Application of Marks to Computation of Ranks and Subdegrees of the Symmetric Group Acting on Ordered Pairs and on Ordered Triples

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Abstract
Ranks and subdegrees can be computed using combinatorial arguments, the Cauchy-Frobenius lemma and use of the concept of marks. However, the concept of marks has been given very little attention. In this paper we will apply the concept of marks to compute the ranks and subdegrees of the symmetric group $S_n(n = 5, 6, 7)$ and $S_n(n = 6, 7, 8)$ acting on ordered pairs and triples respectively.

Keywords: Ranks, Subdegrees, Marks

1. Introduction
Let $G$ be transitive on a set $X$ and let $G_x$ be the stabilizer in $G$ of a point $x \in X$. The orbits of $G_x$ on $X$ are known as suborbits of $G$. The number of these suborbits is known as the rank of $G$ on $X$ and the lengths of the suborbits is known as the subdegrees of $G$ on $X$. If $G$ acts on $X$ transitively, then its action is equivalent to the action on the cosets of $H = G_x$, while that of $H$ on any suborbit is equivalent to its action on the cosets of some subgroup $F$ of $H$.

2. Notations and Preliminary Results

Notation 2.1
In this paper, $X^{[r]} = \{a_1, a_2, ..., a_n\}, R(G)$ and $m(A, B, G)$ denote the set of all ordered $r$-element subsets from the set $X$, ordered $r$-element subset of $X$ the rank of a permutation Group $G$ and the mark of subgroup $A$ in the representation of $G$ on the cosets of subgroup $B$ respectively.

Theorem 2.1 (Rose [5])
Let $G$ act on the set $X$, and let $x \in X$. Then
$$|Orb_G(x)| = |G:Stab_G(x)|.$$

Definition 2.2 (Ivanov [2])
If $G_j \leq G_i \leq G$ and $(G_{j_1}, G_{j_2}, ..., G_{j_n})$ is a complete set of conjugacy class representatives of subgroups of $G$, then $m(G_{j_i}, G, G) = \sum_{k=1}^{n}|N_G(G_{j_k}):N_{G_j}(G_{j_k})|$

Definition 2.3
Let $H_1, H_2, ..., H_t$ be a set of representatives of all distinct conjugacy classes of subgroups of $H$ in $G$, ordered such that $|H_1| \leq |H_2| \leq \cdots \leq |H_t| = |H|$. Form a matrix $M = (m_{ij})$, where $m_{ij} = m(H_j, H_i, H)$. We call $M$ the table of marks of $H$.

If we denote by $Q_i$ the number of suborbits $\Delta_i$ on which the action of $H$ is equivalent to its action on the cosets of $H_i(i = 1, 2, ..., t)$, by computing all $Q_i$ we get the subdegrees of $G$ on $X$.

Theorem 2.4 (Ivanov [2])
The numbers $Q_i$ satisfy the system of linear equations
$$\sum_{i=j}^{t}Q_jm(H_j, H_i, H) = m(H_j, H, G),$$
for each $j = 1, 2, ..., t$.

Theorem 2.5 (Rimberia [4])
Let $G$ be the symmetric group $S_n$, then $G$ acts transitively on $X^{[r]}$.

**Theorem 2.6 (Rimberia [4])**

Let $G = S_n$ be the symmetric group on the set $X = \{a_1, a_2, ..., a_n\}$. The stabilizer of an ordered subset $\{a_1, a_2, ..., a_r\}$ in $G$ is isomorphic to $S_{n-r}$, the symmetric group on the set $\{a_{r+1}, a_{r+2}, ..., a_n\}$.

