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Normal Form for Module of Equivariants

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ABSTACT

The subject of dynamical systems is concerned with:

i. Given an $n \times n$ matrix A describe the behavior , in a neighborhood of the origin the solutions of all systems of differential equations having a rest point at the origin with the linear part Ax
ii. Describe the behavior (near origin) of all systems close to a system of the type described in i above.

The normal form is intended to be the ''simplest ''form in which any system of intended type can be transformed by changing the coordinates in a prescribed manner. But if a normal form is thought as the ''simplest'' form into which a system can be placed there might be disagreement as to what is considered simplest. A systematic policy for deciding what accounts for simplest is called normal form style. The important normal form styles are Semisimple, Innerproduct and SL(2) Or Triad styles.

The unfolding of the normal form is intended to be the simplest form in which all systems close to the original system can be transformed

I will present the procedure for obtaining the normal form using the inner product style and explain the structure of normal form using the language of equivariants over a ring of invariants and give an algorithm suitable for use in symbolic computation systems procedure

1.STANLEY DECOMPOSITION

Give the Grobner basis for the ker φ is $\mathbf{G} \cap k[a,b,c,d,e,f] = \{a^2 e + b^2 d - c^2\}$

We write down the Stanley decomposition of the ring of invariants. We know that the Groebner basis on the ideal of relations

Let $\widetilde{I} = <\alpha,\beta,\gamma,\delta,\xi,\eta>$ be the ideal generated by the leading terms of the Groebner basis for I, which is a monomial ideal. It is a fact that the Stanley decomposition of $\Re[\alpha,\beta,\gamma,\delta,\xi,\eta]/I$ is the same as the Stanley decomposition of $\Re[\alpha,\beta,\gamma,\delta,\xi,\eta]/\widetilde{I}$ and so it suffices to work $\Re[\alpha,\beta,\gamma,\delta,\xi,\eta]/\ \widetilde{I}$ [7]

Where

$$\alpha = x_1,\ \beta = x_2,\ \gamma = x_1y_2 - x_2y_1,\ \delta = y_1^2 - 2x_1z_1,\quad \xi = y_2^2 - 2x_2z_2,\ \eta = y_1y_2 + x_1z_2 - x_2z_1$$

The Stanley decomposition of the normal form with linear part $N = N_{33}$

Where $\chi = x_1\dfrac{\partial}{\partial y_1} + y_2\dfrac{\partial}{\partial z_1} + x_2\dfrac{\partial}{\partial y_2}\,y_2\dfrac{\partial}{\partial z_2}$

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The only generator, which forms a Groebner basis for the ideal of relations among the basic invariants, is

$$\gamma^2 = \alpha^2\xi - \beta^2\delta^2$$

This relation can be used to eliminate the combination γ^2 (the leading term of the relation from any polynomial.

Consider the ring = $\Re[\alpha,\beta,\gamma,\delta,\xi,\eta] \in f_6^6$ of polynomial in the known basic invariants. The representation of an element in $\Re$ is not unique but this equation itself can be used to restore uniqueness by excluding γ^n or any higher power of γ that is where γ^2 appear in the polynomial in $\Re$ it can be replaced by the right hand of the equation and the expression obtained this way will be unique. Thus the Stanley decomposition of $\Re$ is

$$\Re[\alpha,\beta,\delta,\xi,\eta] \oplus \Re[\alpha,\beta,\delta,\xi,\eta]\gamma \qquad \text{(i)}$$

Another way is to say that this polynomials in $\Re$ can be written uniquely as

$$f(\alpha,\beta,\delta,\xi,\eta) \oplus g(\alpha,\beta,\delta,\xi,\eta)\gamma \qquad \text{(ii)}$$

Where f and g are polynomials in five variables. Still another way to say this is that each element $\Re$ is uniquely expressible as a linear combination of standard monomials which are monomials in basic invariants that contain at most one factor of γ (equivalently, not divisible by γ^2). Everything stated in this paragraph remains true if the polynomial is replaced by formal power series provided equation (i) is written as **R**= $\Re[\alpha,\beta,\delta,\xi,\eta]$ $\oplus$ $\Re[\alpha,\beta,\delta,\xi,\eta]\gamma$ and linear combination is understood as infinite linear combinations.

The Stanley decomposition of (ii) can be abbreviated as $f.I + g\gamma$. The function f and g will be referred as coefficients and I and γ being called Stanley basis elements.

