Mathematical Theory and Modeling
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.11, 2014

Trends in Transitive p-Groups and Their Defining Relations

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Abstract

In this paper unless otherwise stated the letter p represents a fixed prime number. The concept of p- groups is fundamental in the theory of groups. Sylow theorems will be assumed known in this paper. In classifying finite groups we know in the Abelian case that the number of groups of order n is influenced largely by the character of the prime factorization of n, and not by the size of n alone. Any finite group G contains the so called Sylow p-subgroups which are p-groups and are closely linked to the structure of G. Recent developments in theory of finite simple groups have brought insights on p-groups and have suggested investigations in diverse areas. In this paper, however, we shall present some of the most basic results on transitive p-groups and their defining relations.

Keywords: transitive p -groups, isomorphism, classification

1. Introduction

Groups of Orders p, p^2, pq, p^3.

A group of prime order p cannot have a proper subgroup and so must be a cyclic group, generated by any element different from the identity. It is well known that a group G without any proper subgroups is cyclic of a prime order. A group G of order p^2, if it is not cyclic, will contain two distinct subgroups of order$p$, say $\{a\}$ and$\{b\}$, where $a^p = 1, b^p = 1,$ and $\{a\} \cap \{b\} = 1,$ since this are both maximal subgroups. Since these are both maximal subgroups they will both be normal, and $G = \{a\} \times \{b\}$; and so, G is an Abelian group with a, b as a basis. Suppose G is of order pq, where $p < q$ are primes. By the third, Sylow theorem, the number of subgroups of order q is of the form $1 + kq$ and divides p, whence it must be 1, and the unique subgroup of order q will be normal, say $\{b\}$, with $b^q = 1$. The number of subgroup of order p is of the form $1 + kp$ and divides q, whence it is 1 or q. If the number is 1, we have for some a a normal subgroup $\{a\}$ with $a^p = 1,$ and G as the direct product of $\{a\}$ and $\{b\}$. But here $c = ab$ is of order pq and G is cyclic. There remains the case with $1 + kp = q$ subgroup order p, where a subgroup $\{a\}$ of order p is not normal. Then we have $a^p = 1, b^q = 1,$ and since $\{b\}$ is normal, $a^{-1}ba = b^r$ for some r. Here if $r = 1, G$ is Abelian and is the cyclic group above. Hence $r \neq 1$. Then $a^{-1}b^i a = b^{ir}$ for any $i,$ and in particular $a^{-1}b^r a = b^{r^2}$, whence $a^{-2}ba^2 = a^{-1}b^r a = b^{r^2}$. More generally we find $a^{-j}ba^j = b^{r^j}$, proceeding by induction. Thus for $j = p$ we have $b = a^{-p}ba^p = b^{r^p}$ whence, $r^p \equiv 1 (mod\ q)$. that this necessary condition on r is also sufficient may be verified by establishing the general rule

$$\left(a^u b^u\right)\left(a^x b^y\right) = a^{u+x} b^{vr^x + y}$$

for multiplying any two elements and proving that this rule defines the group of order pq.

For groups of order p^3, there are three Abelian types, with invariants respectively $\left(p^3\right), \left(p^{2,} p\right)$, and (p, p, p). In finding non-Abelian groups, we handle the cases $p = 2$ and $p - odd$ separately. First let $p = 2$ and consider non-Abelian group of order 8. There can be no element of order 8, since the group would be cyclic. If all elements of order 2, then $(ab)^2 = 1$, or $abab = 1$, $ba = a^2bab^2 = ab$, and the group is Abelian. Hence there must be an element of order 4, say $a^4 = 1$. If $b^2 \notin \{a\} = A$, then $G = A = Ab$ and $b^2 \in A.$. If $b^2 = a$ or a^3, then b is of order 8 and G is cyclic. Hence $b^2 = 1$ or a^2 Also $b^{-1}ab \in A$, since A is normal, and $b^{-1}ab = a$ or a^3, since it is an element of order 4. But with $b^{-1}ab = a, G$ will be Abelian. Hence $b^{-1}ab = a^{3.}$. Thus we have found two non-Abelian groups, the dihedral group with defining relations

$$a^4 = 1, b^2 = 1, b^{-1}ab = a^3,$$

and the quaternion group with defining relations

$$a^4 = 1, b^2 = a^2, b^{-1}ab = a^3,$$

It is easily verified that these relations do define two groups of order 8 and that they are not isomorphic to each other.

Finally, consider non-Abelian groups of order p^3, p an odd prime. Since G is not cyclic, It contains no element of order p^3. Let us first suppose the element G contains an element of order $p^2, a^{p^2} = 1.$ Then $\{a\} = A$, as a maximal subgroup is normal. Let $b \notin A$. Then $G = A + Ab + ... + Ab^{p-1}$, and $b^p \in A, b^{-1}ab = a^r$. Here $r \neq 1$, since G is non-Abelian. Since we find by induction on j that $b^{-j}ab^j = a^{r^j}$, and since b^p as an element of A permutes with a, we have $a = b^{-p}ab^p = a^{r^p}$, whence $r^p \equiv 1 \left(\bmod p^2\right)$. From the Fermat theorem, $r^p \equiv r \left(\bmod p\right)$, and so $r \equiv 1 \left(\bmod p\right)$. Write $r = 1 + sp.$ Then, with j chosen so that $js \equiv 1 \left(\bmod p\right)$, we have

$$b^{-j}ab^j = a^{(1+sp)j} = a^{1+sjp} = a^{1+p}.$$

Since $(j, p) = 1, b^j \notin A$, we may replace b by b^j to get

$$G = A + Ab + ... + Ab^{p-1},$$

Groups of orders p, p^2, p^3

Where $b^{-1}ab = a^{1+p}$.

Now $b^p \in A$, whence $b^p = a^t$. Here t must be a multiple of p since b is not of order p^3. Write $b^p = a^{up}$. Then, using the rule $a^i b = ba^{i(1+p)}$, we calculate and find

$= b^p a^{-up-up(1+2+...+p-1)}$

$= b^p a^{-up} = 1$

Here we use the fact that $1+2+...+p-1 = p(p-1)/2$ is a multiple of p since p is odd. Now with $b_1 = ba^{-u}$, we have the relations $a^{p^2} = 1, b_1^p = 1, b_1^{-1}ab_1 = a^{1+p}$. This last follows since $b_1^{-1}ab_1 = a^u\left(b^{-1}ab\right)a^{-u}$.

