ON THE PRODUCT AND RATIO OF PARETO AND KUMARASWAMY RANDOM VARIABLES

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ABSTRACT. The distributions of the product \( XY \) and the ratio \( X/Y \) are derived when \( X \) and \( Y \) are Pareto and the Kumaraswamy random variables distributed independently of each other.

1. Introduction

For given random variables \( X \) and \( Y \), the distributions of the product \( XY \) and the ratio \( X/Y \) are of interest in many areas of the sciences, engineering and medicine. Examples of \( XY \) include traditional portfolio selection models, relationship between attitudes and behavior, number of cancer cells in tumor biology and stream flow in hydrology. Examples of \( X/Y \) include Mendelian inheritance ratios in genetics, mass to energy ratios in nuclear physics, target to control precipitation in meteorology, inventory ratios in economics and safety factor in engineering. The distributions of \( XY \) and \( X/Y \) have been studied by several authors especially when \( X \) and \( Y \) are independent random variables and come from the same family. With respect to \( XY \), see Sakamoto (1943) for uniform family, Harter (1951) and Wallgren (1980) for Students t family, Springer and Thompson (1970) for normal family, Stuart (1962) and Podolski (1972) for gamma family, Steece (1976), Bhargava and Khatri (1981) and Tang and Gupta (1984) for beta family, AbuSalih (1983) for power function family, and Malik and Trudel (1986) for exponential family (see also Rathie and Rohrer (1987) for a comprehensive review of known results). With respect to \( X/Y \), see Marsaglia (1965) and Korhonen and Narula (1989) for normal family, Press (1969) for Student’s t family, Basu and Lochner (1971) for Weibull family, Shkolnick (1985) for stable family, Hawkins and Han (1986) for non-central chi-squared family, Provost (1989) for gamma family, and Pham-Gia (2000) for beta family. There is relatively little work of this kind when \( X \) and \( Y \) belong to different families. In the applications mentioned above, it is quite possible that \( X \) and \( Y \) could arise from different but similar distributions (see below for examples).

In this paper, we study the exact distributions of \( XY \) and \( X/Y \) when \( X \) and \( Y \) are independent Pareto and Kumaraswamy random variables with pdfs

\[
f_X(x) = \frac{\alpha k^\alpha}{x^{\alpha+1}}, \tag{1.1}
\]

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and
\[ f_Y(y) = aby^{-1}(1-y)^{b-1} \quad (1.2) \]
respectively, for \( k \leq x < \infty, \alpha, k > 0, 0 < y < 1, \alpha > 0, \beta > 0, \lambda > 0 \)

The calculations of this paper involve several special functions, including the gamma function defined by
\[ \Gamma(a) = \int_0^\infty \exp(-t)t^{a-1}dt, \]
and beta function defined by
\[ \text{Beta}(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx. \]

2. Main Results

**Theorem 2.1.** Suppose \( X \) and \( Y \) are distributed according to (1.1) and (1.2), respectively. The cdf of \( Z = XY \) can be expressed as:
\[ F(z) = Pr(Z = XY) = 1 - b\left(\frac{k}{z}\right)^\alpha B\left(1 + \frac{\alpha}{a}, b\right) \quad (2.1) \]
for \( z > 0, \alpha > 2 - a, a > 0, b > 0 \)

**Proof.** The cdf corresponding to (1.1) is \( 1 - \left(\frac{k}{x}\right)^\alpha \). Thus, one can write the cdf of \( X/Y \) as
\[
\Pr(XY \leq z) = \int_0^\infty F_X\left(\frac{z}{y}\right)f_Y(y)dy \\
= 1 - ab \int_0^1 \left(\frac{ky}{z}\right)^\alpha y^{a-1}(1-y)^{b-1}dy \\
= 1 - ab\left(\frac{k}{z}\right)^\alpha \int_0^1 y^{a+\alpha-1}(1-y)^{b-1}dy \\
= 1 - b\left(\frac{k}{z}\right)^\alpha \int_0^1 t^{\frac{\alpha}{a}-1}(1-t)^{b-1}dt \\
= 1 - b\left(\frac{k}{z}\right)^\alpha B\left(1 + \frac{\alpha}{a}, b\right)
\]
\( \square \)

**Theorem 2.2.** The pdf of \( Z = XY \) can be expressed as:
\[ f(z) = \frac{b\alpha k^\alpha}{z^{\alpha+1}} B\left(1 + \frac{\alpha}{a}, b\right) \quad (2.2) \]
for \( z > 0, \alpha > 0, a > 0, b > 0 \).

**Proof.** It is straightforward to show the results of Corollary by taking the differentiation equation (2.1). \( \square \)
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Figure 1. The pdf’s of various values for Z=XY.

