Some Results on the Integral Means of the Derivative of a Univalent Function

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Abstract

The purpose of this research paper to show all the possible values of the pth- power of the integrable function which make the integral means of the derivative of univalent function exists and finite.

Keywords: univalent function, Integral means, self conformal maps.

1. Introduction

Let φ be univalent function to define the integral means

$$I_p(r,\varphi') = \frac{1}{2\pi}\int_{|z|=r} |\varphi'(z)|^p d\theta \quad (0<r<1), \tag{1}$$

where $d\theta$ is the angular measure $\frac{|dz|}{r}$.

One can estimate this kind of integral when r→1 to obtain

$$\lim_{p\to+\infty}\Big(\int_{|z|=r} |\varphi'(z)|^p d\theta\Big)^{\frac{1}{p}} = \max_{|z|=r}|\varphi'(z)|.$$

As known that

$$|\varphi'(z)|^{-1} = O((1-|z|)^{-1}) \quad as\ |z|\to 1-0, \tag{2}$$

is a trivial estimate for $I_{-1}(r,\varphi')$ which obtained by the Koebe distortion theorem, and the first nontrivial result was obtained by [Brennan 1978] through the application of Carleson's method [Carleson 1973]

$$I_{-1}(r,\varphi') = O\big((1-r)^{-1-\zeta}\big) \qquad as\ r\to 1-0, \tag{3}$$

for some absolute constant $\zeta>0$.

[CH. Pommerenke 1985] proved that,

$$I_{-1}(r,\varphi') = \frac{1}{2\pi}\int_0^{2\pi} \left|\varphi'(re^{i\theta})\right|^{-1} d\theta = O((1-r)^{-0.601}). \tag{4}$$

So, the integral means always exist. But how they behave depending on p if r→1.

This paper addresses the following integral means:

$$I_{-p}(r,\varphi') = \frac{1}{2\pi}\int_0^{2\pi} \left|\varphi'(re^{i\theta})\right|^{-p} d\theta, \tag{5}$$

exists for $-1.1697 < p < \frac{2}{3}$.

Then we tried to put some restrictions on univalent function to show how the range of the pth-power integrable function has increased to be $4/3 < p < 5$.

In this regard, we have found how can prove that the boundedness condition on φ needed for expanding the values of the pth-power integrable function as shown by comparing between theorems (2.3), (2.4).

For further information we refer to see [Carleson 1993], [Duren 1970], [Krantz 1990] , [Pommerenke 1990] which were a primary source in this research paper.

Here, we need to recall two comparison theorems from the theory of differential equations which we shall use in proving the next theorem in this paper.

Lemma 1.1.[Sazrski 1965] Let p, q be continuous functions in [a,b), such that p(x) ∈ R, q(x) >0 for a≤ x<b, suppose u is twice differentiable and $u'' < pu' + qu,\ v'' = pv' + qv,$

if

$$u(a) < v(a) \ \ and\ u(a) < v'(a), \qquad then\ \ u(x) < v(x)\ \ for\ a \le x < b. \tag{6}$$

Lemma 1.2. Let q(x) be continuous and positive (q(x) >0) on [a ,b), suppose u is four times differentiable and $u^{(4)} < qu, u^{(4)} = qv,\ u^{(k)}(a) < v^{(k)}(a)\ \ for\ \ k = 0,1,2,3.$

$then\ u(x) < v(x)\ \ for\ \ a \le x < b.$

2. Main Results

Theorem2.1. If φ is an univalent function in unit disk D then, as r→1−0,

$$\mathrm{I}_{-p}(\mathrm{r},\varphi') = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|\varphi'(re^{i\theta})|^p} < \infty \ \ for \qquad -1.1697 < p < \frac{2}{3} \tag{7}$$