As a last case suppose that G contains no elements of order p^2. The center $Z(G)$ must be of order p, since if it were of order p^2, G would be Abelian. $G/Z(G)$ will be of the type $x^p = 1, y^p = 1, yx = xy.$

If in the homomorphism $G \to G/Z(G), a \to x, b \to y,$ then $a^p = 1, b^p = 1, a^{-1}b^{-1}ab = c \in Z(G).$ If $a^{-1}b^{-1}ab = 1$, since a, b and $Z(G)$ generate G, G would be Abelian. Hence $c \neq 1$ is a generator for $Z(G)$ and our relations become

$$a^p = 1, b^p = 1, c^p = 1, ab = ba, ac = ca, bc = cb.$$

1.1.1 TABLE OF DEFINING RELATIONS

$I.G$ order p.

1) Cyclic $a^p = 1$.

$II.G$ order p^2

1) Cyclic. $a^{p^2} = 1$.

2) Elementary Abelian. $a^p = 1, b^2 = 1, ba = ab$.

$III.G$ order $pq, p < q$

Cyclic. $a^{pq}=1.$

Non-Abelian. $a^p=1, b^q=1, a^{-1}ba=b^r,$

$r^p\equiv 1(\bmod q), r\not\equiv 1(\bmod q), p$ divides $q-1$

The solutions of $z^p\equiv 1(\bmod q), z\not\equiv 1(\bmod q)$ are $r, r^2, \ldots r^{p-1},$ and all yield the same group, since replacing a by as a generator of $\{a\}$ replaces r by r^j.

$IV.G$ order p^3.

Abelian.

$a^{p^3}=1.$

$a^{p^2}=1, b^p=1, ba=ab.$

$a^p=b^p=c^p=1, ba=ab, ca=ac, cb=bc.$

Non-Abelian order $2^3=8$.

Dihedral. $a^4=1, b^2=1, ba=a^3b.$

Quaternion $a^4=1, b^2=a^2, ba=a^3b.$

Non –Abelian order p^3, p odd

4) $a^{p^2}=1, b^p=1, b^{-1}ab=a^{1+p}$

5) $a^p=1, b^p=1, c^p=1, ab=bac, ca=ac, cb=bc.$

TRANSITIVE p-GROUPS OF DEGREES p^n (p = 2, 3; n = 2, 3)

2. Results

Let p be a prime number and G be a group acting on a non - empty set Ω of size p^n (n = 2, 3 and p = 2, 3). Here we determine, up to isomorphism, the actual transitive p – groups (abelian and non – abelian) of degrees p^2 and p^3 for p = 2, 3 and achieve a total classification of these according to small degrees. We rely heavily on the algebraic computer software *GAP* (Groups, Algorithms and Programming) to obtain both the presentations and the generators of the afore – mentioned groups.

2.1.1 Lemma

Let Ω be a set, G a group acting on Ω and let H be a transitive subgroup of G on Ω. Then G is transitive on

Ω.

Proof:

Let $\alpha \in \Omega$. Since α^G is an orbit G on Ω, it follows that $\alpha^G \subset \Omega$ (2.1)

Also since H is transitive on Ω, we must have $\alpha^H = \Omega$ (2.2)

Claim: $\Omega \subset \alpha^G$

Let $\beta \in \Omega$, then $\beta \in \alpha^H$ (using (2.2)), thus $\beta = \alpha^h$ for $h \in H \le G$. Hence $\beta = \alpha^h$ for some $h \in G$ and G is transitive on Ω and so $\Omega \subset \alpha^G$ and using (2.1), the result follows.

2.1.2 Lemma

Let $G \le Sym(\Omega)$, where Ω is a set. If $|G| < |\Omega|$, then G is not transitive on Ω.

Proof:

If G is transitive on Ω, then $|\alpha^G| = |\Omega|$, $\forall\ \alpha \in \Omega$, hence

$|G| = |\alpha^G|\,|G_\alpha| = |\Omega|\,|G_\alpha|\ \forall\ \alpha \in \Omega$, thus $|\Omega|\ |\ |G|$ and $|G| \ge |\Omega|$.

2.1.3 Lemma

Let G and K be finite polycyclic groups such that $|G| = |K|$. If G contains all the generators of K, then G = K.

Thus the groups G and K are just two different presentations of the same and one group.

Proof:

Let $\{k_1, k_2, \ldots, k_s\}$ be the set of all generators of K with $k_i^{n_i} = 1$, where

$n_i = o(k_i)$ and $(i = 1, 2, \ldots, s)$; and let $\{g_1, g_2, \ldots, g_r\}$ be the set of the generators of G with $g_i^{m_i} = 1$, $m_i = o(g_i)$, $(i = 1, 2, \ldots, r)$.

Suppose $x \in K$, then we can always write $x = \prod_{i=1}^{s} k_i^{n_i'}$, *where* $n_i' \le n_i$. As each $k_j \in G$ and

$k_j^{n_j} = 1$ we have m_j *divides* n_j and $k_j = \prod_{i=1}^{r} g_i^{m_{ij}}$ *with* $m_{ij} \le m_i$, hence

$$x = \prod_{i=1}^{s} k_i^{n_i'} = \prod_{i=1}^{s}\prod_{j=1}^{r} (g_i^{m_{ij}})^{n_i'} = \prod_{i,j=1}^{r,s} g_i^{m_{ij} n_i'} \ \textit{with}\ \ m_{ij} n_i' \le m_i$$

Consequently $K \subset G$, so K is a subgroup of G and since $|G| = |K|$, the result follows.

2.2 TRANSITIVE 2 – GROUPS OF DEGREE $2^2 = 4$

Let G be a transitive 2 – group of degree $4 = 2^2$ acting on the set $\Omega = \{1, 2, 3, 4\}$. Then $G \le Sym(\Omega)$ and as $|Sym(\Omega)| = 4! = 2^3.3$ we must have $|G| = 2^n$, where n = 1, 2, 3.

Clearly $n \ne 1$, by Lemma2.1.2 and if n = 2, then $|G| = 2^2 = 4$, G is essentially abelian and either $G \cong C_4$ or $G \cong C_2 \times C_2$. For transitivity we must have $\forall \alpha \in \Omega$, $|\alpha^G|\,|G_\alpha| = 4$.

We then have the following possibilities:

$|\alpha^G| = 1, |G_\alpha| = 4,$ (2.3)

$|\alpha^G| = 2, |G_\alpha| = 2,$ (2.4)

$|\alpha^G|=4, |G_\alpha|=1.$ (2.5)

Thus (2.5) holds for transitivity.

If $G \cong C_4$, then we may write $G = \langle a \rangle$ where $a^4 = 1$, $a \in Sym(4)$ and take $a = (1, 2, 3, 4)$.

