The Univalence of Some Integral Operators

Deborah O. Makinde
Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Osun State, Nigeria
E-mail : domakinde.comp@gmail.com; domakinde@oauife.edu.ng

Abstract

In this paper, we obtain the conditions for univalence of the integral operators of the form:

\[ F(z) = \lambda \int_{t}^{z} t^{k-1} \prod_{k=1}^{i} \left( g_i(s) \right) ds \]

And

\[ H(z) = \left\{ \lambda \int_{0}^{z} t^{k-1} \prod_{k=1}^{i} \left( \frac{g_i(s)}{s} \right)^{k-1} ds \right\}^{1/\lambda} \]

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1. Introduction

Let \( A \) be the class of the functions in the unit disk \( U = \{ Z \in C : |z| < 1 \} \) such that \( f(0) = f'(0) - 1 = 0 \). Let \( S \) be the class of functions \( f \in A \) which are univalent in \( U \). Ozaki and Nunokawa [2] showed the condition for the univalence of the function \( f \in A \) as given in the lemma below.

**Lemma 1 [3]:** Let \( f \in A \) satisfy the condition

\[ \left| \frac{z^2 f'(z)}{f(z)} - 1 \right| \leq 1 \] (1)

For all \( z \in U \), then \( f \) is univalent in \( U \)

**Lemma 2 [2]:** Let \( \alpha \) be a complex number, \( \alpha > 0 \), and \( f \in A \). If

\[ \frac{1 - |z|^{2Re\alpha}}{Re\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \]

For all \( z \in U \), then the function \( F_\alpha(z) = [\alpha \int_{0}^{z} t^{a-1} f'(u) du]^{1/\alpha} \) is in the class \( S \).

The Schwartz lemma [1]: Let the analytic function \( f \) be regular in the unit disk and let \( f(0) = 0 \). If \( |f(z)| \leq 1 \), then

\[ |f(z)| \leq |z| \]

For all \( z \in U \), where equality can hold only if \( |f(z)| \equiv Kz \) and \( k = 1 \)

**Lemma 3 [4]:** Let \( g \in A \) satisfies (1) and \( a + b_i \) a complex number, \( a, b \) satisfies the conditions

\[ a \in \left[ \frac{3}{4}, \frac{3}{4} \right] ; b \in [0, \frac{1}{2n2}] \] (2)

\[ 8a^2 + a^8b^2 - 18a + 9 \leq 0 \] (3)

If \( |g(z)| \leq 1 \) for all \( z \in U \) then the function \( G(z) = \left\{ a + b \int_{0}^{z} \left( \frac{g(u)}{u} \right)^{a+b-1} du \right\}^{1/\alpha+b_i} \)
Is univalent in $U$

1. MAIN RESULTS

**Theorem 1:** Let $g \in A$ satisfies (1) and $\lambda$ is a complex number, $\lambda$ satisfies the conditions

\[ \Re \lambda \in (0, \sqrt{k + 2}], k \geq 1 \]

\[ (\Re \lambda)^4 (\Re \lambda)^2 (\Im \lambda)^2 - (k + 2)^2 \geq 0 \]

If $|g(z)| \leq 1$ for all $z \in U$ then the function

\[ F(z) = \lambda \int_0^z t^{\lambda - 1} \prod_{i=1}^k \left( \frac{g_i(s)}{s} \right)^{1/\lambda} \, ds \] is univalent in $U$

**Proof:** Let

\[ F(z) = \lambda \int_0^z t^{\lambda - 1} \prod_{i=1}^k \left( \frac{g_i(s)}{s} \right)^{1/\lambda} \, ds \]

And let $f(z) = \int_0^z \prod_{i=1}^k \left( \frac{g_i(s)}{s} \right)^{1/\lambda} \, ds$

Then $f'(z) = \prod_{i=1}^k \left( \frac{g_i(z)}{z} \right)^{1/\lambda}$

\[ f'(z) = \left( \frac{g_1(z)}{z} \right)^{1/\lambda} \left( \frac{g_2(z)}{z} \right)^{1/\lambda} \ldots \left( \frac{g_k(z)}{z} \right)^{1/\lambda} \]

\[ f'(z) = \frac{1}{\lambda} \left( \frac{z g_1'(z)}{g_1(z)} - 1 \right) + \frac{1}{\lambda} \left( \frac{z g_2'(z)}{g_2(z)} - 1 \right) \ldots + \frac{1}{\lambda} \left( \frac{z g_k'(z)}{g_k(z)} - 1 \right) \]

\[ = \frac{1}{\lambda} \sum_{i=1}^k \left( \frac{z g_i'(z)}{g_i(z)} - 1 \right) \]