Proof. Let $p = -1 - \alpha$, $(\alpha \ge 0)$ with an application Distortion theorem (3.7) to obtain,

$$\frac{1}{2\pi}\int_0^{2\pi} |\varphi'(re^{i\theta})|^{-p} d\theta = \frac{1}{2\pi}\int_0^{2\pi} |\varphi'(re^{i\theta})|^{1+\alpha} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} |\varphi'(re^{i\theta})|(|\varphi'(re^{i\theta})|)^{\alpha} d\theta$$

$$= O(1-r)^{-3\alpha}\underbrace{\frac{1}{2\pi}\int_0^{2\pi} |\varphi'(re^{i\theta})| d\theta}_{\mathrm{I}^*}$$

Apply Hölder inequality on the Integral means I^* yields

$$\frac{1}{2\pi}\int_0^{2\pi} |\varphi'(re^{i\theta})|\, d\theta \le \left(\int_0^{2\pi} |\varphi'(re^{i\theta})|^p d\theta\right)^{\frac{1}{p}} \left(\int_0^{2\pi} |\varphi'(re^{i\theta})|^q d\theta\right)^{\frac{1}{q}}$$

Let $\frac{1}{\mathrm{p}} = \frac{1-\delta}{2}, \Longrightarrow q = \frac{2}{1+\delta}$, *such that*

$$\frac{1}{2\pi}\int_0^{2\pi} |\varphi'(re^{i\theta})|\, d\theta \le \left(\int_0^{2\pi} |\varphi'(re^{i\theta})|^2 d\theta\right)^{\frac{1-\delta}{2}} \left(\int_0^{2\pi} |\varphi'(re^{i\theta})|^{\frac{2\delta}{1+\delta}} d\theta\right)^{\frac{1+\delta}{2}} \tag{8}$$

Estimate two integrals on the right-hand side in equality (8) by lemma (3.3), theorem (1)in [Pommerenke 1985] to obtain

$$\int_0^{2\pi} |\varphi'(re^{i\theta})|\, d\theta = O((1-r))^{\frac{-(1-\delta)}{2}} O(1-r)^{\frac{-\beta(1+\delta)}{2}}$$

$$\int_0^{2\pi} |\varphi'(re^{i\theta})|\, d\theta = O(1-r)^{\frac{-1}{2}(1+\delta)\beta-\frac{1}{2}+\frac{\delta}{2}}$$

$Let\ \beta > -\frac{1}{2}+p+\sqrt{\left(\frac{1}{4}-p+4p^2\right)}$ cf. theorem (3.6) and choose $p=\frac{2\delta}{1+\delta}$such that

When(0<δ<1),one can suppose that δ=0.0364 to yield β>0.0168898.

$$\Rightarrow \int_0^{2\pi} |\varphi'(re^{i\theta})|\, d\theta = O(1-r)^{\frac{-1}{2}(1+\delta)\beta-\frac{1}{2}+\frac{\delta}{2}}$$

$$= O(1-r)^{\frac{-1}{2}(1.0364)(0.0169)-0.5+0.0182}$$

$$= O(1-r)^{-0.00875758-0.4818}$$

$$= O(1-r)^{-0.49055758}$$

$$= O(1-r)^{-0.491}$$

Hence,

$$\int_0^{2\pi} |\varphi'(re^{i\theta})|^{-p} d\theta = O(1-r)^{-0.491}\, O(1-r)^{-3\alpha}$$

$$\int_0^1 \int_0^{2\pi} |\varphi'(re^{i\theta})|^{-p} d\theta dr = \int_0^1 O(1-r)^{-3\alpha-0.491}\, dr$$

If −3α−0.491+1>0 ⇒α<0.1697.

By assumption, $p=-1-\alpha$⇒ p>−1.1697, this implies to $-1.1697 < p < \frac{2}{3}$.

We deduce the following theorem to get more on the values pth- power integrable function.