If $G \cong C_2 \times C_2$, then $G = \{1, a, b, ab\}$ where $a, b \in Sym(4)$ with $a^2 = b^2 = (ab)^2 = 1$ with $a = (1, 3)(2, 4)$ and $b = (1, 2)(3, 4)$.

If $n = 3$, then $|G| = 2^3$ and for transitivity we must have $\forall\ \alpha \in \Omega$, $|\alpha^G||G_\alpha| = 8$. This yields the following possibilities for G:

$|\alpha^G|=1, |G_\alpha|=8,$ (2.6)

$|\alpha^G|=2, |G_\alpha|=4,$ (2.7)

$|\alpha^G|=4, |G_\alpha|=2,$ (2.8)

$|\alpha^G|=8, |G_\alpha|=1$ (2.9)

Thus (2.8) only holds for transitivity and in this case G is non – abelian. Consequently G is either the quaternion group or the octic group.

We notice that the octic group $G = \langle a, b: a^4 = 1, b^2 = 1, ba = a^3b \rangle$, where $a = (1, 2, 3, 4)$, $b = (1, 4)(2, 3)$, is non – abelian transitive on Ω.

We now summarize our finding:

2.2.1 Proposition

There are, up to isomorphism, 3 transitive 2 - groups of degree $2^2 = 4$: the Klein 4 - group, the cyclic group of order 4 and the octic group (see table1).

2.3 TRANSITIVE 2 - GROUPS OF DEGREE $2^3 = 8$

Let G be a transitive 2 - group of degree $2^3 = 8$ acting transitively on the set $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8\}$, then $G \leq Sym(8)$ and since $|Sym(8)| = 8! = 2^7.3^2.5.7$ and G is a 2 – group, we must have $|G| = 2^n$, where n = 1, 2, 3, 4, 5, 6, 7.

Clearly, $n \neq 1$ and $n \neq 2$ by Lemma 2.1.2 and when n = 3, then $|G| = 2^3 = 8$ and so G is regular.

Now it is well - known that there are exactly 5 non – isomorphic groups of order 8, 3 of which are abelian, namely C_8, $C_2 \times C_4$, $C_2 \times K_4$ and 2 non – abelian namely the octic group and quaternion group.

First assume G is abelian. If $G \cong C_8$, then $G_{1,3} = \langle x \rangle$ for some x in $Sym(8)$ with $x^8 = 1$, we may take $x = (1, 2, 3, 4, 5, 6, 7, 8)$.

If $G \cong C_2 \times C_4$, then $G_{2,3} = \langle a, b: a^4 =1, b^2 =1, ab = ba \rangle$ for some distinct a and b in $Sym(8)$. Taking $a = (1, 3, 5, 7)(2, 4, 6, 8)$ and $b = (1, 4)(5, 8)(3, 6)(2, 7)$ do satisfy $G_{2,3}$.

If $G \cong C_2 \times K_4$, then $G_{3,3} = \langle a, b, c: a^2 = b^2 = c^2 =1, ba = ab, ca = ac, cb = bc \rangle$. Taking $a = (1,2)(3, 4)(5, 6)(7, 8)$, $b = (1, 3)(2, 4)(5, 7)(6, 8)$, $c = (2, 5)(1, 6)(3, 8)(4, 7)$ do satisfy $G_{3,3}$.

Next we assume G non – abelian, then either $G \cong D_2 = \langle x, y: x^4 = 1, y^2 = 1, yx = x^3y \rangle$ or $G \cong E = \langle x, y: x^4 = 1, x^2 = y^2, yx = x^3y \rangle$.

Now the elements $x = (1, 4, 6, 8)(3, 5, 7, 2)$ and $y = (8, 5)(1, 3)(2, 4)(6, 7)$ of $Sym(8)$ do satisfy D_2.

And taking $x = (1, 5, 7, 2)(4, 6, 8, 3)$ and $y = (1, 6, 7, 3)(4, 2, 8, 5)$ satisfy the requirements for E.

Clearly all the above – named groups satisfy $|\alpha^G| = 8$, $|G_\alpha| = 1$, $\forall\ \alpha \in \Omega$.

2.3.1 Lemma

There are, up to isomorphism, 5 transitive 2 – groups of degree 2^3 and order $2^3 = 8$, namely the groups $G_{1,3}$, $G_{1,2}$, $G_{3,3}$, D_2 and E described above.

When n = 4, then $|G| = 2^4 = 16$ and for transitivity, we must have $|\alpha^G| = 8$, $|G_\alpha| = 2, \forall\ \alpha \in \Omega$.

In this case G is not abelian and so contains no elements of order 16. Suppose G is of exponent 8, then G contains an element a, say, of order 8 and let H = < a >. Then

$G = H \cup Hb$ for some $b \in G$, so $[G: H] = 2$ and H is normal in G. Clearly $b^2 \in H$ and we have the following possibilities for b^2:

$b^2=1$, (2.10)

$b^2=a$, (2.11)

$b^2=a^2$, (2.12)

$b^2=a^3$, (2.13)

$b^2=a^4$, (2.14)

$b^2=a^5$, (2.15)

$b^2=a^6$ (2.16) ,

$b^2=a^7$. (2.17)

Cases (2.11), (2.13), (2.15) and (2.17) imply that G = H which is impossible since $|G|>|H|$. Thus cases (2.10), (2.12), (2.14), and (2.16) hold.

Now as H is normal in G, $b^{-1}ab \in H$ and as a and $b^{-1}ab$ have the same order, we have

$b^{-1}ab = a$ (2.18)

or $b^{-1}ab=a^3$ (2.19)

or $b^{-1}ab=a^5$ (2.20)

or $b^{-1}ab=a^7$. (2.21)

Clearly $b^{-1} a b \neq a$, otherwise G would be abelian. Thus we have the following possibilities for G:

$G_{1,4}=<a,b:a^8=1,b^2=1,ab=ba^3>$, (2.22)

$G_{2,4}=<a,b:a^8=1,b^2=1,ab=ba^5>$, (2.23)

$G_{3,4}=<a,b:a^8=1,b^2=1,ab=ba^7>$, (2.24)

$G_{1,4}^{/}=<a,b:a^8=1,b^2=a^2,ab=ba^3>$, (2.25)

$G_{2,4}^{/}=<a,b:a^8=1,b^2=a^2,ab=ba^5>$, (2.26)

$G_{3,4}^{/}=<a,b:a^8=1,b^2=a^2,ab=ba^7>$, (2.27)

$G_{1,4}^{//}=<a,b:a^8=1,b^2=a^4,ab=ba^3>$, (2.28)

$G_{2,4}^{//}=<a,b:a^8=1,b^2=a^4,ba=ba^5>$, (2.29)

$G_{3,4}^{//}=<a,b:a^8=1,b^2=a^4,ab=ba^7>$, (2.30)

$G_{1,4}^{///}=<a,b:a^8=1,b^2=a^6,ab=ba^3>$, (2.31)

$G_{2,4}^{///}=<a,b:a^8=1,b^2=a^6,ab=ba^5>$, (2.32)

$G_{3,4}^{///}=<a,b:a^8=1,b^2=a^6,ab=ba^7>$, (2.33)

Now the elements a = (1, 2, 3, 4, 5, 6, 7, 8) and b = (4, 8)(1, 7)(3, 5) satisfy the requirements for $G_{1,4}$.