1.2 Table of functions

To generate the table of functions of Stanley decomposition, we replace each term in (ii) by a rational function P/Q in d and w (symbols standing for degree in x and weight) constructed as follows;

1. For each basic invariant appearing in a coefficient, the dominator Q is to contain the factor $1-d^p w^q$ where p and q are degree and weight of the invariants
2. For each term the numerator p will be $d^p w^q$ where p and q are the degree and weight of the Stanley basis element for that term.

When the table of function is fully expanded as a power series in d and w, it contains the term $md^r w^s$ if and only if there exists m standard monomials that have total degree r and weights. Thus, the table of function is encoded version of that part of the top weight list, carried to all orders that can be generated by $\alpha,\beta,\gamma,\delta$, ξ,η. To prove $\Re$ =ker X, it would suffice to prove that the number of elements in the chains having these top weights in degree j is equal to dim f_j^n.

The following lemma gives a method to check that enough basic invariants have been found.

Lemma 1.2.1 *Let { X, Y, Z} be a triad of* $n \times n$ *matrices, let { X, Y, Z} be the induced triad, and suppose that* I_l *,…,*I_t *is a finite set of polynomials in ker X, let R be a subring of* $R[I_1,\ldots I_t]$*; suppose that the relations among the* $I_1,\ldots I_t$ *have been found, and that the Stanley decomposition and its associated table function* $T(d,w)$ *have been determined.*

Then $R = \ker X \subset P_j(\Re^n,\Re^n)$ if and only if

$$\left.\frac{\partial}{\partial w} wT\right|_{w=1} = \frac{1}{(1-d)_n}$$

By lemma 1.1.1.we will note that

$$\frac{\partial}{\partial w} wT\Big|_{w=1} = \frac{1}{(1-d)_6}$$

Note

We have chosen a term ordering on the symbols $\alpha,\beta,\gamma,\delta,\xi,\eta$ such that γ^2 is the leading term of $\gamma^2+\alpha^2\varsigma$ - $\beta^2\ \delta=0$ we associate the monomial ideal generated by its leading term (γ^2). A standard monomial is any monomial in these symbols that does not belong to this monomial ideal.

2. NORMAL FORM MODULE OF EQUIVARIATS

Here we obtain the Stanley decomposition of the module ker X^r, given the Stanley decomposition of ker x. To do so we shall use the following theorem theorem,

Theorem 2.1 *The image of* $\wp$ is ker $x^{r_1}\oplus,\ldots,\oplus\ker x^{r_k}$ and the mapping $\wp$: ker X $\rightarrow$ $\ker X \rightarrow \ker x^{r_1}\oplus\ldots\oplus\ker x^{r_k}$ is an isomorphism of modules over the ring ker X.

Proof. Observe that if $f\in$ ker X and g $\in P(\mathfrak{R}^n, \mathfrak{R})$, then

$X(fg) = fxg$.

It follows that if g $\in$ ker X^r (for any r) then, $fg\in\ker x^r$:that is ker X^r is a module over ker X .

The image of φ is ker $X^r{}_1\oplus\ker x^r{}_2\oplus,\ldots,$ker $X^r{}_k$ and the mapping.

$\varphi:\ker x\rightarrow\ker x^r{}_1\oplus\ker x^r{}_2\oplus,\ldots,\oplus\ker x^r{}_k$ is an isomorphism of modules over the ring ker x.[5]

2.1 Stanley decomposition for the normal form module ker x

Lemma 2 .1.1 *Let* $\mathfrak{R}$ *be any sub ring of ker x generated by homogenous polynomials I,…,I, in* $x=(x_1,\ldots,x_n)$ *which are weight vectors for the triad* $\{x,y,z\}$ *and let* R_{ik} *be the vector sub space of* $\mathfrak{R}$ *consisting of polynomials homogenous in x of degree I and weight k. Let a Groebner basis for the relations of* $I_I,\ldots$*, Is be be selected. Then:*

1. *The standard monomials* $I_I,\ldots$*, Is(with respect to the given Groebner basis) having degree i(in x) and weight k form a basis for* R_{ik}.
2. *If* $\mathfrak{R}=\ker x$*, the standard monomials of degree I form a set of chain tops for the chains in* $p_i(\mathfrak{R}^n$ $\mathfrak{R})$.