Theorem2.2. Ifφ is an univalent function in D with |φ(z)|≤ 1 for all z and φ(a)= b for some a, b∈ D then,

$$\int\int_D |\varphi'|^p dxdy < \infty,\ exists\ for\ -1-\alpha < \lambda < \frac{2}{3},$$

Proof. An application of Distortion theorem (3.7) and Schwarz-Pick lemma (3.2) yield

$$\int_0^1 \int_0^{2\pi} |\varphi'|^p d\theta dr = \int_0^1 \int_0^{2\pi} |\varphi'|^{-1-\alpha} r dr d\theta$$

$$= \int_0^1 \int_0^{2\pi} |\varphi'|^{-\alpha} |\varphi'|^{-1} r dr d\theta$$

$$= \int_0^1 \int_0^{2\pi} O((1-r)^{-1})^{\alpha} |\varphi'|^{-1}\, r dr d\theta;$$

$$\leq \int_0^1 \int_0^{2\pi} O(1-r)^{-\alpha} \left(\frac{1-|\varphi(z)|^2}{1-|z|^2}\right)^{-1} r dr d\theta;$$

$$\leq \int_0^1 \int_0^{2\pi} O(1-r)^{-\alpha} \left(\frac{1}{1-|z|}\right)^{-1} r dr d\theta$$

$$= \int_0^1 \int_0^{2\pi} O(1-r)^{-\alpha} \frac{1}{(1-r)^{-1}} r dr d\theta$$

$$= C \int_0^1 \frac{r dr}{(1-r)^{\alpha-1}}$$

$$\leq C \int_0^1 \frac{dr}{(1-r)^{\alpha-1}}$$

$$= \int_0^1 (1-r)^{-\alpha+1} dr = \left[\frac{(1-r)^{-\alpha+2}}{-\alpha+2}\right]_0^1$$

if $-\alpha+2>0 \Rightarrow \lambda= -1-\alpha>-3 \Rightarrow -3<\lambda<\frac{2}{3}$

Brennan's conjecture stated that,

$$\int\int_D |\varphi'|^{2-p} dxdy = \int\int_\Omega |\Psi'|^p dxdy < \infty; \ whenever \ \frac{4}{3}<p<4.$$

Hence, $2-p=\lambda \Rightarrow p=2-\lambda \Rightarrow \frac{4}{3}<p<5.$

Theorem 2.3. If φ is holomorphic and univalent in unit disk D, then

$$I(r,\varphi') = O((1-r)^{-2.914}) \quad \text{as } r\to 1-0 \qquad (9)$$

Proof. : Suppose that $\varphi'(z) = \left(\sqrt{\varphi'(z)}\right)^2$ such that $(\varphi'(z))^{\frac{1}{2}} = F(z) = \sum_{n=0}^{\infty} a_n z^n$

be holomorphic function in unit disk and $|\varphi'(z)|=|F(z)|^2$.

$$I(r,\varphi') = \frac{1}{2\pi}\int_0^{2\pi} |\varphi'(re^{i\theta})| d\theta = \frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2 d\theta; \qquad z=re^{i\theta} \qquad (10)$$

An application of Parseval formula (3.1) yields

$$I(r) = \frac{1}{2\pi}\int_0^{2\pi} |\varphi'(re^{i\theta})| d\theta = \frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n} \qquad (11)$$

Differentiate equation (11) to obtain,

$$I'(r) = \sum_{n=0}^{\infty} 2n|a_n|^2 r^{2n-1}$$

$$I''(r) = \sum_{n=0}^{\infty} 2n(2n-1)|a_n|^2 r^{2n-2}$$

$$I^{(3)}(r)=\sum_{n=0}^{\infty}2n(2n-1)(2n-2)|a_n|^2r^{2n-3}$$

$$\mathrm{I}^{(4)}(r)=\sum_{n=0}^{\infty}2n(2n-1)(2n-2)(2n-3)|a_n|^2r^{2n-4}$$

And

$$F'(z)=\sum_{n=0}^{\infty}na_nz^{n-1}$$

$$F''(z)=\sum_{n=0}^{\infty}n(n-1)a_nz^{n-2}$$

$$|F''(z)|^2=\sum_{n=0}^{\infty}n^2(n-1)^2|a_n|^2|z|^{2n-4}$$

$$\mathrm{I}^{(4)}(r)\le K\left|F''\left(re^{i\theta}\right)\right|^2$$

Compare the coefficients between $\mathrm{I}^{(4)}(r)$and $\left|F''\left(r|F''(z)|^2e^{i\theta}\right)\right|^2$to find K as follows:

$$2n(2n-1)(2n-2)(2n-3)|a_n|^2r^{2n-4}\le K\,n^2(n-1)^2|a_n|^2r^{2n-4}$$

$$4(2n-1)(n-1)(2n-3)\le K\,n(n-1)^2$$

$$4(2n-1)(2n-3)\le K\,n(n-1)$$

$$4\left(1-\frac{1}{n}\right)(2-\frac{1}{n-1})\le K$$

So, K=16 is smallest such constant as n→∞.

$$\mathrm{I}^{(4)}(r)=\sum_{n=2}^{\infty}2\mathrm{n}(2\mathrm{n}-1)(2\mathrm{n}-2)(2\mathrm{n}-3)|a_n|^2\mathrm{r}^{2n-4}\le 16\sum_{n=0}^{\infty}n^2(n-1)^2|a_n|^2|z|^{2n-4}$$

$$\mathrm{I}^{(4)}(r)=\sum_{n=2}^{\infty}2\mathrm{n}(2\mathrm{n}-1)(2\mathrm{n}-2)(2\mathrm{n}-3)|a_n|^2\mathrm{r}^{2n-4}\le\frac{16}{2\pi}\int_0^{2\pi}\left|F''\left(re^{i\theta}\right)\right|^2d\theta \quad (12)$$

$$\therefore\ \mathrm{I}^{(4)}(r)\le\frac{16}{2\pi}\int_0^{2\pi}\left|F''\left(re^{i\theta}\right)\right|^2d\theta \quad (13)$$

Differentiate $\mathrm{F}(z)=(\varphi')^{1/2}$ twice with an application of Schwarzian derivative

$\{\mathrm{S}_\varphi\}=[\frac{d}{dz}\left(\frac{\varphi''}{\varphi'}\right)-\frac{1}{2}\left(\frac{\varphi''}{\varphi'}\right)^2]$, yield

$$|F''(z)|\le\frac{1}{2}|\varphi'|^{\frac{1}{2}}\left|\left(\frac{\varphi''}{\varphi'}\right)^2+\{\mathrm{S}_\varphi\}\right|\le\frac{1}{2}|\varphi'|^{\frac{1}{2}}\left[\left|\frac{\varphi''}{\varphi'}\right|^2+\left|\mathrm{S}_\varphi\right|\right]$$

$$|F''(z)|^2 \leq \frac{1}{4}|\varphi'|\left[\left|\frac{\varphi''}{\varphi'}\right|^2 + |S_\varphi|\right]^2$$

$$= \frac{1}{4}|\varphi'|\left[\left|\frac{\varphi''}{\varphi'}\right|^4 + \frac{12}{(1-r^2)^2}\left|\frac{\varphi''}{\varphi'}\right|^2 + \frac{36}{(1-r^2)^4}\right],$$

where $\{S_\varphi\} \leq \frac{6}{(1-|z|^2)^2}$ (cf.[Nehari 1949],[4,pp.261-263])[1].

$$|F''(z)|^2 \leq \frac{1}{4}|\varphi'|\left[\frac{1325.25}{(1-r^2)^4}\right] = |\varphi'|\frac{331.3125}{(1-r^2)^4}$$

$$\int_0^{2\pi} |F''(z)|^2\, d\theta \leq \frac{331.3125}{(1-r^2)^4}\int_0^{2\pi} |\varphi'(re^{i\theta})|d\theta.$$