Next taking a = (1, 2, 3, 4, 5, 6, 7, 8), b = (6, 2)(4, 8) satisfy the requirements for $G_{2,4}$. Taking a = (1, 2, 3, 4, 5, 6, 7, 8), b = (2, 8)(3, 7)(4, 6) satisfy the requirements for $G_{3,4}$.

It is easy to see that $G_{2,4} = G_{2,4}^{/} = G_{2,4}^{///}$ (by Lemma 2.1.3).

We easily see, by *Gap* – programmes, that the groups $G_{1,4}^{/}$, $G_{3,4}^{/}$, $G_{2,4}^{//}$, $G_{3,4}^{//}$, $G_{1,4}^{///}$ and $G_{3,4}^{///}$ do not exist as transitive permutation groups of degree 8.

If G is of exponent 4, then we obtain the following groups:

$G_{5,4} = < a, b, c: a^4 =1, b^2 =1, ab = ba, c^2=1, ac = ca^3, bc = ca^2b >$ with generators

a = (1, 3, 5, 7)(2, 4, 6, 8), b = (2, 6)(4, 8) and c = (1, 2)(3, 8)(4, 7)(5, 6)(see *GAP* – programme 4).

$G_{6,4} = < a, b, c, d: a^2 = b^2 = c^2 =1, ba = ab, ca = ac, cb = bc, d^2=1, ad = db, bd = db,$

$cd = dabc >$ with generators a, b, c the same as those of $G_{3,3}$ and d = (2, 3)(6, 7).

Note that the group $G_{7,4} = < a, b: a^4 =1, b^4 =1, ab = ba^3 >$, with $|G_{7,4}|=16$ does not exist.

If G is of exponent 2, then G does not exist as a permutation group.

We summarize our finding into the following:

2.3.2 Lemma

There are, up to isomorphism, 5 non – abelian transitive 2 - groups of degree 2^3 and order $2^4=16$, namely the groups $G_{1,4}$, $G_{2,4}$, $G_{3,4}$, $G_{5,4}$ and $G_{6,4}$ described above.

When n =5, then $| G | = 2^5 = 32$ and for transitivity we must have:

$|\alpha^G| = 8$, $|G_\alpha| = 4$, $\forall\ \alpha\in\Omega$, in this case G is non – regular and must be non – abelian.

Since G is of degree 8, it contains no elements of order 16 and 32.

Let $a \in G$ be such that $a^8 =1$ and let $A = < a >$, set $H = A \cup A b$ for some $b \in H – A$. Since

$| N_H (A) | = | H | = 16$, then $A \trianglelefteq H$ and $b^{-1} a b \in A$, this yields the following valid equations:

$$ab=ba^3 \tag{2.34}$$

or $$ab=ba^5 \tag{2.35}$$

or $$ab=ba^7. \tag{2.36}$$

Thus we have the following possibilities for H:

$$H_1 = < a, b: a^8 =1, b^2 =1, ab = ba^3 >, \tag{2.37}$$

$$H_2=<a,b:a^8=1,b^2=1,ab=ba^5>, \tag{2.38}$$

$$H_3=<a,b:a^8=1,b^2=1,ab=ba^7>, \tag{2.39}$$

$$H_1^{/}=<a,b:a^8=1,b^2=a^2,ab=ba^3>, \tag{2.40}$$

$$H_2^{/}=<a,b:a^8=1,b^2=a^2,ab=ba^5>, \tag{2.41}$$

$$H_3^{/}=<a,b:a^8=1,b^2=a^2,ab=ba^7>, \tag{2.42}$$

$$H_1^{//}=<a,b:a^8=1,b^2=a^4,ab=ba^5>, \tag{2.43}$$

$$H_2^{//}=<a,b:a^8=1,b^2=a^4,ab=ba^5>, \tag{2.44}$$

$$H_3^{//}=<a,b:a^8=1,b^2=a^4,ab=ba^7>, \tag{2.45}$$

$$H_1^{///}=<a,b:a^8=1,b^2=a^6,ab=ba^3>, \tag{2.46}$$

$$H_2^{///}=<a,b:a^8=1,b^2=a^6,ab=ba^5>, \tag{2.47}$$

$$H_3^{///}=<a,b:a^8=1,b^2=a^6,ab=ba^7> \tag{2.48}$$

But these are the groups obtained when treating the case n = 4, hence only the groups H_1, H_3, H_3 are valid. Now $G_i = H_i \cup H_i c$ (i = 1, 2, 3) for some $c \in G – H_i$. Clearly $c^2 \in H_i$ as

$[G: H_i] = 2$ (for i = 1, 2, 3). Also there is no c in $G – H_i$ such that $c^2 \in A b$. Hence we must have $c^2\in A$, and from above we have the following possibilities:

$$c^2=1 \tag{2.49}$$

or $$c^2=a^2 \tag{2.50}$$

or $$c^2=a^4 \tag{2.51}$$

or $$c^2=a^6 \tag{2.52}$$

Also as $H_i \trianglelefteq G$, $c^{-1}ac \in H_i = A \cup Ab$ and $c^{-1}bc \in H_i = A \cup Ab$, for each i =1, 2, 3.

We first treat case H_1:

Since $o(c^{-1}ac) = o(a) = 8$, we must have:

$ac=ca$ (2.53)

or $ac=ca^3$ (2.54)

or $ac=ca^5$ (2.55)

or $ac=ca^7$ (2.56)

Also since $o(c^{-1}bc) = o(b) = 2$, we must have:

$bc=cb$ (2.57)

or $bc=ca^4$ (2.58)

or $bc=ca^2b$ (2.59)

or $bc=ca^4b$ (2.60)

or $bc=ca^6b$ (2.61)

Combining the equations (2.49) – (2.52) with equations (2.53) – (2.56) and with equations (2.57) - (2.61) yields 80 different presentations of the same non – abelian group containing H_1 as a subgroup and for reference purposes, we choose one of them, say,

$G_{1,5} = < a, b, c: a^8 = 1, b^2 = 1, ab = ba^3, c^2 = 1, bc = cb, ac = ca^7 >$ with

$a = (1, 2, 3, 4, 5, 6, 7, 8)$, $b = (4, 8)(1, 7)(3, 5)$ and $c = (1, 3)(4, 8)(5, 7)$(here c determined from *Gap* – programme 1).

