On Contra $S_5$-Continuous Functions

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Abstract
In this paper, we apply the notion of $S_5$-open set in topological spaces to introduce and investigate the concept of contra $S_5$-continuous which is a subclass of the class of contra semi continuous functions.

Keywords: $S_5$-closed, contra $S_5$-continuous, contra $S_5$-closed and strongly contra $S_5$-closed.

1. Introduction
In 1966, Dontchev [3], introduced the notion of contra continuity and established some results about $S$-closedness and strongly $S$-closedness. Subsequently, Dontchev and Noiri [4], introduced and studied contra semi continuity and gave several properties about these functions. Later Jafari and Noiri [5], investigated contra $\alpha$- continuous and contra pre-continuous. Recently authors introduced $S_5$-open set for topological spaces where $S_5$-continuity had been investigated. For a subset $A$ of $X$, $Cl(A)$ and $Int(A)$ represent the closure and interior respectively. A subset $A$ of $X$ is called semi-open[9]\((\alpha\text{-open}[11], \text{pre-open}[10], \text{regular open}[17])\) set if $A \subseteq Cl(Int(A))$, $A \subseteq Int(Cl(A))$, $A \subseteq Int(Cl(A)) = Int(Cl(A))$. The complement of semi-open($\alpha$-open, pre-open, regular open) set is called semi-closed ($\alpha$- closed, pre-closed, regular closed)set. A subset $A$ of topological space $(X, \tau)$ is called $\theta$-open set [16] if for each $x \in A$, there is an open set $U$ such that $x \in U \subseteq Cl(U) \subseteq A$. A subset $A$ is called semi regular [2] if it is both semi-open and semi-closed. The main purpose of this paper is to introduce the notion of contra $S_5$-continuous functions and obtained some its properties. Also, we defined and studied the concept of contra $S_5$-closed and strongly $S_5$-closed.

2. Preliminaries
The following definitions and results are needed.

Definition 2.1. A topological space $X$ is called:

1) locally indiscrete [3], if every open set in $X$ is closed.
2) extremally disconnected [6], if the closure of every open subset of $X$ is open or the interior of every closed subset of $X$ is closed.
3) semi-$T_1$ [1], If for every two distinct points $x$, $y$ in $X$, there exist two semi open sets, one containing $x$ but not $y$ and the another containing $y$ but not $x$.

Lemma 2.2[1]. A space $X$ is semi-$T_1$, if and only if, the singleton $\{x\}$ is semi-closed for any point $x \in X$.

Definition 2.3. A function $f: X \rightarrow Y$ is called:

1) Contra continuous [3], if the inverse image of every open set in $Y$ is closed set in $X$.
2) Semi-continuous [9] (resp., contra semi continuous [4]) if the inverse image of every open set in $Y$ is semi-open (resp., semi-closed) set in $X$. 

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3) Perfectly continuous [15] (SR-continuous [12], RC-continuous [12]) if the pre image of every open set in \( Y \) is clopen (semi regular, regular closed) set in \( X \).

4) pre-closed [5], if the image of every closed set in \( X \) is pre-closed set in \( Y \).

**Theorem 2.4** [1]. For any spaces \( X \) and \( Y \), if \( A \subseteq X \) and \( B \subseteq Y \), then

1) \( s\text{Int}_{XY}(A \times B) = s\text{Int}_X(A) \times s\text{Int}_Y(B) \)

2) \( s\text{Cl}_{XY}(A \times B) = s\text{Cl}_X(A) \times s\text{Cl}_Y(B) \)

The following definitions and results are from [7].

**Definition 2.5** A semi-open subset \( A \) of a space \( X \) is called \( S_\alpha \)-open if for each \( x \in A \), there exist semi-closed \( F \) such that \( x \in F \subseteq A \).

The complement of \( S_\alpha \)-open set in \( X \) is called \( S_\alpha \)-closed set in \( X \).

**Proposition 2.6.** Let \( \{ A_\alpha; \alpha \in \Delta \} \) be collection of \( S_\alpha \)-closed sets in topological space \( X \), then \( \bigcap \{ A_\alpha; \alpha \in \Delta \} \) is \( S_\alpha \)-closed.