This implies that

\[ \frac{1-|\lambda|^2 \Re \lambda}{\Re \lambda} \left| \frac{f'(z)}{f(z)} \right| = \frac{1-|\lambda|^2 \Re \lambda}{|\lambda|} \left| \sum_{i=1}^k \frac{z g_i'(z)}{g_i(z)} - 1 \right| \]

\[ \leq \frac{1-|\lambda|^2 \Re \lambda}{|\lambda|} \left| \sum_{i=1}^k \frac{z g_i'(z)}{g_i(z)} \right| + 1 \]

\[ \leq \frac{1-|\lambda|^2 \Re \lambda}{|\lambda|} \left| \sum_{i=1}^k \frac{z^2 g_i'(z)}{g_i(z)} \right| + 2 \]

Using Schwartz-lemma in (7), we have

\[ \frac{1-|\lambda|^2 \Re \lambda}{\Re \lambda} \left| \frac{f'(z)}{f(z)} \right| \leq \frac{1-|\lambda|^2 \Re \lambda}{|\lambda|} \left| \sum_{i=1}^k \frac{z^2 g_i'(z)}{g_i(z)} - 1 \right| + 2 \]

But $g$ satisfies (1), thus

\[ \left| \sum_{i=1}^k \frac{z^2 g_i'(z)}{g_i(z)} - 1 \right| \leq k \]
This implies that
\[
\left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1-|z|^{2 \Re \lambda}}{\Re \lambda} \frac{1}{|\lambda|} (k + 2) \leq \frac{k+2}{\Re \lambda|\lambda|} \tag{8}
\]

From (4) and (5), we have
\[
\frac{k+2}{\Re \lambda|\lambda|} \leq 1 \tag{9}
\]

Using (9) in (8), we obtain
\[
\frac{1-|z|^{2 \Re \lambda}}{\Re \lambda} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1
\]

And from (6) \( f'(z) = \prod_{i=1}^{k} \left( \frac{g_i(s)}{s} \right)^{\lambda-1} \)

And by lemma 2 \( F(z) \) is univalent

**Theorem 2** Let \( g \in A \) satisfies (1) and let \( \lambda \) be a complex number with \( \Re \lambda, \Im \lambda \) satisfying (2) and (3). If \( |g(z)| \leq 1 \) for all \( z \in U \) then the function
\[
H(z) = \left\{ \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^{k} \left( \frac{g_i(s)}{s} \right)^{\lambda-1} \right\}^{1/\lambda}
\]
Is univalent in \( U \).

**Proof:** Let
\[
H(z) = \left\{ \lambda \int_0^z t^{\lambda-1} \prod_{i=1}^{k} \left( \frac{g_i(s)}{s} \right)^{\lambda-1} \right\}^{1/\lambda} \tag{10}
\]

Following the procedure of the proof of theorem 1, we obtain
\[
\left| \frac{1-|z|^{2 \Re \lambda}}{\Re \lambda} \frac{zh''(z)}{h'(z)} \right| \leq \frac{1-|z|^{2 \Re \lambda}}{\Re \lambda} |\lambda - 1| \left( \left| \sum_{i=1}^{k} \frac{z^2 g_i'(z)}{g_i(z)} \right| - 1 \right) + 2
\]

But \( g \) satisfies (1), thus
\[
\left| \frac{1-|z|^{2 \Re \lambda}}{\Re \lambda} \frac{zh''(z)}{h'(z)} \right| \leq \frac{1-|z|^{2 \Re \lambda}}{\Re \lambda} |\lambda - 1|(k + 2) \leq \frac{|\lambda - 1|(k + 2)}{\Re \lambda}
\]

From (2) and (3) we have
\[
\frac{|\lambda - 1|(k + 2)}{\Re \lambda} \leq 1
\]

This implies that
\[
\left| \frac{1-|z|^{2 \Re \lambda}}{\Re \lambda} \frac{zh''(z)}{h'(z)} \right| \leq 1 \text{ for all } z \in U
\]

From (10), \( h'(z) = \prod_{i=1}^{k} \left( \frac{g_i(s)}{s} \right)^{\lambda-1} \)

And by lemma 2, \( H(z) \) is univalent in \( U \).

**Remark:** Theorem 1 and theorem 2 gives a generalization of theorem 1[4] and lemma 3 respectively.
References


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