Next we consider the case H_2, and following the above arguments, we get 96 different presentations of the same non – abelian group containing H_2 as a subgroup. As a representative of these groups we choose the following:

$G_{2,5} = < a, b, c: a^8 = 1, b^2 =1, ab = ba^5, c^2 =1, cb = bc, ac = ca^7 >$ with

$a = (1, 2, 3, 4, 5, 6, 7, 8)$, $b = (2, 6)(4, 8)$ and $c = (2, 8)(3, 7)(4, 6)$ (here c is determined by some adjustments to *Gap* – programme 1). But we see that the set of generators of $G_{2,5}$ is contained in $G_{1,5}$ and by Lemma 2.1.3, $G_{2,5} = G_{1,5}$.

Lastly we examine the case H_3:

Here we obtain 144 different presentations of the same non – abelian group $G_{3,5}$ containing H_3 as a subgroup. Again for reference purposes we choose one of these, say,

$G_{3,5} = < a, b, c: a^8 =1, b^2 =1, ab = ba^7, c^2 =1, bc = cb, ac = ca^3 >$ with $a = (1, 2, 3, 4, 5, 6, 7, 8)$, $b = (2, 8)(3, 7)(4, 6)$, $c = (2, 4)(3, 7)(6, 8)$ (here c is obtained from *Gap* – programme 1).

Again the generators of $G_{3,5}$ are in $G_{1,5}$ and by Lemma 2.1.3, $G_{3,5} = G_{1,5}$.

There is no elements a, b in *Sym* (8) with $G = < a, b: a^8 =1, b^4 =1, ab = ba^3 >$ and $|G| = 2^5 = 32$.

If G is of exponent 4, we have the following groups:

$G_{4,5} = < a, b, c: a^4 =1, b^4 =1, ab = ba, c^2 =1, ac = cab^2, bc = cb^3 >$ with generators

$a = (1, 3, 5, 7)(2, 4, 6, 8)$, $b = (2, 4, 6, 8)$ and $c = (4, 8)$,

$G_{5,5} = < a, b, c, d: a^4 =1, b^2 =1, ab = ba, c^2 =1, ac = ca^3, bc = ca^2b, d^2=1, ad = da^3$, $bd = db, \quad cd = dc >$ with the same generators a, b, c for $G_{5,4}$ and $d = (3, 7)(4, 8)$ (see *Gap* - programme 2).

From $G_{6,4}$, we obtain a group of exponent 4 isomorphic to $G_{3,5}$.

If G is of exponent 2, then G does not exist as a permutation group.

We now summarize our findings into the following:

2.3.3 Lemma

There are, up to isomorphism, 3 non – abelian transitive 2 - groups of degree 2^3 and order $2^5 = 32$, namely the groups $G_{1,5,}$ $G_{2,5}$ and $G_{3,5}$ described above.

Mathematical Theory and Modeling www.iiste.org
ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online)
Vol.4, No.11, 2014

When n = 6, then $|G| = 2^6 = 64$ and for G to be transitive, we must have:
$|\alpha^G| = 8$, $|G_\alpha| = 8$, $\forall\ \alpha\in\Omega$.
Consequently G is not abelian. Clearly G contains no elements of order 64, 32 and 16, hence G is essentially of exponent 8. If G contains two elements of order 8, then G must be of the form:
$G_{0,6} = < a, b: a^8 =1, b^8 =1, ab = ba^3 >$ or $G_{00,6} = < a, b: a^8 =1, b^8 =1, ab = ba^5 >$ or
$G_{000,6} = < a, b: a^8 =1, b^8 =1, ab = ba^7 >$.
But computations confirm the non – existence of such permutations a and b satisfying the requirements of $G_{0,6}$ or $G_{00,6}$ or $G_{000,6}$. Consequently G contains exactly one generator of order 8.
Now let $a \in G$ with $a^8 =1$ and let $A = < a >$. For $b \in G – A$, let $H = < b, A >$, then
$H = A \cup A b$ and $[H: A] = 2$, so that $A \trianglelefteq H$. Consequently, $b^{-1}ab \in A$ and since
$o(b^{-1}ab) = o(a) = 8$ and G is non – abelian, it follows that:

$ab=ba^3$ (2.62)
or $ab=ba^5$ (2.63)
or $ab=ba^7$ (2.64)

Also as $[H: A] = 2$, $b^2 \in A$. Hence either:

$b^2=1$ (2.65)
or $b^2=a^2$ (2.66)
or $b^2=a^4$ (2.67)
or $b^2=a^6$ (2.68)

Combining the equations (2.62) – (2.64) and the equations (2.65) – (2.68) yields the already obtained groups H_1, H_2, … (see the case n = 4 above) of which only 3 are valid.
Let H_i (i =1, 2, 3) be any such group and let $K_i = < c, H_i >$, for some $c \in G - H_i$ then
$K_i = H_i \cup H_i c$ and $[K_i: H_i] = 2$.Thus $H_i \trianglelefteq K_i$ and $c^2 \in H_i$, $c^{-1}ac \in H_i$ and $c^{-1}bc \in H_i$ (since
$a \in A \trianglelefteq H_i$, $b\in Ab \subseteq H_i$). Thus the group K_i so constructed is just the unique group $G_{1,5}$ obtained in the case n = 5.
Finally let $G_{1,6} = < d, G_{1,5} >$ for some $d \in G_{1,6} - G_{1,5}$, then let $G_{1,6} = G_{1,5} \cup G_{1,5}d$,
$[G_{1,6} : G_{1,5}] = 2$ and $G_{1,5} \trianglelefteq G_{1,6}$. Consequently, $d^2 \in G_{1,5}$, $d^{-1}ad$, $d^{-1}bd$, $d^{-1}cd \in G_{1,5}$ (since $a\in H_i \subseteq G_{1,5}$, $b\in H_i \subseteq G_{1,5}$, $c\in G_{1,5}$) for each i. The elements a, b, c are already known and we look for the generator d satisfying the above requireme nts.