3. **Ranks And Subdegrees Of The Symmetric Group $S_n$ ($N = 5, 6, 7$) Acting On $X^{[2]}$**

3.1 **Rank and Subdegrees of $S_5$ acting on $X^{[2]}$**

We first construct the table of marks of $H = Stab_G([1,2])$, where $G = S_5$. Then by Theorem 2.6, $H$ is isomorphic to $S_3$, the symmetric group on the set $\{3,4,5\}$. We will use GAP software to determine the conjugacy classes of subgroups and construct the table of marks by using the algorithms shown on the Appendix 1. The conjugacy classes of subgroups of $S_3$ are as shown below and its table of marks is as shown in Table 1.

- i) $H_1$ Identity
- ii) $H_2$ 3 conjugate subgroups of order 2
- iii) $H_3$ A normal cyclic group of order 3
- iv) $H_4$ $H$

For $F \leq H$, with the order they are represented in Table 1, the values of $m(F,H,G) = m(F)$ are displayed in Table 3.1.1.

**Table 3.1.1: Marks of $F$ in $G/H$, where $G = S_5$ and $H = Stab_G([1,2])$**

<table>
<thead>
<tr>
<th>F</th>
<th>$H_1$</th>
<th>$H_2$</th>
<th>$H_3$</th>
<th>$H_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>m(F)</td>
<td>20</td>
<td>6</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Let $Q = (Q_1, Q_2, Q_3, Q_4)$ and the matrix obtained from Table 1 be $M$. By Theorem 2.3 and using Table 1 we obtain the following system of linear equations,

$$
6Q_1 + 3Q_2 + 2Q_3 + Q_4 = 20
$$
$$
Q_2 + Q_4 = 6
$$
$$
2Q_3 + Q_4 = 2
$$
$$
Q_4 = 2
$$

On solving (1), we get $Q = (1, 4, 0, 2)$.

Therefore the subdegrees of $G$ acting on $X^{[2]}$ are as shown in Table 3.1.2.

**Table 3.1 2: subdegrees of $S_5$ acting on $X^{[2]}$**

<table>
<thead>
<tr>
<th>Suborbit length</th>
<th>1</th>
<th>3</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of suborbits</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

From Table 3.1.2 we obtain, the rank of $G$, $R(G) = 1 + 4 + 2 = 7$.

We now apply the same procedure for the other actions.

3.2. **Rank and Subdegrees of $G = S_6$ acting on $X^{[2]}$**

Let $H = Stab_G([1,2])$, where $G = S_6$. By Theorem 3.1.1 $H$ is isomorphic to $S_4$ which has 11 conjugacy classes of subgroups. These are:

- i). $H_1$ identity.
- ii). $H_2$ conjugate subgroups of order 2 generated by permutations of the form $(ab)(cd)$.
- iii). $H_3$ conjugate subgroups of order 2 generated by permutations of the form $(ab)$.
- iv). $H_4$ conjugate cyclic subgroups of order 3.
- v). $H_5$ a normal cyclic group of order 4 generated by permutations of the form $(ab)(cd)$ and $(ac)(bd)$.
- vi). $H_6$ 3 conjugate subgroups of order 4 generated by permutations of the form $(ab)(cd)$ and $(cd)$.
- vii). $H_7$ 3 conjugate cyclic subgroups of order 4.
- viii). $H_8$ 4 conjugate subgroups of order 6 isomorphic to $S_3$ generated by permutations of the form $(abc)$ and $(bd)$.
- ix). $H_9$ 3 conjugate subgroups of order 8 generated by permutations of the form $(ab)(cd)$, $(ac)(bd)$ and $(bd)$.
\[ m(F) = 30 \]

<table>
<thead>
<tr>
<th>F</th>
<th>H_1</th>
<th>H_2</th>
<th>H_3</th>
<th>H_4</th>
<th>H_5</th>
<th>H_6</th>
<th>H_7</th>
<th>H_8</th>
<th>H_9</th>
<th>H_{10}</th>
<th>H_{11}</th>
</tr>
</thead>
<tbody>
<tr>
<td>m(F)</td>
<td>30</td>
<td>12</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