Acoording to this lemma the chain tops of $p(\mathfrak{R}^n\ \mathfrak{R})$ under the triad $\{x,y,z\}$ can be taken to be the standard monomials in the basis invariants, say $I_1,I_2,\ldots,I_s$ with respect to the Stanley decomposition of ker X. The chains under these chain tops are the Jordan chains of y and can be obtained by repeated application of y(regarded as a mapping downward), a vector space basis foe ker X is obtained by computing the y iterates of depth r.

We now describe how to obtain the Stanley decomposition of ker X^r for any Stanley decomposition of ker X.

We recall here the definition of prefix ring. Let $m_1,\ldots,m_p$ be the leading monomials of the Groebner basis $\gamma_1,\ldots,\gamma_p$ of the ideal of relations. Let g be a stripped suffix or suffixes and let $I_i(g)$ be the first basic invariant (from the left) appearing in g. if f is a Standard monomial with respect to the ideal

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$j_s = \langle m'_1,\dots,m'_p, I_i(g)_{+1},\dots,I_s\rangle$ where $m'_1 = m'_i/\gcd(m_i, g)$, then the suffix monomial f associated with the given stripped suffix g are the standard monomials with respect to the (new) ideal $j_s = \langle m'_1,\dots,m'_p, I_i(g)_{+1},\dots,I_s\rangle$ (ordering the basic invariants by $I_i < I_j$ if j<i). Now let f be a prefix monomials associated with a given suffix S, then the collection of polynomials which are linear combination of such prefix monomials for a given suffix s is a ring called prefix ring for s which has a Stanley decomposition(defined by its standard prefix monomials).

We now consider N_{33} to illustrate the above steps.

Ordering the invariants by $\alpha < \beta < \delta < \zeta < \eta < \gamma$.

These are related by the single relation $\gamma^2 + \alpha^2\xi - \beta^2\delta = 0$ with γ^2 as the leading monomial. To compute ker x^3 it is necessary to apply y to depth 3.

A monomial of the first type has the form Iγ where I is an arbitrary monomial not involving γ that is I $\in \Re[\alpha,\beta,\delta,\zeta,\eta]$. The entries under the Chain top of this form will be I $y\gamma$,I $y^2\gamma$ and I $y^3\gamma$ such an entry will be said to have prefix I and suffix $y\gamma$, $y^2\gamma$ or $y^3\gamma$. A suffix always begins with y. The prefix ring for a given suffix is the set of admissible prefixes that can appear with that suffix; the prefix ring for each suffix $y\gamma, y^2\gamma$ and $y^3\gamma$.

A monomial of the second type has the form Iα with I $\in \Re[\alpha,\beta,\delta,\zeta,\eta]$, the entries under it have prefix I suffix $y\gamma, y^2\gamma$ and $y^3\gamma$. The monomial of the third type has the form Iβ with I$\in[\beta,\delta,\zeta,\eta]$. the entries under it have the prefix I and suffix $y\beta, y^2\beta$ or $y^3\beta$

Ker $x^3 = \Re[\alpha,\beta,\delta,\zeta,\eta] \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]\gamma$

$\oplus \Re[\alpha,\beta,\delta,\zeta,\eta]y\gamma \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]y^2\gamma$

$\oplus \Re[\alpha,\beta,\delta,\zeta,\eta]y^3\gamma \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]y\alpha$

$\oplus \Re[\alpha,\beta,\delta,\zeta,\eta]y^2\alpha \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]y^3\alpha$

$\oplus \Re[\beta,\delta,\zeta,\eta]\, y\beta \oplus \Re[\beta,\delta,\zeta,\eta]\, y^2\beta$

$\oplus \Re[\beta,\delta,\zeta,\eta]\, y^3\beta$

Having obtained the Stanley decomposition for ker x^3 the normal form for ker x that is the module of equivariants is obtained by Theorem 2.1.i.e. ker $x \cong \ker x^{r_1} \oplus \ker x^{r_2} \oplus \dots \oplus \ker x^{rk}$

Hence for N_{33} we have ker $x^3 \oplus$ ker x^3

Ker x = ($\Re[\alpha,\beta,\delta,\zeta,\eta] \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]\gamma$

$\oplus\ \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{1, y\gamma\} \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{1, y^2\gamma\}$