**Proposition 2.7.** Let \( X \) be topological space and \( A, B \subseteq X \). If \( A \in S_\delta O(X) \) and \( B \) is both \( \alpha \)-open and semi-closed, then \( A \cap B \in S_\delta O(X) \).

**Proposition 2.8.** Let \( (Y, \tau_Y) \) be an subspace of \( (X, \tau) \) and \( A \subseteq Y \), then the following properties are true:

1) If \( A \in S_\delta O(Y, \tau_Y) \) and \( Y \) is semi-regular, then \( A \in S_\delta O(X, \tau) \).

2) If \( A \in S_\delta O(X, \tau) \) and \( Y \) is \( \alpha \)-open, then \( A \in S_\delta O(Y, \tau_Y) \).

**Proposition 2.9.** If \( (X, \tau) \) is a semi-\( T_1 \) space, then \( S_\delta O(X, \tau) = SO(X, \tau) \).

**Definition 2.10.** A function \( f:X \rightarrow Y \) is called \( S_\alpha \)-continuous at a point \( x \in X \), if for each an open set \( V \) of \( Y \) containing \( f(x) \), there exist an \( S_\delta \)-open set \( U \) of \( X \) containing \( x \) such that \( f(U) \subseteq V \).

**Proposition 2.11.** For a function \( f:X \rightarrow Y \), the following statements are equivalent:

1) \( f \) is \( S_\alpha \)-continuous,

2) The inverse image of every open set in \( Y \) is an \( S_\alpha \)-open set in \( X \),

3) The inverse image of every closed set in \( Y \) is an \( S_\alpha \)-closed set in \( X \).

**Definition 2.12** [8]. A function \( f:X \rightarrow Y \) is called weakly \( S_\alpha \)-continuous. If for each \( x \in X \) and each open set \( H \) in \( Y \) containing \( f(x) \), there is an \( S_\delta \)-open set \( G \) containing \( x \) such that \( f(G) \subseteq Cl(H) \).

### 3. Contra \( S_\alpha \)-continuous function

**Definition 3.1.** A function \( f:X \rightarrow Y \) is called contra \( S_\alpha \)-continuous if \( f^{-1}(U) \) is \( S_\alpha \)-closed in \( X \) for each open set \( U \) in \( Y \).

**Theorem 3.2** for a function \( f:X \rightarrow Y \) the following conditions are equivalent.

1) \( f \) is contra \( S_\alpha \)-continuous.

2) The inverse image of every closed set in \( Y \) is \( S_\alpha \)-open set in \( X \).

3) For each \( x \in X \), and each closed subset \( F \) of \( Y \) containing \( f(x) \), there is \( S_\alpha \)-open \( U \) containing \( x \) such that \( f(U) \subseteq F \).

**Proof.** (1)\( \Rightarrow \)(2). Let \( F \) be closed subset of \( Y \), then \( Y - F \) is an open set in \( Y \). since \( f \) is contra \( S_\alpha \)-continuous, then \( f^{-1}(Y - F) = X - f^{-1}(F) \) is \( S_\alpha \)-closed set in \( X \). Hence \( f^{-1}(F) \) is \( S_\alpha \)-open set in \( X \).
(2)⇒(3). Let \( F \) be closed subset of \( Y \) containing \( f(x) \), then by (2) \( f^{-1}(F) \) is \( S_5 \)-open set in \( X \) containing \( x \). since \( f^{-1}(F) \subseteq F \). Take \( U=f^{-1}(F) \). Hence \( \phi(U) \subseteq F \).

(3)⇒(1). Let \( x \in X \) and let \( H \) be an open set in \( Y \), therefore \( Y - H \) is closed subset of \( Y \) containing \( f(x) \). Then by (3) there exist \( S_5 \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq Y - H \) implies that \( U \subseteq f^{-1}(Y - H) = X - f^{-1}(H) \). Hence \( f^{-1}(H) \) is \( S_5 \)-closed set in \( X \).