Let \( Q = (Q_1, Q_2, \ldots, Q_{11}) \) By Theorem 2.3 and using Table 3, we obtain the following system of linear equations in matrix form,

\[
\begin{pmatrix}
24 & 12 & 12 & 8 & 6 & 6 & 4 & 3 & 2 & 1 \\
0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 & 1 \\
0 & 0 & 4 & 0 & 2 & 6 & 2 & 0 & 3 & 2 & 1 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 3 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
Q_1 \\
Q_2 \\
Q_3 \\
Q_4 \\
Q_5 \\
Q_6 \\
Q_7 \\
Q_8 \\
Q_9 \\
Q_{10} \\
Q_{11}
\end{pmatrix}
= \begin{pmatrix}
30 \\
12 \\
2 \\
6 \\
2 \\
2 \\
2 \\
2 \\
2
\end{pmatrix}
\]

On solving (2), we obtain \( Q = (0, 1, 0, 0, 0, 0, 4, 0, 0, 2) \).

Therefore subdegrees of \( S_6 \) acting on \( X^{[2]} \) are as shown in Table 3.2.2.

<table>
<thead>
<tr>
<th>Suborbit length</th>
<th>1</th>
<th>4</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of suborbits</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

From Table 4 we see that, the rank of \( G, R(G) = 2 + 4 + 1 = 7 \).

### 3.3 Rank and Subdegrees of \( G = S_7 \) acting on \( X^{[2]} \)

Let \( H = \text{Stab}_G([1,2]) \), where \( G = S_7 \). By Theorem 2.6 \( H \) is isomorphic to \( S_5 \) which has 19 conjugacy classes of subgroups (See Appendix 3). Its table of marks is as shown by Table 3.

For \( F \leq H \), with the order they are represented in Table 3, the values of \( m(F, H, G) = m(F) \) are displayed in Table 3.3.1.

<p>| Table 3.3.1: Marks of ( F ) in ( G(H) ) where ( G = S_7 ) and ( H = \text{Stab}_G([1, 2]) ) |</p>
<table>
<thead>
<tr>
<th>F</th>
<th>H_1</th>
<th>H_2</th>
<th>H_3</th>
<th>H_4</th>
<th>H_5</th>
<th>H_6</th>
<th>H_7</th>
<th>H_8</th>
<th>H_9</th>
<th>H_{10}</th>
</tr>
</thead>
<tbody>
<tr>
<td>m(F)</td>
<td>42</td>
<td>20</td>
<td>6</td>
<td>12</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>2</td>
<td>12</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>F</th>
<th>H_{11}</th>
<th>H_{12}</th>
<th>H_{13}</th>
<th>H_{14}</th>
<th>H_{15}</th>
<th>H_{16}</th>
<th>H_{17}</th>
<th>H_{18}</th>
<th>H_{19}</th>
</tr>
</thead>
<tbody>
<tr>
<td>m(F)</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>6</td>
<td>2</td>
<td>2</td>
<td></td>
</tr>
</tbody>
</table>

Let \( Q = (Q_1, Q_2, \ldots, Q_{19}) \). By Theorem 2.3 and using Table 5, we obtain the following system of linear equations in matrix form,
On solving (3) we obtain \( Q = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 4, 0, 2) \).

Therefore, the Subdegrees of \( G \) acting on \( X[2] \) are as shown in Table 3.3.2.

**Table 3.3.2: Subdegrees of \( G = S_7 \) acting on \( X[2] \)**

<table>
<thead>
<tr>
<th>Suborbit length</th>
<th>1</th>
<th>5</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>No. of suborbits</td>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

From Table 6 we see that the Rank of \( G, R(G) = 2 + 4 + 1 = 7 \).