Finally

$$I^{(4)}(r) \leq 331.3125(1-\mathrm{r})^{-4} I(r).$$

Then

$\nu(\mathrm{r})=A(1-\mathrm{r})^{-\beta}$; where A is a constant and $\beta > 0$,

Satisfies the comparison equation

$$v^{(4)}(r) = (1-\mathrm{r})331.3125^{-4} v(r) \tag{14}$$

Differentiate $\nu(\mathrm{r})$ four times yields

$$\beta(\beta+1)(\beta+2)(\beta+3)=331.3125,$$

Such that

$$v(r) = \mathrm{A}(1-\mathrm{r})^{-2.914}$$

is the solution of the equation (14). If we choose A sufficiently large, then $I^{(4)}(r_0) > v^{(4)}(r_0)$ for k=0,1,2,3.at $r_0 \leq r < 1$.

$$\mathrm{I(r)} < v(r), \quad \forall\, r, \quad r_0 \leq r < 1$$

$$\mathrm{I(r)} < v(r) = \mathrm{A}(1-\mathrm{r})^{-2.914}$$

$$\mathrm{I(r)} = O(1-\mathrm{r})^{-2.914}$$

Theorem 2.4. If φ is bounded and univalent in D then,

$$\iint_D |\varphi'(z)|dxdy = O(1-r)^{-0.497} \tag{15}$$

Proof.:-

Let $\delta > 0$. From Cauchy-Schwarz inequality, we get the bound

$$\int_0^{2\pi} |\varphi'|d\theta \leq \underbrace{\left(\int_0^{2\pi} |\varphi'|^{2\delta} d\theta\right)^{\frac{1}{2}}}_{\mathrm{I}_1}\underbrace{\left(\int_0^{2\pi} |\varphi'|^{2} d\theta\right)^{\frac{1}{2}}}_{\mathrm{I}_2}$$

Estimate two integrals in the right-hand side as follows:

$$I_2(r) = \int_0^{2\pi} |\varphi'|^2 d\theta = O(1-r)^{-1} \text{see lemma (3.3)}$$

Define,

$$I_1(r) = \int_0^{2\pi} |\varphi'|^{2\delta} d\theta,$$

where $|\varphi'|^{2\delta} = F(z)$. Rewrite

$$I_1(r) = \int_0^{2\pi} |F(z)|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |c_n|^2 r^{2n}$$

Differentiate above equation yields

$$I''_1(r) \leq 8\pi \sum_{n=1}^{\infty} n^2 |c_n|^2 r^{2n-2} \tag{16}$$

As known

$$2\pi \sum_{n=1}^{\infty} n^2 |c_n|^2 r^{2n-2} = \int_0^{2\pi} \left|F'(re^{i\theta})\right|^2 d\theta$$

$$= \delta^2 \int_0^{2\pi} \left|\frac{\varphi''}{\varphi'}\right|^2 |\varphi'|^{2\delta} d\theta.$$

Hence

$$2\pi \sum_{n=1}^{\infty} n^2 |c_n|^2 r^{2n-2} \leq \frac{36\delta^2}{(1-r)^2} \int_0^{2\pi} |\varphi'|^{2\delta} d\theta \tag{17}$$

Combine an inequalities (16) and (17) yield

$$\frac{I_1''(r)}{I_1(r)} \leq \frac{36\delta^2}{(1-r)^2}$$

$$(\log I_1(r))'' = \frac{I_1''(r)}{I_1(r)} - \left(\frac{I_1'(r)}{I_1(r)}\right)^2 \leq \frac{I_1''(r)}{I_1(r)} \leq \frac{144\delta^2}{(1-r)^2} \tag{18}$$

Integrating twice yields;

$$I_1(r) = O(1-r)^{-144\delta^2}$$

Consequently, we obtain

$$\int_0^{2\pi} |\varphi'|^{1+\delta} d\theta = O(1-r)^{-\frac{1}{2}} O(1-r)^{-72\delta^2}$$

$$= O(1-r)^{-\frac{1}{2}-72\delta^2} \qquad as\ r \to 1-0.$$

3. Auxiliary results

In this section we collect some brief recalls on the main topics involved with this work.