Theorem 2.3. Suppose X and Y are distributed according to (1.1) and (1.2), respectively. The rth moment of Z = XY, say \( E[Z^r] \), is

\[
E[Z^r] = \frac{b \alpha k^r}{(\alpha - r)} B \left( 1 + \frac{\alpha}{a}, b \right)
\]  

(2.3)

Proof.

\[
E[Z^r] = \int_k^\infty z^r f(z) dx = \int_k^\infty \frac{b \alpha k^r}{z^{\alpha - r + 1}} B \left( 1 + \frac{\alpha}{a}, b \right) dz
\]

\[
= b \alpha k^r B \left( 1 + \frac{\alpha}{a}, b \right) \int_k^\infty \frac{dx}{z^{\alpha - r + 1}}
\]

\[
= b \alpha k^r B \left( 1 + \frac{\alpha}{a}, b \right) \left[ -\frac{1}{(\alpha - r)z^{\alpha - r}} \right]_k^\infty
\]

\[
= b \alpha k^r \left( 1 + \frac{\alpha}{a}, b \right)
\]
Theorem 2.4. Suppose $X$ and $Y$ are distributed according to (1.1) and (1.2), respectively. The cdf of $Z = \frac{X}{Y}$ can be expressed as:

$$F(Z) = 1 - b \left( \frac{k}{z} \right)^{\alpha} B \left( \frac{a - \alpha}{a}, b \right)$$  \hspace{1cm} (2.4)$$

Proof. The cdf corresponding to (1.1) is $1 - \left( \frac{k}{z} \right)^{\alpha}$. Thus, one can write the cdf of $X/Y$ as

$$\Pr \left( \frac{X}{Y} \leq z \right) = \int_0^\infty F_X(zy)f_Y(y)dy$$

$$= 1 - ab \int_0^1 \left( \frac{k}{zy} \right)^{\alpha} y^{a-1}(1-y^{a})^{b-1}dy$$

$$= 1 - ab \left( \frac{k}{z} \right)^{\alpha} \int_0^1 y^{a-\alpha-1}(1-y^{a})^{b-1}dy$$

$$= 1 - b \left( \frac{k}{z} \right)^{\alpha} \int_0^1 t^{\frac{a-\alpha}{a}-1}(1-t)^{b-1}dt$$

$$= 1 - b \left( \frac{k}{z} \right)^{\alpha} B \left( a - \frac{\alpha}{a}, b \right)$$

\[ \Box \]

Theorem 2.5. The pdf of $Z = \frac{X}{Y}$ can be expressed as:

$$f(z) = \frac{b a k^{\alpha} z^{\alpha+1}}{z^{\alpha+1}} B \left( \frac{a - \alpha}{a}, b \right)$$  \hspace{1cm} (2.5)$$

for $z > 0$, $\alpha > a$, $a > 0$, $b > 0$.

Proof. It is straightforward to show the results of Corollary by taking the differentiation equation (2.1). \[ \Box \]

Theorem 2.6. Suppose $X$ and $Y$ are distributed according to (1.1) and (1.2), respectively. The $r$th moment of $Z = \frac{X}{Y}$, say $E[Z^r]$, is

$$E[Z^r] = \frac{b a k^{\alpha} z^{\alpha+1}}{z^{\alpha+1}} B \left( \frac{a - \alpha}{a}, b \right)$$  \hspace{1cm} (2.6)$$
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![Figure 2. The pdf's of various values for Z=X/Y.](image)

**Proof.**

\[
E[Z^r] = \int_0^\infty z^r f(z) dx = \int_0^\infty \frac{b \alpha k^\alpha}{z^{a+1}} \frac{\Gamma(a - \alpha)}{\Gamma(a) \Gamma(b)} \left( \frac{z}{a} \right)^{a-\alpha} \left( 1 - \frac{z}{a} \right)^{b-1} \frac{dz}{z^{a+r+1}}
\]

\[
= b \alpha k^\alpha \frac{\Gamma(a - \alpha)}{\Gamma(a) \Gamma(b)} \left[ - \frac{1}{(\alpha - r)z^{\alpha-r}} \right]_k^\infty
\]

\[
= \frac{b \alpha k^\alpha}{(\alpha - r)} \frac{\Gamma(a - \alpha)}{\Gamma(a) \Gamma(b)}
\]

\[\square\]

3. Order statistics

Suppose \(Z_1, Z_2, \ldots, Z_N\) is a random sample from (2.2). Let \(Z_{1:N} < Z_{2:N} < \ldots < Z_{N:N}\) denote the corresponding order statistics. It is well known that the pdf and
the cdf of the kth order statistic, say \( H = Z_{k,N} \) are given by

\[
f_H(h) = \frac{N!}{(k-1)!((N-k)!)} \sum_{l=0}^{N-k} \binom{N-j}{l} (-1)^l F_Z^{k-1+l}(h) f_Z(h)
\]

