Gamma(1-\beta)}\sum_{r=1}^{n}\Delta x_r\, b_r^n\left(u^r-u^{r-1}\right)-\Delta x_n\, {}_0^C D_x^\beta u^n,\ \nabla v_h^n\Big). \tag{18}$$

Choosing $v_h^n = e^n$, by using Cauchy-Schwartz inequality (13), from the property (2) and the definition 6.2 , we get

$$\left|b_n^n\right|\left\|e^n\right\|_{L^2(\Omega)}^2 +\varepsilon\, L_g L_{G_\sigma}\frac{\Gamma(1-\alpha)}{\Gamma(1-\beta)}\sum_{r=1}^{n}\Delta x_r\left|b_r^n\right|\left\|e^r-e^{r-1}\right\|_{L^2(\Omega)}\left\|e^n\right\|_{L^2(\Omega)}\le$$

$$\left|b_1^n\right|\left\|e^0\right\|_{L^2(\Omega)}\left\|e^n\right\|_{L^2(\Omega)}+\sum_{k=2}^{n}\left|b_k^n-b_{k-1}^n\right|\left\|e^{k-1}\right\|_{L^2(\Omega)}\left\|e^n\right\|_{L^2(\Omega)}$$

$$+\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}b_k^n\left(u^k-u^{k-1}\right)-{}_0^C D_t^\alpha u^n\right\|_{L^2(\Omega)}\left\|e^n\right\|_{L^2(\Omega)}$$

$$+\varepsilon\, L_g L_{G_\sigma}\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\beta)}\sum_{r=1}^{n}\Delta x_r\, b_r^n\left(u^r-u^{r-1}\right)-\Delta x_n\, {}_0^C D_x^\beta u^n\right\|_{L^2(\Omega)}\left\|e^n\right\|_{L^2(\Omega)},$$

We divide the last equation by the quantity $\left\|e^n\right\|_{L^2(\Omega)}$, by using assumption (a) and since the term

$\varepsilon\, L_g L_{G_\sigma}\dfrac{\Gamma(1-\alpha)}{\Gamma(1-\beta)}\displaystyle\sum_{r=1}^{n}\Delta x_r\left|b_r^n\right|\left\|e^r-e^{r-1}\right\|_{L^2(\Omega)}\left\|e^n\right\|_{L^2(\Omega)}>0$, we obtain

$$\left|b_n^n\right|\left\|e^n\right\|_{L^2(\Omega)}\le\left|b_1^n\right|\left\|e^0\right\|_{L^2(\Omega)}+\sum_{k=2}^{n}\left|b_k^n-b_{k-1}^n\right|\left\|e^{k-1}\right\|_{L^2(\Omega)}$$

$$+\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}b_k^n\left(u^k-u^{k-1}\right)-{}_0^C D_t^\alpha u^n\right\|_{L^2(\Omega)}$$

$$+\varepsilon\, L_g L_{G_\sigma}\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\beta)}\sum_{r=1}^{n}\Delta x_r\, b_r^n\left(u^r-u^{r-1}\right)-\Delta x_n\, {}_0^C D_x^\beta u^n\right\|_{L^2(\Omega)},$$

by using lemma 6.1, we have

$$\left\|e^n\right\|_{L^2(\Omega)}\le\left\|e^0\right\|_{L^2(\Omega)}+\frac{1}{\left|b_1^n\right|}\Bigg\{\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}b_k^n\left(u^k-u^{k-1}\right)-{}_0^C D_t^\alpha u^n\right\|_{L^2(\Omega)}$$

$$+\varepsilon\, L_g L_{G_\sigma}\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\beta)}\sum_{r=1}^{n}\Delta x_r\, b_r^n\left(u^r-u^{r-1}\right)-\Delta x_n\, {}_0^C D_x^\beta u^n\right\|_{L^2(\Omega)}\Bigg\}, \tag{19}$$