Lemma 3.1.(Parseval formula) ([Kammler 2000, pp.74)

Let $\varphi(z)$ be an holomorphic function in unit disk D such that it is represented there by Taylor series expansion $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ then

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} \left| \varphi'(re^{i\theta}) \right|^2 d\theta$$

Lemma 3.2. [Krantz 1999] Let φ be holomorphic on the unit disk D, and assume that $|\varphi(z)| \leq 1$ for all z, and $\varphi(a) = b$ for some a, b in D, then

$$|\varphi'(z)| \leq \frac{1-|b|^2}{1-|a|^2}$$

Lemma3.3. [Pommerenke 1975] Let the an open subset of $[0,2\pi]$ and $0 \leq r < 1$. If $\varphi \in S$, then

$$\int_T \left| \varphi'(re^{i\theta}) \right|^2 \left| \varphi(re^{i\theta}) \right|^{p-2} d\theta \leq \begin{cases} K(p)(1-r)^{-1}(M_T(r))^p & \text{if } p > 0, \\ K(p)(1-r)^{-1}(M_T(r))^p & \text{if } p < 0. \end{cases}$$

Such that

$$M_T(r) = \max_{\theta \in T} \left| \varphi(re^{i\theta}) \right| \ ; \quad 0 \leq r < 1 \ ; \ T \subset [0,2\pi].$$

Theorem3.4.(Nehari's Theorem)[Nehari 1949]

Let φ be a regular function (holomorphic and single valued function (1-1) in D and suppose its Schwarzian derivative satisfies

$$\left| S_\varphi \right| \leq 2(1-|z|^2)^2 \ , \quad |z| < 1. \qquad (19)$$

Thenφ is univalent in D.

Remark3.5. Inequality (19) is a necessary condition for univalent function by inversion of the function

$$F(z) = \frac{\varphi(\frac{z+\varsigma}{1+\varsigma z}) - \varphi(\varsigma))}{(1-|\varsigma|^2)\varphi'(\varsigma)} = z + A_2 z^2 + A_3 z^3 + \cdots$$

To replace a constant 2 by 6. This result was rediscovered by [Nehari 1949].

Theorem3.6.[Pommerenke 1985] Ifφ is holomorphic and univalent in unit disk D, then

$$I_p(r, \varphi') = O\left((1-r)^{-\beta}\right) \quad \text{as } r \to 1-0 \quad \text{for } p \in R \qquad (20)$$

And

$$\beta > -\frac{1}{2} + p + \sqrt{\frac{1}{4} - p + 4p^2} \qquad (21)$$

Theorem3.7.(Distortion Koebetheorem)[Duren 1983]For each $\varphi \in S$ defined on unit disc D,

$$\frac{1-r}{(1+r)^3} \leq |\varphi'(z)| \leq \frac{1+r}{(1-r)^3}, \qquad |z| = r < 1.$$

Equality holds if and only if φ is a suitable rotation of the Koebefunction.

4. Conclusion

This research paper revealed that:

1. The bounded condition for the univalent function φ play the main role in an increasing the values of the integral means of the derivative of a univalent function once by a boundedness condition for the function φ.

2. We addressed a generalization of earlier result by [Pommerenke 1985] as the follows

$$\mathrm{I}_{-p}(\mathrm{r},\varphi') = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\theta}{|\varphi'(re^{i\theta})|^{-p}} < \infty \qquad for \ -1.1697 < p < \frac{2}{3} \tag{22}$$

3. We shown that the rang of the integral mean

$$\mathrm{I}_p(\mathrm{r},\varphi') = \frac{1}{2\pi}\int_0^{2\pi} |\varphi'|^p \, dxdy,$$

Can be extended to be $\frac{4}{3} < p < 5$ by putting some restriction on the function φ.

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