**Remark 3.3.** Every contra \( S_5 \)-continuous is contra semi continuous.

But the converse is not true as showing in the following example,

**Example 3.4.** Let \( X = \{a, b, c\} \) with the topologies \( = \{\emptyset, X, \{c\}\} \) and \( = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\} \). If \( f: (X, \tau) \to (X, \sigma) \) defined by \( f(a) = f(b) = b \) and \( f(c) = c \). Then \( f \) is contra semi continuous but it is not contra \( S_5 \)-continuous because \( f^{-1}(\{b\}) = \{a, b\} \) which is not \( S_5 \)-closed set in \( (X, \tau) \).

**Proposition 3.5.** Let \( f: X \to Y \) be a semi continuous function, then \( f \) is contra semi continuous if and only if it is contra \( S_5 \)-continuous.

**Proof.** Sufficiently, obvious.

Necessity, let \( F \) be a closed subset of \( Y \). Since \( f \) is both contra semi continuous and semi continuous, then \( f^{-1}(H) \) is semi clopen subset of \( X \), so it is \( S_5 \)-open. Therefore \( f \) is contra \( S_5 \)-continuous.

**Proposition 3.6.** If \( f: X \to Y \) is contra semi continuous and \( X \) is semi \( T_1 \)-space, then \( f \) is contra \( S_5 \)-continuous.

**Proof.** Let \( F \) be an open subset of \( Y \). Since \( f \) is contra semi continuous, then \( f^{-1}(F) \) is a semi-closed subset of \( X \). Thus \( X - f^{-1}(F) \) is semi-open in \( X \) and since \( X \) is semi \( T_1 \)-space, then by Proposition 2.9, \( X - f^{-1}(F) \) is \( S_5 \)-open. Therefore, \( f^{-1}(F) \) is \( S_5 \)-closed and hence \( f \) is contra \( S_5 \)-continuous.

**Corollary 3.7.** A function \( f: X \to Y \) is contra \( S_5 \)-continuous if it is one of the following:

1) \( f \) is strongly continuous
2) \( f \) is perfectly continuous
3) \( f \) is RC-continuous
4) \( f \) is SR-continuous

**Proof.** Straightforward.

**Proposition 3.8.** If a function \( f: X \to Y \) is contra \( S_5 \)-continuous, then \( f \) is weakly \( S_5 \)-continuous.

**Proof.** Let \( V \) be an open subset of \( Y \), then \( Cl(V) \) is closed set in \( Y \). Since \( f \) is contra \( S_5 \)-continuous, then by Theorem 3.2, \( f^{-1}(Cl(V)) \) is \( S_5 \)-open in \( X \) and since \( f^{-1}(Cl(V)) \subseteq Cl(V) \). Take \( U = f^{-1}(Cl(V)) \), therefore \( f(U) \subseteq Cl(V) \). Hence \( f \) is weakly \( S_5 \)-continuous.

The converse of the above proposition is not true as it is shown in the next example.

**Example 3.9.** Let \( X = \{a, b, c\} \) and let \( = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}, \{a, c\}\} \). If \( \sigma = \{\emptyset, X, \{a\}\} \) be two topologies on \( X \). Then the function \( f: (X, \tau) \to (X, \sigma) \) defined by \( f(a) = a, f(b) = f(c) = c \) is weakly \( S_5 \)-continuous but it is not contra \( S_5 \)-continuous because \( f^{-1}(\{a\}) \) is not \( S_5 \)-closed.

**Proposition 3.10.** Let \( f: X \to Y \) be any function and \( Y \) be extremally disconnected, then \( f \) is contra \( S_5 \)-continuous if and only if the inverse image of each clopen subset of \( Y \) is \( S_5 \)-open subset of \( X \).

**Proof.** Sufficiently, straightforward.

Necessity, suppose the inverse image of clopen subset in \( Y \) is \( S_5 \)-open. Let \( F \) be a closed subset of \( Y \)
containing $f(x)$. Since $X$ is extremally disconnected then $\text{Int}(F)$ is clopen set in $Y$. So by hypothesis, $f^{-1}(\text{Int}(H))$ is $S_5$-open set in $X$. Since $f(f^{-1}(\text{Int}(F))) \subseteq \text{Int}(F)$, take $U = f^{-1}(\text{Int}(F))$ then $f(U) \subseteq \text{Int}(F) \subseteq F$. Therefore by Theorem 3.2, $f$ is contra $S_5$-continuous.

