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Analytical Solution for Telegraph Equation by Modified of Sumudu Transform "Elzaki Transform"

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Abstract

In this work modified of Sumudu transform [10,11,12] which is called Elzaki transform method (new integral transform) is considered to solve general linear telegraph equation, this method is a powerful tool for solving differential equations and integral equations [1, 2, 3, 4, 5]. Using modified of Sumudu transform or Elzaki transform, it is possible to find the exact solution of telegraph equation. This method is more efficient and easier to handle as compare to the Sumudu transform method and variational iteration method. To illustrate the ability of the method some examples are provided.

Keywords: modified of Sumudu transform- Elzaki transform - Telegraph equation - Partial Derivatives

1. Introduction

Telegraph equations appear in the propagation of electrical signals along a telegraph line, digital image processing, telecommunication, signals and systems.

The general linear telegraph equation is

$$U_{tt} + aU_t + bU = c^2 U_{xx} \tag{1}$$

With the initial conditions:

$$U(x,0) = \alpha \quad , \quad U_t(x,0) = \beta \tag{2}$$

Where α, β are functions of x .

The basic definitions of modified of Sumudu transform or Elzaki transform is defined as follows [1, 2], Elzaki transform of the function $f(t)$ is

$$E\left[f(t)\right]=u\int_0^{\infty}f(t)\,e^{-\frac{t}{u}}dt\ ,\qquad t>0$$

(3)

Tarig M. Elzaki and Sailh M. Elzaki in [1,2,3,4,5,6], showed the modified of Sumudu transform [10,11,12] or Elzaki transform was applied to partial differential equations, ordinary differential equations, system of ordinary and partial differential equations and integral equations.

In this paper, Elzaki transform is applied to solve telegraph equations, which the solution of this equation have a major role in the fields of science and engineering.

To obtain Elzaki transform of partial derivative we use integration by parts, and then we have:

$$E\left[\frac{\partial f(x,t)}{\partial t}\right]=\frac{1}{u}T(x,u)-u \qquad E\left[\frac{\partial^2 f(x,t)}{\partial t^2}\right]=\frac{1}{u^2}T(x,u)-f(x,0)-u\frac{\partial f(x,0)}{\partial t}$$

$$E\left[\frac{\partial f(x,t)}{\partial x}\right]=\frac{d}{dx}\left[T(x,u)\right] \qquad E\left[\frac{\partial^2 f(x,t)}{\partial x^2}\right]=\frac{d^2}{dx^2}\left[T(x,u)\right]$$

Proof:

To obtain ELzaki transform of partial derivatives we use integration by parts as follows:

$$\mathrm{E}\left[\frac{\partial f}{\partial t}(x,t)\right]=\int_0^{\infty}u\frac{\partial f}{\partial t}e^{\frac{-t}{u}}dt=\lim_{p\to\infty}\int_0^{p}ue^{\frac{-t}{u}}\frac{\partial f}{\partial t}dt=\lim_{p\to\infty}\left\{\left[ue^{\frac{-t}{u}}f(x,t)\right]_0^p-\int_0^p e^{\frac{-t}{u}}f(x,t)dt\right\}$$

$$=\frac{T(x,u)}{u}-uf(x,0)$$

We assume that f is piecewise continuous and it is of exponential order.

Now

$$\mathrm{E}\left[\frac{\partial f}{\partial x}\right]=\int_0^{\infty}ue^{\frac{-t}{u}}\frac{\partial f(x,t)}{\partial x}dt=\frac{\partial}{\partial x}\int_0^{\infty}ue^{\frac{-t}{u}}f(x,t)dt,\quad (u\sin g\ \ the\ \ Leibnitz\ \ rule)$$

$$=\frac{\partial}{\partial x}\left[T(x,u)\right]\quad and\quad \mathrm{E}\left[\frac{\partial f}{\partial x}\right]=\frac{d}{dx}\left[T(x,u)\right]$$

Also we can find: $\mathrm{E}\left[\frac{\partial^2 f}{\partial x^2}\right]=\frac{d^2}{dx^2}\left[T(x,u)\right]$. To find: $\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]$

Let $\frac{\partial f}{\partial t}=g\ ,\ then$ we have $\mathrm{E}\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]=\mathrm{E}\left[\frac{\partial g(x,t)}{\partial t}\right]=\mathrm{E}\frac{\left[g(x,t)\right]}{u}-ug(x,0)$

$$E\left[\frac{\partial^2 f}{\partial t^2}(x,t)\right]=\frac{1}{u^2}T(x,u)-f(x,0)-u\frac{\partial f}{\partial t}(x,0)$$

(4)

We can easily extend this result to the nth partial derivative by using mathematical induction.

2. Applications

In this section, modified of Sumudu transform or Elzaki transform method will be applied for solving some equations of linear forms. The results reveal that the method is very effective and simple.

To find the solution of equation (1), applying Elzaki transform of that equation and making use the initial conditions to find:

$$T(x,u)-u^2\alpha-u^3\beta+auT(x,u)-au^2\alpha+u^2bT(x,u)+u^2c^2\frac{d^2}{dx^2}T(x,u)=0$$

$$\Rightarrow u^2c^2\frac{d^2}{dx^2}T(x,u)+(1+au+u^2b)T(x,u)=u^2\alpha+u^3\beta+a\alpha u^2$$

This is the second order linear differential equation. The particular solution of this equation is obtained as:

$$T(u,x)=\frac{\alpha u^2+\beta u^3+a\alpha u^2}{c^2u^2D^2+(1+au+bu^2)}=F(u).G(x) \quad , \quad D=\frac{d}{dx}$$

Where $F(u),G(x)$ are functions of u,x respectively.