$\oplus\ \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{1, y^3\gamma\}$

$\oplus\ \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{1, y\alpha\} \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{1, y^2\alpha\}$

$\oplus\ \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{1, y^3\alpha\} \oplus \Re[\beta,\delta,\zeta,\eta]\ v\{1, y\beta\}$

$\oplus \Re[\beta,\delta,\zeta,\eta]\ v\{1, y^2\beta\} \oplus \Re[\beta,\delta,\zeta,\eta]\ v\{1, y^3\beta\}$

$$\oplus (\Re[\alpha,\beta,\delta,\zeta,\eta] \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]\,\gamma)\ v\{2.2\}$$

$$\oplus\ \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{2, y\gamma\} \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{2, y^2\ \gamma\}$$

$$\oplus\ \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{2, y^3\ \gamma\} \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{2, y\alpha\}$$

$$\oplus\ \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{2, y^2\ \alpha\} \oplus \Re[\alpha,\beta,\delta,\zeta,\eta]\ v\{2, y^3\ \alpha\}$$

$$\oplus \Re\ [\beta,\delta,\zeta,\eta]\ v\{2, y\beta\} \oplus \Re\ [\beta,\delta,\zeta,\eta]\ v\{2, y^2\ \beta\}$$

$$\oplus \Re\ [\beta,\delta,\zeta,\eta]\ v\{2, y^3\ \beta\}$$

Where

$$V\{1,1\} = \begin{bmatrix} 0\\0\\1\\0\\0\\0 \end{bmatrix},\quad v\{1, y\gamma\} = \begin{bmatrix} 0\\ xy\gamma\\ y\gamma\\0\\0\\0 \end{bmatrix},\quad v\{1, y^2\ \gamma\} = \begin{bmatrix} x^2y^2\\ xy^2\\ y^2\gamma\\0\\0\\0 \end{bmatrix},\quad v\{1, y^3\ \gamma\} = \begin{bmatrix} x^2y^3\gamma\\ xy^3\gamma\\ y^3\gamma\\0\\0\\0 \end{bmatrix}$$

$$v\{1, y\alpha\} = \begin{bmatrix} 0\\ xy\alpha\\ y\alpha\\0\\0\\0 \end{bmatrix},\quad v\{1, y^2\ \alpha\} = \begin{bmatrix} x^2y^2\gamma\\ xy^2\alpha\\ y^2\alpha\\0\\0\\0 \end{bmatrix},\ v\{1, y^3\ \alpha\} = \begin{bmatrix} x^2y^3\alpha\\ xy^3\alpha\\ y^3\alpha\\0\\0\\0 \end{bmatrix},\ v\{1, y\beta\} = \begin{bmatrix} 0\\ xy\beta\\ y\beta\\0\\0\\0 \end{bmatrix},$$

$$v\{1, y^2\ \beta\} = \begin{bmatrix} x^2y^2\beta\\ xy^2\beta\\ y^2\beta\\0\\0\\0 \end{bmatrix},\quad v\{1, y^3\ \beta\} = \begin{bmatrix} x^2y^3\beta\\ xy^3\beta\\ y3\beta\\0\\0\\0 \end{bmatrix},\ v\{2,2\} = \begin{bmatrix} 0\\0\\0\\0\\0\\2 \end{bmatrix},\ v\{2, y\gamma\} = \begin{bmatrix} 0\\0\\0\\0\\ xy1\gamma\\ y\gamma \end{bmatrix},$$

$$v\{2, y^2\gamma\} = \begin{bmatrix} 0\\0\\0\\ x^2y^2\gamma\\ xy^2\gamma\\ y^2\gamma \end{bmatrix},\ v\{2, y^3\gamma\} = \begin{bmatrix} 0\\0\\0\\ x^2y^3\gamma\\ xy^3\gamma\\ y^3\gamma \end{bmatrix},\ v\{2, y\alpha\} = \begin{bmatrix} 0\\0\\0\\0\\ xy\alpha\\ y\alpha \end{bmatrix},\ v\{2, y^2\alpha\} = \begin{bmatrix} 0\\0\\0\\ x^2y^2\alpha\\ xy^2\alpha\\ y^2\alpha \end{bmatrix},$$

$$v\{2, y^3\alpha\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^3y^3\alpha \\ xy^3\alpha \\ y^3\alpha \end{bmatrix}, v\{2, y\beta\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ xy\beta \\ y\beta \end{bmatrix}, v\{2, y^2\beta\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^2y^2\beta \\ xy^2\beta \\ y^2\beta \end{bmatrix}, v\{2, y^3\beta\} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x^2y^3\beta \\ xy^3\beta \\ y^3\beta \end{bmatrix}$$