Clearly that contra $S_5$-continuity and $S_5$-continuity are independent

**Proposition 3.11.** If a function $f : X \to Y$ is contra $S_5$-continuous and $Y$ is regular space then $f$ is $S_5$-continuous.

**Proof.** Let $V$ be an open set in $Y$ containing $f(x)$ for $x \in X$. Since $Y$ is regular, then there is an open set $W$ in $Y$ such that $f^{-1}(W) \subseteq \text{Cl}(W) \subseteq V$. Since $f$ is contra $S_5$-continuous then by Theorem 3.2, there is an $S_5$-open set $U$ in $X$ containing $x$ such that $f^{-1}(U) \subseteq \text{Cl}(W) \subseteq V$. Hence $f$ is $S_5$-continuous.

**Corollary 3.12.** If a function $f : (X, \tau) \to (R, \tau_0)$ is contra $S_5$-continuous, then $f$ is $S_5$-continuous.

**Proposition 3.13.** If a function $f : X \to Y$ is $S_5$-continuous and $Y$ is locally indiscrete, then $f$ is contra $S_5$-continuous.

**Proof.** Straightforward.

**Definition 3.14.** A topological space $(X, \tau)$ is called SCC-space if every $S_5$-closed subset of $X$ is closed.

**Example 3.15.** Let $X = \{a, b, c\}$ with $\tau = \{\emptyset, \{a\}, \{b, c\}\}$ then clearly $(X, \tau)$ is SCC-space.

**Proposition 3.16.** Let $f : X \to Y$ be surjective, pre-closed and contra $S_5$-continuous. If $X$ is SCC-space, then $Y$ is extremally disconnected.

**Proof.** Let $V$ be an open set in $Y$, then $f^{-1}(V)$ is $S_5$-closed subset of $X$. But $X$ is SCC-space, then $V$ is closed in $X$. $f$ is pre-closed, then $f^{-1}(V)$ is pre-closed in $Y$ which implies that $\text{Cl}(f^{-1}(V)) = \text{Cl}(V) \subseteq V$. And so $\text{Cl}(V)$ is an open set in $Y$. Hence $Y$ is extremally disconnected.

**Proposition 3.17.** If a function $f : X \to Y$ is contra $S_5$-continuous, then for any subset $A$ of $X$, $f(S_5 - \text{Int}(A)) \subseteq \text{Cl}(f(A))$.

**Proof.** Let $A \subseteq X$, then $\text{Cl}(f(A))$ is closed subset of $Y$. Since $f$ is contra $S_5$-continuous, then $f^{-1}(\text{Cl}(f(A)))$ is $S_5$-open set in $X$. Therefore, $S_5 - \text{Int}(\text{Cl}(f(A))) = f^{-1}(\text{Cl}(f(A)))$. Since $A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\text{Cl}(f(A)))$ implies that $S_5 - \text{Int}(A) \subseteq S_5 - \text{Int}(f^{-1}(\text{Cl}(f(A)))) = f^{-1}(\text{Cl}(f(A)))$. Hence $f(S_5 - \text{Int}(A)) \subseteq \text{Cl}(f(A))$.

**Definition 3.18.** A topological space $(X, \tau)$ is called $S_5$-locally indiscrete if every $S_5$-open subset of $X$ is closed.

**Example 3.19.** Let $X = \{a, b, c\}$ with topology $\tau = \{\emptyset, \{a\}, \{b, c\}\}$, then $(X, \tau)$ is $S_5$-locally indiscrete.

**Proposition 3.20.** If a function $f : X \to Y$ is contra $S_5$-continuous and $X$ is $S_5$-locally indiscrete space, then $f$ is continuous.

**Proof.** Let $F$ be any closed subset of $Y$. Since $f$ is contra $S_5$-continuous, then $f^{-1}(F)$ is $S_5$-open subset of $X$. But $X$ is $S_5$-locally indiscrete, then $f^{-1}(F)$ is closed. Hence $f$ is continuous.