Clearly $|G_{1,6}| = 2^6$ and the group $G_{1,6}$ contains $G_{1,5}$ as a subgroup and is the group we are after. As $d^2 \in G_{1,5}$, then $o (d^2) = 1$ or 2 or 4, that is,

$d^2=1$ (2.69)
or $d^2=a^4$ (2.70)
or $d^2=b$ (2.71)
or $d^2=a^2b$ (2.72)
or $d^2=a^4b$ (2.73)
or $d^2=a^6b$ (2.74)
or $d^2=c$ (2.75)
or $d^2=ac$ (2.76)
or $d^2=a^2c$ (2.77)
or $d^2=a^3c$ (2.78)
or $d^2=a^4c$ (2.79)
or $d^2=a^5c$ (2.80)

or $d^2=a^6c$ (2.81)
or $d^2=a^7c$ (2.82)
or $d^2=bc$ (2.83)
or $d^2=a^4bc$ (2.84)
or $d^2=a^2$ (2.85)
or $d^2=a^6$ (2.86)
or $d^2=ab$ (2.87)
or $d^2=a^3b$ (2.88)
or $d^2=a^5b$ (2.89)
or $d^2=a^7b$ (2.90)
or $d^2=a^2bc$ (2.91)
or $d^2=a^6bc$ (2.92)

If o (d^2) =1 then d^2 =1. But $d^{-1}ad \in G_1$ and since $o(d^{-1}ad) = o(a) = 8$, it follows that

$ad=da$ (2.93)
or $ad=da^3$ (2.94)
or $ad=da^5$ (2.95)
or $ad=da^7$ (2.96)
or $ad=dabc$ (2.97)
or $ad=da^3bc$ (2.98)
or $ad=da^5bc$ (2.99)
or $ad=da^7bc$ (2.100)

Also $d^{-1}bd \in G_1$ and as o $(d^{-1}bd)$ = o (b) = 2, if follows that

$bd=db$ (2.101)
or $bd=da^4$ (3.102)
or $bd=dc$ (2.103)
or $bd=da^2b$ (2.104)
or $bd=da^4b$ (2.105)
or bd=da6b (2.106)
or $bd=dac$ (2.107)
or $bd=da^2c$ (2.108)
or $bd=da^3c$ (2.109)
or $bd=da^4c$ (2.110)
or $bd=da^5c$ (2.111)
or $bd=da^6c$ (2.112)
or $bd=da^7c$ (2.113)

As $d^{-1}cd \in G$ and $o(d^{-1}cd) = o(c) = 2$, it follows that

$cd=dc$ (2.114)
or $cd=da^4$ (2.115)
or $cd=dc$ (2.116)
or $cd=da^2b$ (2.117)
or $cd=da^4b$ (2.118)
or cd=da6b (2.119)
or $cd=dac$ (2.120)
or $cd=da^2c$ (2.121)

or $cd=da^3c$ (2.122)
or $cd=da^4c$ (2.123)
or $cd=da^5c$ (2.124)
or $cd=da^6c$ (2.125)
or $cd=da^7c$ (2.126)

Combining the equations (2.69) – (2.92) with the equations (2.93) – (2.100) with the equations (2.101) – (2.113) with the equations (2.114) - (2.126) yields 45 valid different representations of the group $G_{1,6}$ and we choose one of them as a representative of $G_{1,6}$, thus $G_{1,6} = < a, b, c, d: a^8 = 1, b^2 = 1, ab = ba^3, c^2 = 1, bc = cb, ac = ca^7, d^2 = 1, bd = da^2c,$
$cd = da^6b, ad = da^3bc >$, where a = (1, 2, 3, 4, 5, 6, 7, 8), b = (1, 7)(3, 5)(4, 8),
c = (1, 3)(4, 8)(5, 7) and d = (1, 6)(2, 5)(3, 8)(4, 7) (here we use *Gap* – programme 2 to obtain d) and so and $G_{1,6}$ is transitive on Ω.

If G is of exponent 4, then we have the following presentations:
$G_{2,6} = < a, b, c: a^4 =1, b^4 =1, ab = ba, c^4 =1, ac = ca^3b^2, bc = ca^3b >$, with generators
a = (1, 3, 5, 7)(2, 4, 6, 8), b = (2, 4, 6, 8) and c = (1, 2, 3, 4)(5, 6, 7, 8).
$G_{3,6} = < a, b, c, d, e: a^4 =1, b^2 =1, ab = ba, c^2 =1, ac = ca^3, bc = ca^2b, d^2 =1, ad = da^3,$
$bd = db, cd = dc, e^2=1, ae = eab, be = eb, ce = ecd, de = ed >$, where a, b, c, d are the same generators as those of $G_{3,5}$ and e = (4, 8)
Thus, we have:

2.3.4 Lemma
There are, up to isomorphism, 3 non – abelian transitive 2 – groups of degree 2^3 and order $2^6 = 64$, namely the groups $G_{1,6}$, $G_{2,6}$ and $G_{3,6}$ described above.

When n = 7, then $|G| =2^7 = 128$ and for G to be transitive, we must have:
$|\alpha^G| = 8, |G_\alpha| =16, \forall\alpha \in\Omega.$
Arguing in a similar fashion as in the case n = 6, we see that G contains $G_{1,6}$ as a subgroup and that a presentation of G of exponent 8 is
$G_{1,7} = < a, b, c, d, e: a^8 =1, b^2 =1, ab = ba^3, c^2 =1, bc = cb, ac = ca^7, d^2 =1, bd = da^2c,$
$cd = da^6b, ad = da^3bc, e^2 =1, be = ea^4c, ce = ea^4b, de = ea^3c, ae = ebd >$ where a, b, c, d are the same generators as those of $G_{1,6}$ and e = (2, 4)(6, 8).

If G is of exponent 4, then we have the following presentations:
$G_{2,7} = < a, b, c, d: a^4=1, b^4=1, ab = ba, c^4=1, ac = ca^3b^2, bc = ca^3b, d^2 =1, ad = dab^2,$
$bd = db^3, cd = dac^3 >$, where a, b, c are the same generators as those of $G_{2,6}$ and d = (4, 8).
$G_{3,7} = < a, b, c, d, e, f: a^4 =1, b^2 =1, ab = ba, c^2 =1, ac = ca^3, bc = ca^2b, d^2 =1, ad = da^3,$
$bd = db, cd = dc, e^2 =1, ae = eab, be = eb, ce = ecd, de = ed, f^2 =1, af = facd, bf = fbd,$
$cf = fc, df = fd, ef = fde >$, where a, b, c, d, e are the same generators as those of $G_{3,6}$ and
f = (3, 4)(7, 8).

