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A Fixed Point Result in Banach Spaces

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Abstract
In the present paper we establish some fixed point theorems in Banach space taking rational expression. Our Result Generalize the result of many authors.

Key words: Fixed point, common fixed point, rational expressions, Banach spaces

Introduction: Fixed point has drawn the attentions of the authors working in non-linear analysis, the study of non-expansive mapping and the existence of fixed point. The non-expansive mappings include contraction as well as contractive mappings. Browder [1] was the first mathematician to study non-expansive mappings; he applied these results for proving the existence of solutions of certain integral equations.

It is well known that differential and the integral equations that arise in physical problems are generally non-linear, therefore the fixed point technique provides a powerful tool for obtaining the solutions of these equations which otherwise are difficult to solve by ordinary methods. No doubt, it is also true that some qualitative properties of the solution of related equations is lost by functional analysis approach. Many attempts have been made in this direction to formulate fixed point theorems. Schauder, J. formulated the well known Schauder's fixed point principle in 1930.

Browder [1], Gohde [6] and Kirk [10] have independently proved a fixed point theorem for non-expansive mappings defined on a closed bounded and convex subset of a uniformly convex Banach space and in the spaces with richer generalizations of non-expansive mappings, prominent being Datson [2], Emmanuele [3], Goebel [4], Goebel and Zlotkienwicz [5], Iseki [7], Sharma & Rajput [11], Singh and Chatterjee [13]. They have derived valuable results with non-contraction mapping in Banach space.

Our object in this chapter is to prove some fixed and common fixed point theorems using Banach space.

Our results include the results of Goebel and Zlotkiewicz [5], Iseki [7], Sharma and Bajaj [12], Khan [9], Jain and Jain [8]. We shall prove:-

Theorem-1 :

Let F be a mapping of a Banach space x into itself. If F satisfies the following conditions;

1. $F^2 = I$, where I is the identity mapping. ……….(1.1)
2. $\|F(x) - F(y)\|$ ……….(1.2)
   $\leq a_1 [\|x - F(x)\| + \|y - F(y)\|] + a_2 [\|x - y\|]$
   $+ a_3 \text{ Max } \{P, Q\}$
   Where
   $P = \frac{\|x-F(x)\|\|y-F(y)\| + \|x-F(y)\| \|y-F(x)\|}{\|x-y\|}$ and $Q = \frac{\|x-F(x)\| \|x-F(y)\| + \|y-F(y)\| \|y-F(x)\|}{\|x-y\|}$
   For every $x,y \in X$, where $0 < a_1,a_2,a_3$ and $4a_1 + a_2 + 8a_3 < 2$, then F has a fixed point, if $a_2 + a_3 < 1$,then F has a unique fixed point.

Proof: Suppose x be a point in Banach space X. Taking

$y = \frac{1}{2}(F+I)(x)$

$z = F(y)$ and

$u = 2y-z$

We have

$\|z - x\| = \|F(y) - F^2(x)\| = \|F(y) - F(F(x))\|$

$\leq a_1 [\|y - F(y)\| + \|F(x) - F^2(x)\|] + a_2 \|y - F(x)\| + a_3 \max [P, Q]$

$P = \frac{\|x-F(x)\| \, \|y-F(y)\| + \|x-F(y)\| \, \|y-F(x)\|}{\|x-y\|}$ and $Q = \frac{\|x-F(x)\| \, \|x-F(y)\| + \|y-F(y)\| \, \|y-F(x)\|}{\|x-y\|}$

Case 1: when max [P, Q] = P

Then $\|z - x\|$

$\leq a_1 [\|y - F(y)\| + \|F(x) - F^2(x)\|] + a_2 \|y - F(x)\| + a_3 \left[\frac{\|y-F(y)\| \|F(x) - F^2(x)\| + \|y-F^2(x)\| \|F(x) - F(y)\|}{\|y-F(x)\|}\right]$

$= a_1 [\|y - F(y)\| + \|F(x) - x\|] + a_2 \|y - F(x)\| + a_3 \left[\frac{\|y-F(y)\| \|F(x) - x\| + \|y-x\| \|F(x) - F(y)\|}{\|y-F(x)\|}\right]$

$= a_1 [\|y - F(y)\| + \|F(x) - x\|] + a_2 \left\|\frac{1}{2}(F + I)(x) - F(x)\right\| + a_3 \left[\frac{\|y-F(y)\| \|F(x) - x\| + \left\|\frac{1}{2}(F+I)(x) - x\right\| \|F(x) - F(y)\|}{\left\|\frac{1}{2}(F+I)(x) - F(x)\right\|}\right]$

$= a_1 [\|y - F(y)\| + \|F(x) - x\|] + \frac{a_2}{2} \|x - F(x)\| + 2a_3 \|y - F(y)\| + a_3 \|F(x) - F(y)\|$