**Proposition 3.21.** If $f : X \to Y$ is contra $S_5$-continuous, then $S_5 - \text{Cl}(f^{-1}(V)) \subseteq f^{-1}(\text{Cl}_G(V))$.

**Proof.** If $x \not\in f^{-1}(\text{Cl}_G(V))$ implies that $f(x) \not\in \text{Cl}_G(V)$, then there is an open set $G$ containing $f(x)$ such that $\text{Cl}(G) \cap V = \emptyset$. Since $f$ is contra $S_5$-continuous, then there is an $S_5$-open set $U$ such that $f(U) \subseteq \text{Cl}(G)$ and hence $U \cap f^{-1}(V) = \emptyset$. This shows that $x \not\in S_5 - \text{Cl}(f^{-1}(V))$.

**Proposition 3.22.** If a function $f : X \to Y$ is contra $S_5$-continuous and $U$ is $\alpha$-open and semi-closed subset of $X$ then $f|U : U \to Y$ is contra $S_5$-continuous.

**Proof.** Let $H$ be a closed set in $Y$. Since $f$ is contra $S_5$-continuous, then by Theorem 3.2, $f^{-1}(H)$ is $S_5$-open set in $X$ and since $U$ is $\alpha$-open and semi-closed subset of $X$, then by Proposition 2.7, $f(U)^{-1}(H) = f^{-1}(H) \cap U$ is $S_5$-open in $X$. By Proposition 2.8(2), $(f|U)^{-1}(H)$ is $S_5$-open set in $U$. This shows that $f|U$ is contra
Proposition 3.23. A function \( f:X \rightarrow Y \) is contra \( S_\delta \)-continuous if for each \( x \in X \), there exist semi regular set \( A \) of \( X \) containing \( x \) such that \( f[A]: A \rightarrow Y \) is contra \( S_\delta \)-continuous

Proof. Let \( x \in X \), then there exist semi regular set \( A \) of \( X \) containing \( x \). Let \( F \) be closed subset of \( Y \) containing \( f(x) \), then by Theorem 3.2, there exist \( S_\delta \)-open set \( U \) in \( X \) containing \( x \) such that \( (f[A])(U) \subseteq F \). Since \( A \) is semi regular set in \( X \), by Proposition 2.8(1), \( U \) is an \( S_\delta \)-open set in \( X \) and hence \( f(U) \subseteq F \). Thus \( f \) is contra \( S_\delta \)-continuous.

Proposition 3.24. Let \( f:X \rightarrow Y \) and \( g:Y \rightarrow Z \) be any two functions then

1) \( g \circ f:X \rightarrow Z \) is contra \( S_\delta \)-continuous if \( f \) is contra \( S_\delta \)-continuous and \( g \) is contra continuous.

2) \( g \circ f:X \rightarrow Z \) is contra \( S_\delta \)-continuous if \( f \) is contra \( S_\delta \)-continuous and \( g \) is continuous

Proof.(1) Let \( H \) be an open set in \( Z \), since \( g \) is contra continuous, then \( g^{-1}(H) \) is closed subset of \( Y \) and since \( f \) is contra \( S_\delta \)-continuous, then by Proposition 2.11, \( f^{-1}(g^{-1}(H)) = (g \circ f)^{-1}(H) \) is \( S_\delta \)-closed. Hence \( g \circ f \) is contra \( S_\delta \)-continuous.

Proof. If \( f:X \rightarrow Y \) is contra \( S_\delta \)-irresolute and \( g:Y \rightarrow Z \) is contra \( S_\delta \)-continuous, then \( g \circ f:X \rightarrow Z \) is contra \( S_\delta \)-continuous.

Definition 3.25. A function \( f:X \rightarrow Y \) is called \( S_\delta \)-irresolute if \( f^{-1}(U) \) is \( S_\delta \)-open in \( X \) for each \( S_\delta \)-open set \( U \) in \( Y \).

Example 3.26. Let \( X=[a,b,c] \) with topology \( \tau=\{\phi,X,[c]\} \) and \( \sigma=\{\phi,X,[a]\} \) then clearly \( f:(X,\tau) \rightarrow (X,\sigma) \) defined by \( f(a)=a, f(b)=c \) and \( f(c)=c \) is \( S_\delta \)-irresolute function.