(3.1)

and

\[
F_H(h) = \sum_{j=k}^{N} \sum_{l=0}^{N-j} \binom{N-j}{l} (-1)^l f_Z^{j+l}(h)
\]

respectively, for \( k = 1, 2, \ldots, N \). It follows from (2.4) and (2.5) that

\[
f_H(h) = \frac{N!}{(k-1)!((N-k)!)} \sum_{l=0}^{N-k} \binom{N-j}{l} (-1)^l \left[ 1 - b \left( \frac{k}{h} \right)^{\alpha} B \left( 1 + \frac{\alpha}{a}, b \right) \right]^{k-1+l}
\]

\[
\times \frac{b\alpha k^\alpha}{h^{\alpha+1}} B \left( 1 + \frac{\alpha}{a}, b \right)
\]

(3.3)

\[
F_H(h) = \sum_{j=k}^{N} \sum_{l=0}^{N-j} \binom{N-j}{l} \left[ 1 - b \left( \frac{k}{h} \right)^{\alpha} B \left( 1 + \frac{\alpha}{a}, b \right) \right]^{j+l}
\]

(3.4)

\( R_1, R_2, \ldots, R_N \) is a random sample from (2.5). Let \( R_{1:N} < R_{2:N} < \ldots < R_{N:N} \) denote the corresponding order statistics. Let \( H = R_{k:N} \), so the pdf and cdf of \( H \) follow from (2.4) and (2.5):

\[
f_H(h) = \frac{N!}{(k-1)!((N-k)!)} \sum_{l=0}^{N-k} \binom{N-j}{l} (-1)^l \left[ 1 - b \left( \frac{k}{h} \right)^{\alpha} B \left( 1 - \frac{\alpha}{a}, b \right) \right]^{k-1+l}
\]

\[
\times \frac{b\alpha k^\alpha}{h^{\alpha+1}} B \left( 1 - \frac{\alpha}{a}, b \right)
\]

(3.5)

and

\[
F_H(h) = \sum_{j=k}^{N} \sum_{l=0}^{N-j} \binom{N-j}{l} (-1)^l \left[ 1 - b \left( \frac{k}{h} \right)^{\alpha} B \left( 1 - \frac{\alpha}{a}, b \right) \right]^{j+l}
\]

(3.6)

4. Application

An application of the result in Theorem 2.1 can be illustrated by deriving the percentage points \( z_q \) associated with \( Z = XY \). These values can be obtained by numerically solving the equation

\[
1 - b \left( \frac{k}{z} \right)^{\alpha} B \left( 1 + \frac{\alpha}{a}, b \right) = z_q.
\]

(4.1)
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Table 1. Percentage points of \( z_q \) for \( \alpha = 1 \), \( k = 1, 2, \ldots, 5 \), \( a = 1, 2, \ldots, 5 \); \( b = 0.1, 0.2, \ldots, 0.5 \)

Evidently, this involves computation of the gamma function, and routines for this are widely available. One can use the \( \text{GAMMA}() \) function in MAPLE, the algebraic manipulation package. A MAPLE procedure for solving Equation (4.1) is:

```maple
alpha := 1;
k := 1;
a := 1;
b := .1;
f := 1-b*k^alpha*GAMMA(1+alpha/a)*GAMMA(b)/(z^alpha*GAMMA(1+alpha/a+b));
p1 := fsolve(f = 0.1e-1, z);
p2 := fsolve(f = 0.5e-1, z);
p3 := fsolve(f = .1, z);
p4 := fsolve(f = .90, z);
p5 := fsolve(f = .95, z);
p6 := fsolve(f = .99, z);
print(p1, p2, p3, p4, p5, p6)
```

Table 1 provides numerical values of \( z_q \) for \( \alpha = 1 \), \( k = 1, 2, \ldots, 5 \), \( a = 1, 2, \ldots, 5 \); \( b = 0.1, 0.2, \ldots, 0.5 \)
An application of the result in Theorem 2.4 can be illustrated by deriving the percentage points $z_q$ associated with $Z = \frac{X}{Y}$. These values can be obtained by numerically solving the equation

$$1 - b \left( \frac{k}{z} \right)^\alpha B \left( \frac{a - \alpha}{a}, b \right) = z_q.$$  

(4.2)

A MAPLE procedure for solving Equation (4.2) is:

```maple
alpha := 1;
k := 5; a := 5;
b := .5;
f := 1-b*k^alpha*GAMMA(1-alpha/a)*GAMMA(b)/(z^alpha*GAMMA(1-alpha/a+b));
p1 := fsolve(f = 0.1e-1, z);
p2 := fsolve(f = 0.5e-1, z);
p3 := fsolve(f = .1, z);
p4 := fsolve(f = .90, z);
p5 := fsolve(f = .95, z);
p6 := fsolve(f = .99, z);
print(p1, p2, p3, p4, p5, p6)
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