Thus, we have:
2.3.5 Lemma
There are, up to isomorphism, 3 non – abelian transitive 2 – groups of degree 2^3 and order $2^7 =128$, namely the groups $G_{1,7}$, $G_{2,7}$ and $G_{3,7}$ described above.
We now summarize our findings in table2 and we have:

2.3.6 Proposition

There are, up to isomorphism, 19 transitive 2 – groups of degree $2^3 = 8$, three of these are abelian. Of the remaining 16 non – abelian, 6 are of exponent 8, 10 are of exponent 4 and none is of exponent 2.

2.4 TRANSITIVE 3 - GROUPS OF DEGREES 3^2 AND 3^3

Here we are working in the groups *Sym* (9) and *Sym* (27) of large orders. Especially the order of *Sym* (27) is too large for the memory of any computer since the *Gap* software was written/ manufactured with such limitations. Consequently, it is impossible to run *Gap* – based computer programmes that search through the whole of *Sym* (27) for results and such programmes are rather run on subgroups of *Sym* (27) of orders relatively small enough for the computer memory.

In order to determine the transitive 3 – groups of degrees 3^2 and 3^3 up to equivalence, we partly make use of the method exposed in determining the transitive 2 – groups of degrees 2^2 and 2^3, Lemmas 2.1.1, 2.1.2, 2.1.3, some facts from Number Theory and the following results:

2.4.1 Lemma

Let p be an arbitrary but fixed prime. Let

$$G = \langle a, b : a^{p^3} = 1,\ b^{p^2} = 1,\ ab = ba^r \rangle \ and\ G_* = \langle a, b_* : a^{p^3} = 1,\ b_*^{\ p} = 1,\ ab_* = b_* a^r \rangle,$$
$$where\ r^{p^2} \equiv 1 \bmod (p^3)\ and\ distinct\ a, b, b_* \in Sym(p^3).$$

Then G = G_* and there is no element b $\in Sym\ (p^3)$ satisfying G where | G | = p^5.

Proof:

We first show that | G | = | G_* | = p^4. Clearly | G_* | = p^4 and let A= < a >, $a^{p^3} = 1$. Then

b $\notin$ A and $G = A \bigcup Ab \bigcup Ab^2 \bigcup \ldots \bigcup Ab^{p^2-1}$, where |A b^i | = p^3, for each

i = 0, 1, . . . , p^2 - 1.

Clearly a^{p^2} is an element of A of order p, while b^p is an element of Ab^p of order p, hence

$$|\langle a^{p^2} \rangle| = |\langle b^p \rangle| = p\ and\ so\ \langle a^{p^2} \rangle \cong \langle b^p \rangle\ and\ \text{since} < a^{p^2} > and < b^p > are\ subgroups\ of$$
$$the\ same\ group, it\ follows\ that\ a^{p^2} = b^p\ or\ a^{p^2} b^{(p-1)p} = 1.\ Now\ as\ g.c.d(1, p-1) = 1, it\ follows$$
$$that\ <b> = <b^{p-1}>, so\ that\ 1 = a^{p^2} b^{(p-1)p} \in Ab^p$$

Thus A = Ab^p and A b^i = Ab^{p+i} for each i = 0, 1, . . . , p^2 – 1. Consequently,

$$Ab^{p^2-1} = Ab^{p^2-1+p} = Ab^{p-1},\ Ab^{p^2-2} = Ab^{p^2-2+p} = Ab^{p-2}, \ldots, Ab^{p^2-(p^2-p-1)} = Ab^{p+1} = Ab^p$$ So

that

$$G = A \bigcup Ab \bigcup Ab^2 \bigcup \ldots \bigcup Ab^{p-1} \bigcup Ab^p \bigcup Ab^{p+1} \bigcup \ldots \bigcup Ab^{p^2-1}$$
$$= A \bigcup Ab \bigcup Ab^2 \bigcup \ldots \bigcup Ab^{p-1}$$

Thus | G | = | A$\cup$Ab $\cup$ … Ab^{p-1} | = | A | + | Ab | + … + | Ab^{p-1} | = p^3. p = p^4 = | G_* |.

Since $b_*^p = 1$, $we\ have\ that\ b_*^{p^2} = 1$ and so b_* $\in$G. Hence the result by lemma 2.1.3

2.4.2 Lemma

Let p be an arbitrary but fixed prime. Let

$$G = \langle a, b : a^{p^3} = 1, b^{p^3} = 1, ab = ba^r \rangle_* \ and \ G_* = \langle a, b_* : a^{p^3} = 1, b_*^{\ p} = 1, ab_* = b_* a^r \rangle,$$
$$where \ r^{p^3} \equiv 1 \bmod(p^3) \ and \ distinct \ a, b, b_* \in Sym(p^3).$$

Then G = G_* and there is no element b $\in Sym\ (p^3)$ satisfying G where | G | = p^6.

Proof:

This follows a similar argument used in the proof of Lemma 2.4.1. Here a^{p^2} is an element of A of order p while b^{p^2} is an element of Ab^p of order p, where A = < a > with $a^{p^3} = 1$ and $b \notin$A.

2.4.3 Lemma

Let p be an arbitrary but fixed prime. Let

$$G = \langle a, b : a^{p^2} = 1, b^{p^2} = 1, ab = ba^r \rangle_* \ and \ G_* = \langle a, b_* : a^{p^2} = 1, b_*^{\ p} = 1, ab_* = b_* a^r \rangle,$$
$$where \ r^{p^2} \equiv 1 \bmod(p^2) \ and \ distinct \ a, b, b_* \in Sym(p^3).$$

Then G = G_* and there is no element $b \in Sym\ (p^3)$ satisfying G where | G | = p^4.

Proof:

This follows a similar argument used in the proof of Lemma 2.4.1. Here a^p is an element of A of order p while b^p is an element of Ab^p of order p, where A = < a > with $a^{p^2} = 1$ and b $\notin$A.

Our Main Result

2.4.4 Theorem

Let p be an arbitrary but fixed prime number and G a non – abelian transitive p – group of degree p^3, exponent p and rank 5, then every p – group of degree p^3 and rank 6 containing G as a normal subgroup, is of exponent p^2.

Proof:

Let $G^/$ be a p – group containing G as a normal subgroup, then by Lemma 2.1.1, $G^/$ is transitive of degree p^3 and by Lemma 2.1.2 and as a consequence of Lagrange's theorem, $G^/$ contains elements of orders p, p^2 and p^3. Since G is of exponent p and rank 5, G is generated by 5 generators each of order p, hence $G^/$ is generated by 6 generators each of order p and $|G^/| = p^6$.