Proposition 3.27. If \( f:X \rightarrow Y \) is \( S_\delta \)-irresolute and \( g:Y \rightarrow Z \) is contra \( S_\delta \)-continuous, then \( g \circ f:X \rightarrow Z \) is contra \( S_\delta \)-continuous.

Proof. Let \( F \) be closed subset of \( Z \), since \( g \) is contra \( S_\delta \)-continuous, then \( g^{-1}(F) \) is an \( S_\delta \)-open set in \( Y \) and since \( f \) is \( S_\delta \)-irresolute, thus \( f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F) \) is an \( S_\delta \)-open subset of \( X \). Hence \( g \circ f \) is contra \( S_\delta \)-continuous.

Proposition 3.28. If a function \( f:X \rightarrow \prod_{\alpha \in \Delta} Y_\alpha \) is contra \( S_\delta \)-continuous then \( P_{\alpha} \circ f:X \rightarrow Y_\alpha \) is contra \( S_\delta \)-continuous for each \( \alpha \) in the index set \( \Delta \) where \( P_{\alpha}:\prod_{\alpha \in \Delta} Y_\alpha \rightarrow Y_\alpha \) is projection map from \( \prod_{\alpha \in \Delta} Y_\alpha \) onto \( Y_\alpha \).

Proof. Let \( H_\alpha \) be an closed set in \( Y_\alpha \) for each \( \alpha \in \Delta \) since \( P_{\alpha} \) is continuous function then \( P_{\alpha}^{-1}(H_\alpha) \) is an closed set in \( \prod_{\alpha \in \Delta} Y_\alpha \) for each \( \alpha \) but \( f \) is contra \( S_\delta \)-continuous, then we have \( P_{\alpha} \circ f)^{-1}(H_\alpha) = f^{-1}(P_{\alpha}^{-1}(H_\alpha)) \) is \( S_\delta \)-open for each \( \alpha \in \Delta \) therefore by Theorem 3.2, we have \( P_{\alpha} \circ f \) is contra \( S_\delta \)-continuous function.

Proposition 3.29. If \( f_i:X_1 \rightarrow Y_1 \) is contra \( S_\delta \)–continuous for \( i=1,2 \). Let \( f:X_1 \times X_2 \rightarrow Y_1 \times Y_2 \) be a function defined as follows \( f(x_1,x_2) = (f_1(x_1),f_2(x_2))) \) then \( f \) is contra \( S_\delta \)–continuous.

Proof. Let \( U_1 \times U_2 \subseteq Y_1 \times Y_2 \) where \( U_i \) is an open set in \( Y_i \) for \( i=1,2 \). Then \( f^{-1}(U_i) \) is \( S_\delta \)–closed subset of \( X_i \), since \( f_i \) is contra \( S_\delta \)–continuous for \( i=1,2 \). Therefore \( f^{-1}(U_1 \times U_2) = f^{-1}(U_1) \times f^{-1}(U_2) \) is \( S_\delta \)–closed subset of \( X_1 \times X_2 \). Hence \( f \) contra \( S_\delta \)–continuous.

Proposition 3.30. Let \( h:X \rightarrow X_1 \times X_2 \) be a contra \( S_\delta \)–continuous function defined as follows: \( h(x) = (h_1(x),h_2(x)) \) then \( h_i:X \rightarrow X_i \) is contra \( S_\delta \)–continuous for \( i=1,2 \).

Proof. Let \( U_i \) be an open set in \( X_i \). Then \( U_1 \times U_2 \) is an open set in \( X_1 \times X_2 \) and then \( h_i^{-1}(U_i) = h_i^{-1}(U_1 \times U_2) \) is \( S_\delta \)–closed set in \( X_i \) hence \( h_i:X \rightarrow X_i \) is contra \( S_\delta \)–continuous. Similarly for \( h_i \) for \( i=2 \).

Proposition 3.31. Let \( f:X \rightarrow Y \) be any function. If \( g:X \rightarrow X \times Y \) defined by \( g(x) = (x,f(x)) \) is contra \( S_\delta \)–continuous, then \( f \) contra \( S_\delta \)–continuous.