4. **Ranks and Subdegrees Of \( S_n \) (\( n = 6, 7, 8 \)) Acting On \( X[3] \)**

4.1 **Rank and Subdegrees of \( G = S_6 \) acting on \( X[3] \)**

Let \( H = \text{Stab}_G([1,2,3]) \), where \( G = S_6 \). By Theorem 2.6, \( H \) is isomorphic to \( S_3 \), the symmetric group on the set \{4, 5, 6\}. Its table of marks is as shown by Table 1.

For \( F \leq H \), with the order they are represented in Table 1, the values of \( m(F, H, G) = m(F) \) are displayed in Table 4.1.1.

**Table 4.1.1: Marks of \( F \) in \( G/\text{H} \), where \( G = S_6 \) and \( H = \text{Stab}_G([1, 2, 3]) \)**

<table>
<thead>
<tr>
<th>( F )</th>
<th>( H_1 )</th>
<th>( H_2 )</th>
<th>( H_3 )</th>
<th>( H_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m(F) )</td>
<td>120</td>
<td>24</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Let \( Q = (Q_1, Q_2, Q_3, Q_4) \). By Theorem 2.3 and using Table 4.1.1, we obtain the following system of linear equations.

\[
6Q_1 + 3Q_2 + 2Q_3 + Q_4 = 120 \\
Q_2 + Q_4 = 24 \\
2Q_3 + Q_4 = 6 \\
Q_4 = 6 \quad \text{………………….(4)}
\]
On solving (4) we obtain \( Q = (10,18,0,6) \).

Therefore the subdegrees of \( G \) acting on \( X^{[3]} \) are as shown in Table 4.1.2.

**Table 4.1.2: subdegrees of \( G = S_6 \) acting on \( X^{[3]} \)**

| No. of suborbits | 6 | 18 | 10 |

From Table 8 we see that the rank of \( G, R(G) = 10 + 18 + 6 = 34 \).

### 4.2. Rank and Subdegrees of \( G = S_7 \) acting on \( X^{[3]} \)

Let \( H = \text{Stab}_G([1,2,3]) \), where \( G = S_7 \). By Theorem 2.6 \( H \) is isomorphic to \( S_6 \), the symmetric group on the set \{4,5,6,7\}. Its table of marks is as shown by Table 2.

For \( F \leq H \), with the order they are represented in Table 2. The values of \( m(F,H,G) = m(F) \) are displayed in Table 4.2.1.

**Table 4.2.1: Marks of \( F \) in \( G(/H) \), where \( G = S_7 \) and \( H = \text{Stab}_G([1,2,3]) \)**

<table>
<thead>
<tr>
<th>( F )</th>
<th>( H_1 )</th>
<th>( H_2 )</th>
<th>( H_3 )</th>
<th>( H_4 )</th>
<th>( H_5 )</th>
<th>( H_6 )</th>
<th>( H_7 )</th>
<th>( H_8 )</th>
<th>( H_9 )</th>
<th>( H_{10} )</th>
<th>( H_{11} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m(F) )</td>
<td>210</td>
<td>60</td>
<td>6</td>
<td>24</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>24</td>
<td>6</td>
<td>6</td>
<td>6</td>
</tr>
</tbody>
</table>

Let \( Q = (Q_1, Q_2, \ldots, Q_{11}) \). By Theorem 2.3 and using Table 4.2.1, we obtain the following system of linear equations in matrix form.

\[
\begin{pmatrix}
24 & 12 & 12 & 8 & 6 & 6 & 4 & 3 & 2 & 1 \\
0 & 2 & 0 & 0 & 0 & 0 & 2 & 2 & 1 & 0 \\
0 & 0 & 4 & 0 & 2 & 6 & 2 & 0 & 3 & 2 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 3 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
Q_1 \\
Q_2 \\
Q_3 \\
Q_4 \\
Q_5 \\
Q_6 \\
Q_7 \\
Q_8 \\
Q_9 \\
Q_{10} \\
Q_{11}
\end{pmatrix}
= 
\begin{pmatrix}
210 \\
60 \\
6 \\
24 \\
6 \\
6 \\
24 \\
6 \\
6 \\
6 \\
6
\end{pmatrix}
\]

\[ \text{......(5)} \]

On solving (5) we obtain \( Q = (1,9,0,0,0,0,0,18,0,0,6) \).