We have that $G^/$ = < G, f >, for some f $\in Sym\ (p^3)$ such that $f^{\ p} = 1$, f $\notin$ G and G $\trianglelefteq G^/$. Now let y $\in G^/$ - G such that y is of order p^3. Then $y^p \neq 1$ is an element of $G^/$ of order p^2. As $\left[G' : G\right] = p$, we have $G^/ = G \cup G f \cup \ldots \cup G f^{\ p-1}$, and so $y^p \in$ G or $y^p \in$ G $f^{\ k}$, for some integer k with $1 \le k \le p$ -1. Now as G is of index p in $G^/$, it follows that $y^p \notin$ G, and hence $y^p \in$ G $f^{\ k}$, thus $y^p f^{-k} \in G$ but $y^p \notin$ G and $f^{-k} \notin G$ for any k with $1 \le k \le p$ -1. Consequently we must have $y^p f^{-k} = 1 \ and \ y^p = f^{\,k}$. Thus $1 = f^{\,p} = (f^{\,p})^k = (f^{\,k})^p = y^{p^2}$, but this is impossible since y is an element of $G^/$ of order p^3. Thus $G^/$ contains no elements of order p^3, but an element of order p^2 and the exponent of G is p^2.

2.4.5 Remark

(i) It is well – known that the solutions of $z^{\ p} \equiv 1$ (mod q), where z is not congruent to 1 modulo q are

$r, r^2, ..., r^{p-1}$ and all yield the same group, since by replacing a by a^j as a generator of < a > replaces r by r^j. Consequently, the groups

< a, b: $a^m = 1$, $b^n = 1$, a b = b a^r >, where r is such that $r^n \equiv 1$ (mod m) and r ≠1, are the same.

(ii) In the light of (i) above, we easily see that the only instance that the group

$$G = \langle a, b : a^{p^2} = 1, b^{p^2} = 1, ab = ba^r \rangle \text{ where } r^{p^2} \equiv 1 \bmod(p^2) \text{ and distinct } a, b \in Sym(p^3),$$

with | G | = p^4 exists as a permutation group of degree p^3 is that it be abelian.

The same remark applies to the group

$$G = \langle a, b, c : a^{p^2} = 1, b^{p^2} = 1, ab = ba^r, c^{p^2} = 1, ac = ca^s, bc = cb^t \rangle \text{ where } r^{p^2} \equiv 1 \bmod(p^2), s^{p^2} \equiv 1 \bmod(p^2),$$
$$t^{p^2} \equiv 1 \bmod(p^2) \text{ and distinct } a, b, c \in Sym(p^3).$$

PROGRAMME 1:

```
gap>s8:=Group((1,2),(1,2,3,4,5,6,7,8));;
gap> a:=(1,2,3,4,5,6,7,8);;b:=(1,7)(3,5)(4,8);;
gap> h:=Subgroup(s8,[a,b]);;
gap> diff:=Difference(s8,h);;
gap> req:=[];;
gap> for c in diff do
> if c^2 = () then
> if b^c = b then
> if a^c = a^7 then
> Add(req,c);
> fi;
> fi;
> fi;
> od;
gap> req;
[ (1,3)(4,8)(5,7), (1,7)(2,6)(3,5) ]
```

PROGRAMME 2:

```
Gap> s8:=SymmetricGroup(8);;
gap>a:=(1,2,3,4,5,6,7,8);;b:=(1,7)(3,5)(4,8);;c:=(1,3)(4,8)(5,7);;
gap> H:=Subgroup(s8,[a,b,c]);;
gap> diff:=Difference(s8,H);;
gap> req:=[];;
gap> for r in diff do
> if r^2=() then
> if Order(s8,r)<>4 then
```

```
> if Order(s8,r)<>8 then
> if a^r in H then
> if b^r in H then
> if c^r in H then
> if Size(Subgroup(s8,[a,b,c,r]))=64 then
> Add(req,r);
> fi;
> fi;
> fi;
> fi;
> fi;
> fi;
> fi;
> od;
gap> req;
[(3,7)(4,8), (2,6)(3,7), (1,2)(3,4)(5,6)(7,8), (1,3)(2,4)(5,7)(6,8), (1,3)(2,8)(4,6)(5,7), (1,4)(2,7)(3,6)(5,8),
(1,5)(4,8), (1,5)(2,6),
  (1,6)(2,5)(3,8)(4,7), (1,7)(2,4)(3,5)(6,8), (1,7)(2,8)(3,5)(4,6),
(1,8)(2,3)(4,5)(6,7) ]
```

PROGRAMME 4

```
gap> s8:=SymmetricGroup(8);;
gap> a:=(1,3,5,7)(2,4,6,8);;b:=(2,6)(4,8);;
gap> H:=Subgroup(s8,[a,b]);;
gap> diff:=Difference(s8,H);;
gap> req:=[];;
gap> for c in diff do
> if c^2=() then
> if Order(s8,c)<>4 then
> if Order(s8,c)<>8 then
> if a^c in H then
> if b^c in H then
> if Size(Subgroup(s8,[a,b,c]))=16 then
> Add(req,c);
> fi;
> fi;
> fi;
> fi;
> fi;
> fi;
> od;
gap> req;;
gap> Size(req);
40
```

```
gap> req;
[ (4,8), (3,7), (3,7)(4,8), (2,4)(6,8), (2,4)(3,7)(6,8), (2,6), (2,6)(3,7),
  (2,6)(3,7)(4,8), (2,8)(4,6), (2,8)(3,7)(4,6), (1,2)(3,4)(5,6)(7,8),
  (1,2)(3,8)(4,7)(5,6), (1,3)(5,7), (1,3)(4,8)(5,7), (1,3)(2,4)(5,7)(6,8),
  (1,3)(2,6)(5,7), (1,3)(2,6)(4,8)(5,7), (1,3)(2,8)(4,6)(5,7),
  (1,4)(2,3)(5,8)(6,7), (1,4)(2,7)(3,6)(5,8), (1,5), (1,5)(4,8),
  (1,5)(3,7)(4,8), (1,5)(2,4)(6,8), (1,5)(2,4)(3,7)(6,8), (1,5)(2,6),
  (1,5)(2,6)(4,8), (1,5)(2,6)(3,7), (1,5)(2,8)(4,6), (1,5)(2,8)(3,7)(4,6),
  (1,6)(2,5)(3,4)(7,8), (1,6)(2,5)(3,8)(4,7), (1,7)(3,5), (1,7)(3,5)(4,8),
  (1,7)(2,4)(3,5)(6,8), (1,7)(2,6)(3,5), (1,7)(2,6)(3,5)(4,8),
  (1,7)(2,8)(3,5)(4,6), (1,8)(2,3)(4,5)(6,7), (1,8)(2,7)(3,6)(4,5)
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