we can solve the quantity $\left|b_1^n\right|$, as follows

$$\left|b_1^n\right|=\left|\frac{1}{\Delta t}\int_{t_0}^{t_1}\frac{ds}{\left(t_n-s\right)^\alpha}\right|\le\frac{1}{(T-0)}\left|\frac{-\left(t_n-s\right)^{1-\alpha}}{1-\alpha}\right|_{t_0}^{t_1}\le\frac{1}{T}\left|\frac{\left(t_n-t_0\right)^{1-\alpha}-\left(t_n-t_1\right)^{1-\alpha}}{1-\alpha}\right|, \text{ then}$$

$$\frac{1}{\left|b_1^n\right|} \leq T \left|\frac{1-\alpha}{(t_n - t_0)^{1-\alpha} - (t_n - t_1)^{1-\alpha}}\right| \leq T \ .$$

Then (19) become

$$\left\|e^n\right\|_{L^2(\Omega)} \leq \left\|e^0\right\|_{L^2(\Omega)} + \ T\,\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n} b_k^n\left(u^k - u^{k-1}\right) - {}_0^C D_t^\alpha u^n\right\|_{L^2(\Omega)}$$

$$+T\ \varepsilon\ L_g L_{G_\sigma}\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\beta)}\sum_{r=1}^{n}\Delta x_r\ b_r^n\left(u^r - u^{r-1}\right) - \Delta x_n\ {}_0^C D_x^\beta u^n\right\|_{L^2(\Omega)} . \qquad (20)$$

First, we will estimate the term $\left\|e^0\right\|_{L^2(\Omega)}$, by subtracting (10) from (12) and taking $v^n = v_h^n = e^0$ we get

$$(e^0,\ e^0) = 0,$$

by using Cauchy-Schwartz inequality (13), gives

$$\left\|e^0\right\|_{L^2(\Omega)}^2 \leq 0 .$$

Equation (20) become

$$\left\|e^n\right\|_{L^2(\Omega)} \leq \underbrace{T\,\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n} b_k^n\left(u^k - u^{k-1}\right) - {}_0^C D_t^\alpha u^n\right\|_{L^2(\Omega)}}_{A(1)}$$

$$\underbrace{+T\ \varepsilon\ L_g L_{G_\sigma}\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\beta)}\sum_{r=1}^{n}\Delta x_r\ b_r^n\left(u^r - u^{r-1}\right) - \Delta x_n\ {}_0^C D_x^\beta u^n\right\|_{L^2(\Omega)}}_{A(2)} . \qquad (21)$$

To estimate the term $\left|A(1)\right|$, we use assumption (a), (b), we get

$$\left|A(1)\right| = T\,\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{1}{(t_n - s)^\alpha}\frac{(u^k - u^{k-1})}{\Delta t_k}ds - \frac{1}{\Gamma(1-\alpha)}\int_{t_{n-1}}^{t_n}\frac{1}{(t_n - s)^\alpha}u_t^n ds\right\|_{L^2(\Omega)}$$

$$= T\,\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\alpha)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}\frac{1}{(t_n - s)^\alpha}\,u_t ds - \frac{1}{\Gamma(1-\alpha)}\int_{t_{n-1}}^{t_n}\frac{1}{(t_n - s)^\alpha}\,u_t^n ds\right\|_{L^2(\Omega)}$$

$$\leq T\,\Gamma(1-\alpha)\left\|\frac{1}{\Gamma(1-\alpha)}\int_{t_{n-1}}^{t_n}\frac{1}{(t_n - s)^\alpha}u_t ds - \frac{1}{\Gamma(1-\alpha)}\int_{t_{n-1}}^{t_n}\frac{1}{(t_n - s)^\alpha}\,u_t^n ds\right\|_{L^2(\Omega)}$$

$$\leq T\ \Delta t_n\left|\int_{t_{n-1}}^{t_n}\frac{1}{(t_n - s)^\alpha}\,ds\right|\left\|\frac{u_t - u_t^n}{\Delta t_n}\right\|_{L^2(\Omega)}$$

$$\leq T\ \Delta t_n\left|\int_{t_{n-1}}^{t_n}\frac{1}{(t_n - s)^\alpha}\,ds\right|\ \left\|\ u_{tt}^n\right\|_{L^2(\Omega)}$$