Now apply the inverse Elzaki transform to find the solution of the general telegraph equation (1) in the form

$$U(x,t)=G(x)E^{-1}\left(F(u)\right)=G(x)f(t)$$

Assume that the inverse Elzaki transform is exists.

Example 2.1:

Consider the telegraph equation:

$$\frac{\partial^2 U}{\partial x}=\frac{\partial^2 U}{\partial t^2}+2\frac{\partial U}{\partial t}+U \tag{5}$$

With the initial conditions:

$$U(x,0)=e^x \quad , \quad U_t(x,0)=-2e^x \tag{6}$$

Appling Elzaki transform to equation (5), and making use the initial conditions (6), to find:

$$\frac{d^2}{dx^2}[T(x,u)] = \frac{T(x,u)}{u^2} - e^x + 2ue^x + 2\frac{T(x,u)}{u} - 2ue^x + T(x,u)$$

$$\Rightarrow u^2T'' - (1+u)^2T = -u^2e^x, \ and \ \ T(x,u) = \frac{u^2e^x}{1+2u}. \ then: U(x,t) = e^{-2t}e^x = e^{x-2t}$$

This is the exact solution of equation (5).

Example 2.2:

Consider the telegraph equation:

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2} + 4\frac{\partial U}{\partial t} + 4U \tag{7}$$

With the initial conditions:

$$U(x,0) = 1 + e^{2x} \quad , \quad U_t(x,0) = -2 \tag{8}$$

Take Elzaki transform of (7), and use (8) to find that:

$$\frac{d^2}{dx^2}T(x,u) = \frac{1}{u^2}T(x,u) - (1+e^{2x}) + 2u + \frac{4}{u}T(x,u) - 4u - 4ue^{2x} + 4T(x,u)$$

And $\quad u^2T''(x,u) - (1+2u)^2T(x,u) = -(2u^3+u^2) - (4u^3+u^2)e^{2x}$

The solution of equation is:

$T(x,u) = \dfrac{u^2}{1+2u} + u^2e^{2x}$, and the inverse of this equation gives the solution of equation (7) in the form:

$$U(x,t) = e^{-2t} + e^{2x}$$

Example 2.3:

Let us consider the telegraph equation:

$$\frac{\partial^2 U}{\partial x^2} = \frac{\partial^2 U}{\partial t^2} + 4\frac{\partial U}{\partial t} + 4U \tag{9}$$

With the initial conditions:

$$U(x,0) = e^x \quad , \quad U_t(x,0) = -e^x \tag{10}$$

Take Elzaki transform of (9), and use (10) to find that:

$u^2T'' = (4u^2+4u+1)T - (u^2+3u^3)e^x$, and the solution of this equation is:

$T(x,u)=\dfrac{u^2e^x}{1+u}$, and the inverse of this equation gives the solution of equation (9) in the form:

$$U(x,t)=e^{-t}.e^{x}=e^{x-t}$$

Example 2.4:

Consider the telegraph equation:

$$\frac{\partial^2 U}{\partial x^2}=\frac{\partial^2 U}{\partial t^2}+\frac{\partial U}{\partial t}+U \tag{11}$$

With the initial conditions:

$$U(x,0)=e^{x} \quad , \quad U_t(x,0)=-e^{x} \tag{12}$$

Take Elzaki transform of (11), and use (12), and use the same method to find the solution of equation (11) in the form: $U(x,t)=e^{-t}.e^{x}=e^{x-t}$

3. Conclusion:

In this work, Elzaki transform is applied to obtain the solution of general linear telegraph equation. It may be concluded that Elzaki transform is very powerful and efficient in finding the analytical solution for a wide class of initial boundary value problems.

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Table 1. Modified of Sumudu transform or ELzaki transform of some functions

$f(t)$	$\mathrm{E}\left[f(t)\right]=T(u)$
1	u^2
t	u^3
t^n	$n!\,u^{n+2}$
$t^{a-1}\big/\Gamma(a), a>0$	u^{a+1}
e^{at}	$\dfrac{u^2}{1-au}$
te^{at}	$\dfrac{u^3}{(1-au)^2}$
$\dfrac{t^{n-1}e^{at}}{(n-1)!}$, $n=1,2,....$	$\dfrac{u^{n+1}}{(1-au)^n}$
$\sin at$	$\dfrac{au^3}{1+a^2u^2}$
$\cos at$	$\dfrac{u^2}{1+a^2u^2}$
$\sinh at$	$\dfrac{au^3}{1-a^2u^2}$
$\cos\mathrm{h}\, at$	$\dfrac{au^2}{1-a^2u^2}$
$e^{at}\sin bt$	$\dfrac{bu^3}{(1-au)^2+b^2u^2}$
$e^{at}\cos bt$	$\dfrac{(1-au)u^2}{(1-au)^2+b^2u^2}$

$t \sin at$	$\dfrac{2au^4}{1+a^2u^2}$
$J_0(at)$	$\dfrac{u^2}{\sqrt{1+au^2}}$
$H(t-a)$	$u^2 e^{-\frac{a}{u}}$
$\delta(t-a)$	$u\,e^{-\frac{a}{u}}$

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