Therefore the subdegrees of \( G \) acting on \( X^{[3]} \) are as shown in Table 4.2.2.

**Table 4.2.2: Subdegrees of \( G = S_7 \) acting on \( X^{[3]} \)**

| No. of suborbits | 6 | 18 | 9 | 1 |

From Table 4.2.2 we see that the rank of \( G, R(G) = 6 + 18 + 9 + 1 = 34 \).

### 4.3 Rank and Subdegrees of \( G = S_8 \) acting on \( X^{[3]} \)

Let \( H = \text{Stab}_G([1,2,3]) \), where \( G = S_8 \). By Theorem 2.6 \( H \) is isomorphic to \( S_7 \), the symmetric group on the set \{4,5,6,7\}. Its table of marks is as shown by Table 3.

For \( F \leq H \), with the order they are represented in Table 3, the values of \( m(F,H,G) = m(F) \) are as displayed in Table 4.3.1.

**Table 4.3.1: Marks of \( F \) in \( G(/H) \), where \( G = S_8 \) and \( H = \text{Stab}_G([1,2,3]) \)**

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Appendix 1: GAP algorithms used to generate conjugacy classes and table of marks of $H = S_3$
To generate $H = S_3$ the symmetric group on the set {3,4,5} on the GAP, we have used its generators; (3,4,5) and (3,4) and the command;

```gap
H := Group([ (3,4,5), (3,4) ]);
```

To get the conjugacy classes of subgroups of $H$ we use the command;

```gap
ConjugacyClassesSubgroups(H);
```

To construct the table of marks of $H$ we use the command;

```gap
Tom1 := TableOfMarks(H);
```

To view it we use the command;

```gap
Display(Tom1);
```

Table 1: Table of marks of $S_3$

To get the sizes number of conjugate subgroups in each conjugacy class of subgroups we use the following command;

```gap
LengthsTom(Tom1);
```

To get the orders of the conjugacy classes of subgroups we use the following command;

```gap
OrdersTom(Tom1);
```

Appendix 2: GAP algorithms used to generate conjugacy classes and table of marks of $H = S_4$

$S_4$ can be generated by (ab) and (abcd). To generate $H = S_4$ the symmetric group on the set {3,4,5,6} on the GAP, we have used its generators; (3,4,5,6) and (3,4) and the command;

```gap
H := Group([ (3,4,5,6), (3,4) ]);
```

To solve the system of linear equation we use the command;

```gap
SolutionMat(M,[20,6,2,2]);
```

For the other cases the similar algorithms were used to generate the other results.

```gap
M := MatTom(Tom2);
```

Table 2: Table of marks of $S_4$
Appendix 3: GAP algorithms used to generate conjugacy classes and table of marks of $H = S_5$

$S_5$ can be generated by $(ab)$ and $(abcd,e)$. To generate $H = S_5$ the symmetric group on the set $\{3,4,5,6,7\}$ on the GAP, we have used its generators; $(3,4,5,6,7)$ and $(3,4)$ using the command:

```gap
H:=Group([[3,4,5,6,7],[3,4]]);
Group([ (3,7), (4,7), (5,7), (6,7) ])^G,
ConjugacyClassesSubgroups(H):
Group([ (3,7), (4,7), (5,7), (6,7) ])^G,
Group([ (5,6,7) ]^G,
Group([ (4,5)(6,7), (4,6,5,7) ]^G,
Group([ (3,4,5,6,7) ]^G,
Group([ (5,6,7) ]^G,
Group([ (6,7), (3,4,5) ]^G,
TableOfMarks(H):
TableOfMarks(Sym([3..7]));
Display(T);
LengthsTom(Tom3);
OrdersTom(Tom3);
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