3.0 APPLICATION

3.1 Sositaisvilis theorem

Consider the parameter dependent vector field.

$$x = Ax + f(\text{x,y,z}, \varepsilon,)$$

$$y = By = g(\text{x,y,z}, \varepsilon,)$$

$$z = Cz + h(\text{x,y,z}, \varepsilon,)$$

$$(x, y, z, \varepsilon)\_ \in \Re^{C}\ x\Re^{S}\ x\ \Re^{U}\ x\Re^{p}$$

Where

$$f\ (0, 0, 0) = 0$$

$$g(0,0,0) = 0$$
$$h(0,0,0) = 0$$
$$Df(0,0,0) = 0$$
$$Dg(0,0,0) = 0$$
$$Dh(0,0,0) = 0$$

And f, g and h are $C^r\ (r \geq 2)$ in some neighbours of the origin A, is C x C matrix having eigenvalues with zero real part, B is a S x S matrix having eigenvalues with positive real parts.

The centre manifold Theorem tells us that near the origin in $\Re^{C}\ x\Re^{S}\ x\ \Re^{U}\ x\Re^{p}$, the flow generated by equations above is C^0 conjugated to the flow generated by the following vector field.

$$x = w(x, \varepsilon)$$

$$y = -y$$

$$z = z$$

$$(\text{x,y,z}, \varepsilon,)\ \in \Re^{C}\ x\Re^{S}\ x\ \Re^{U}\ x\Re^{p}$$

Where w(x, ε) represents the C^r vector fields on the centre manifold. The results are due to Sositaisvili(1975)[2].

A wide area where this knowledge is also necessary is in Poincare maps where the idea is to reduce continuous time systems (flows) to discrete time system (maps)

Under these we have:

a. Poincare map near s periodic orbit.
b. Poincare maps of a time periodic ordinary differential equation.
c. The Poincare map near homoclinic orbit.
d. The Poincare map associated with two degree of freedom Hamiltonian systems.

3.2 Areas for Further Research

- Generalize to $N_{33\ldots\ldots3}$ and mixed problems like $N_{233.}$
- The role of transvectants in N_{33} in particular,
    - Deriving the relation among the basic invariants from the trasvectant expression.
    - Is there a way to obtain the Groebner basis work using the transvectants?
    - Relation between the transvectant structure and the bracket aldgebra that Cushman and Sanders use.
- Obtaining the invariants using the cross – section method.

4.0 CONCLUSION

The method of normal forms provides a way of finding a coordinate system in which the dynamical systems takes the “simplest” form. Three important characteristics are apparent.

1. The method is local in the sense that the coordinate transformations are generated in a neighbourhood of a known solution.
2. In general, the coordinate transformations will be non linear functions of the dependent variables. However, the important point is that these coordinate transformations are found by solving a sequence of linear problems.
3. The structure of the normal form is determined entirely by the nature of linear part of the vector field.

REFERENCES

1) Cushman R. and Sanders J. (1990). *A Survey of Invariants Theory Applied to Normal Form of Vector Fields with Nilpotent Linear Part* , invariant Theory and Tableaux, D. Stanton, Ed. Springer- Verlag. New York.
2) Hartley B. and Hawkes. T. (1970) *Rings, modules and linear algebra*, University press, Campbridge
3) Lewis D.W. (1991). *Matrix theory*, WorldScience publishing
4) Malonza D.M. (2003). *Normal forma of coupled Takens- Bogdanov systems* Ph.D. thesis, Iowa State University
5) Morris W. and Smale S. (1974). *Differential equations, dynamical systems and linear algebra*, Academic Press, New York.
6) Murdock J.(2002). *Normal forms and unfolding for local dynamical system*, Springer Verlag, New York.
7) Sturmfels B. and white N. (1991). *Computing Combinatorial Decomposition of ring*. Combinatorica II, 275-293.
8) Wiggins S. (2003). *Introduction to applied non linear dynamical systems and chaos*, Springer Verlag, New York.
9) William W. and Loustaunau P. (1994). *An introduction to Groebner bases*. American mathematical society, Providence.