Proof. Let \( F \) be closed subset of \( Y \), then \( X \times F \) is closed subset of \( X \times Y \). Since \( g \) is contra \( S_\delta \)–continuous, then \( g^{-1}(X \times F) = f^{-1}(F) \) is an \( S_\delta \)–open subset of \( X \). Hence \( f \) contra \( S_\delta \)–continuous.
4. Functions with contra $S_S$–closed and strongly contra $S_S$–closed graphs

**Definition 4.1.** The graph $G(f)$ of the function $f: X \to Y$ is said to be contra $S_S$–closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist an $S_S$–open set $U$ containing $x$ and a closed set $V$ in $Y$ containing $y$ such that $(U \times V) \cap G(f) = \phi$.

**Proposition 4.2.** The graph $G(f)$ of the function $f: X \to Y$ is contra $S_S$–closed if for each $(x, y) \in (X \times Y) - G(f)$, there exist an $S_S$–open set $U$ containing $x$ and a closed set $V$ in $Y$ containing $y$ such that $f(U) \cap V = \phi$.

**Proof.** Follows from the definition.

**Theorem 4.3.** If a function $f: X \to Y$ is contra $S_S$–continuous and $Y$ is Urysohn then $G(f)$ is contra $S_S$–closed.

**Proof.** Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and since $Y$ is Urysohn, there exist two open sets $A$ and $B$ in $Y$ such that $\text{Cl}(A) \cap \text{Cl}(B) = \phi$. Since $f$ is contra $S_S$–continuous, then there exist an $S_S$–open set $U$ containing $x$ such that $f(U) \subseteq \text{Cl}(A)$ implies that $f(U) \cap \text{Cl}(B) = \phi$. Therefore by Proposition 4.2, $G(f)$ is contra $S_S$–closed.

**Theorem 4.4.** If a function $f: X \to Y$ is $S_S$–continuous and $Y$ is $T_1$–space, then $G(f)$ is contra $S_S$–closed.

**Proof.** Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and since $Y$ is $T_1$–space, there exists an open set $H$ in $Y$ such that $f(x) \in H, y \notin H$. Since $f$ is $S_S$–continuous, then there exists an $S_S$–open set $U$ containing $x$ such that $f(U) \subseteq H$ which implies that $f(U) \cap (Y - H) = \phi$ where $Y - H$ is a closed set in $Y$ containing $y$. Hence by Proposition 4.2, we obtain that $G(f)$ is contra $S_S$–closed.

**Definition 4.5.** The graph $G(f)$ of the function $f: X \to Y$ is strongly contra $S_S$–continuous if for each $(x, y) \in (X \times Y) - G(f)$, there exist an $S_S$–open set $U$ containing $x$ and a regular closed set $V$ in $Y$ containing $y$ such that $(U \times V) \cap G(f) = \phi$.

**Proposition 4.6.** The graph $G(f)$ of the function $f: X \to Y$ is strongly contra $S_S$–continuous if for each $(x, y) \in (X \times Y) - G(f)$, there exist an $S_S$–open set $U$ containing $x$ and a regular closed set $V$ in $Y$ containing $y$ such that $f(U) \cap V = \phi$.

**Proof.** Follows from the definition.

**Theorem 4.7.** If a function $f: X \to Y$ is contra $S_S$–continuous and $Y$ is Urysohn, then $G(f)$ is strongly $S_S$–closed in $X \times Y$.

**Proof.** Let $(x, y) \in (X \times Y) - G(f)$. Then $y \neq f(x)$ and since $Y$ is Urysohn, there exist two open sets $A$ and $B$ in $Y$ such that $\text{Cl}(A) \cap \text{Cl}(B) = \phi$. Since $f$ is contra $S_S$–continuous, then there exist an $S_S$–open set $U$ containing $x$ such that $f(U) \subseteq \text{Cl}(A)$ implies that $f(U) \cap \text{Cl}(\text{int}(B)) = f(U) \cap \text{Cl}(B) = \phi$. Since $\text{Cl}(\text{int}(B))$ is regular closed in $Y$. Hence by Proposition 4.6, $G(f)$ is strongly $S_S$–closed in $X \times Y$.

References


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