$$
\begin{aligned}
&\le T\ \Delta t_n \left| -(t_n - t_n)^{1-\alpha} + (t_n - t_{n-1})^{1-\alpha} \right| \ \| u_{tt}^n \|_{L^2(\Omega)} \\
&\le T \max_{1\le n\le j} \left\{ (\Delta t_n)^{2-\alpha} \| u_{tt}^n \|_{L^2(\Omega)} \right\} \\
&\le \varepsilon\ (\Delta t)^{2-\alpha} \| u_{tt}^n \|_{L^\infty(L^2(\Omega))} \\
&\le \varepsilon\ (\Delta t)^{2-\alpha} . \qquad (22)
\end{aligned}
$$

To estimate the term $|A(2)|$, we use assumption (a), (c), we get

$$
\begin{aligned}
|A(2)| &= T\ \varepsilon\ L_g L_{G_\sigma} \Gamma(1-\alpha) \left\| \frac{1}{\Gamma(1-\beta)} \sum_{r=1}^{n} \Delta x_r \int_{x_{r-1}}^{x_r} \frac{1}{(x_n - s)^\beta} \frac{(u^r - u^{r-1})}{\Delta x_r} ds - \frac{1}{\Gamma(1-\beta)} \Delta x_n \int_{x_{n-1}}^{x_n} \frac{1}{(x_n - s)^\beta} u_x^n ds \right\|_{L^2(\Omega)} \\
&= T\ \varepsilon\ L_g L_{G_\sigma} \frac{\Gamma(1-\alpha)}{\Gamma(1-\beta)} \left\| \sum_{r=1}^{n} \Delta x_r \int_{x_{r-1}}^{x_r} \frac{1}{(x_n - s)^\beta}\ u_x ds - \Delta x_n \int_{x_{n-1}}^{x_n} \frac{1}{(x_n - s)^\beta} u_x^n ds \right\|_{L^2(\Omega)} \\
&\le \varepsilon \left\| \Delta x_n \int_{x_{n-1}}^{x_n} \frac{1}{(x_n - s)^\beta}\ u_x ds - \Delta x_n \int_{x_{n-1}}^{x_n} \frac{1}{(x_n - s)^\beta}\ u_x^n ds \right\|_{L^2(\Omega)} \\
&\le \varepsilon\ (\Delta x_n)^2 \left| \int_{x_{n-1}}^{x_n} \frac{1}{(x_n - s)^\beta}\ ds \right| \left\| \frac{u_x - u_x^n}{\Delta x_n} \right\|_{L^2(\Omega)} \\
&\le \varepsilon\ (\Delta x_n)^2 \left| \int_{x_{n-1}}^{x_n} \frac{1}{(x_n - s)^\beta}\ ds \right| \| u_{xx}^n \|_{L^2(\Omega)} \\
&\le \varepsilon\ (\Delta x_n)^2 \left| -(x_n - x_n)^{1-\beta} + (x_n - x_{n-1})^{1-\beta} \right| \ \| u_{xx}^n \|_{L^2(\Omega)} \\
&\le \varepsilon \max_{1\le n\le j} \left\{ (\Delta x_n)^{3-\beta} \| u_{xx}^n \|_{L^2(\Omega)} \right\} \\
&\le \varepsilon\ (\Delta x)^{3-\beta} \| u_{xx}^n \|_{L^\infty(L^2(\Omega))} \\
&\le \varepsilon\ (\Delta x)^{3-\beta} . \qquad (23)
\end{aligned}
$$

Combining (22) and (23) into (21) then the proof is complete. ■

7. Conclusion

In this paper, we paid attention to study of Perona-Malik equation by the concept of the Caputo derivative for the first-order time and space derivatives of orders $(0<\alpha<1)$ and $(0<\beta<1)$, respectively on a finite domain. Therefore we defined the left Caputo time-space fractional Perona-Malik equation on a finite domain. The new equation can be solved analytically by using the finite element method and we show that the order of the error estimate in $L^2 -$ norm is proved of $O\left(\left(\Delta t\right)^{2-\alpha}+\left(\Delta x\right)^{3-\beta}\right)$.

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