$= a_1 [\|x - F(x)\| + \|y - F(y)\|] + \frac{a_2}{2} \|x - F(x)\| + 3a_3 \|y - F(y)\| + a_3 \|F(x) - y\|$

$= a_1[\|x - F(x)\| + \|y - F(y)\|] + \frac{a2}{2} \|x - F(x)\|$

$= (a_1 + 3a_3) [\|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + \frac{a3}{2}\right) \|x - F(x)\|$

Therefore ,

$\|z - x\| \leq (a_1 + 3a_3) [\|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + \frac{a3}{2}\right) \|x - F(x)\|$

Also

$\|u - x\| = \|2y - z - x\| = \|(F + I)(x) - F(y) - x\| = \|F(x) - F(y)\|$

$\leq a_1 [\|x - F(x)\| + \|y - F(y)\|] + a_2 \|x - y\| + a_3 \left[\frac{\|x-F(x)\| \|y - F(y)\| + \|x-F(y)\| \|y - F(x)\|}{\|x-y\|}\right]$

$= a_1 [\|x - F(x)\| + \|y - F(y)\|] + a_2 \left\|x - \frac{1}{2}(F + I)(x)\right\|$

$+ a_3 \left[\frac{\|x-F(x)\| \|y - F(y)\| + \|x-F(y)\| \left\|\frac{1}{2}(F+I)(x) - F(x)\right\|}{\left\|x - \frac{1}{2}(F+I)(x)\right\|}\right]$

$= a_1 [\|x - F(x)\| + \|y - F(y)\|] + \frac{a_2}{2} \|x - F(x)\| + 2a_3 \|y - F(y)\| + a_3 \|x - y\|$

$+ a_3 \|y - F(y)\|$

$= (a_1 + 3a_3) [\|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + \frac{a3}{2}\right) \|x - F(x)\|$.

Therefore,

$\|u - x\| \leq (a_1 + 3a_3) [\|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + \frac{a3}{2}\right) \|x - F(x)\|$. ……. (1.3)

Now

$\|z - u\| \leq \|z - x\| + \|x - u\|$

$= (a_1 + 3a_3) [\|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + \frac{a3}{2}\right) \|x - F(x)\| + (a_1 + 3a_3) [\|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + \frac{a3}{2}\right) \|x - F(x)\|$

$= 2(a_1 + 3a_3)\,[\,\|y - F(y)\|] + 2\left(a_1 + \frac{a_2}{2} + \frac{a3}{2}\right)\|x - F(x)\|$

Thus, $\|z - u\| \leq 2(a_1 + 3a_3)\,[\,\|y - F(y)\|] + 2\left(a_1 + \frac{a_2}{2} + \frac{a3}{2}\right)\|x - F(x)\|$ ……. (1.4)

Also, $\|z - u\| = \|F(y) - (2y - z)\|$

$= \|F(y) - 2y + z\|$

$= 2\|F(y) - y\|$

Combining (1.3) and (1.4), we have

$\|Y - F(y)\| \leq 2[(a_1+3a_3)\,\|y - F(y)\| + (a_1 + a_2/2 + a_3/2)\,\|x - F(x)\|\,]$

Therefore $\|y - F(y)\| \leq q\,\|x - F(x)\|$

Where

$q = \frac{(2a1+a2+a3)}{(1-3a3-a1)} < 1,$

since $4a_1 + a_2 + 7a_3 < 2$

on taking

$G = \frac{1}{2}(F + I)$ then for every $x \in X$

$$\left\|G^2\,(x) - G(x)\right\| = \|G(y) - y\|$$

$$= \frac{1}{2}\|y - F(y)\|$$

$$< {}^{q}/_{2}\,\|x - F(x)\|$$

By the definition of q, we claim that $\{G^n(x)\}$ is a Cauchy sequence in X. Therefore, by the property of completeness, $G^n(x)\}$ converges to some element x_0 in X.

i.e. $\lim_{n\to\infty} G^n(x) = x_0$
Which implies $G(x_0) = x_0$
Hence $F(x_0) = x_0$
i.e. x_0 is a fixed point of F.
For the uniqueness, if possible let $y_0\ (\neq x_0)$ be another fixed point of F. Then
$\|x_0 - y_0\| = \|F(x_0) - F(y_0)\|$
$\leq a_1\,[\,\|x_0 - F(x_0)\| + \|y_0 - F(y_0)\|\,] + a_2\,\|x_0 - y_0\|$
$+ a_3\,\frac{\|x0-F(x0)\|\,\|y0-F(y0)\|+\|x0-F(y0)\|\|y0-F(x0)\|}{\|x0-yo\|}$
$= a_2\|x_0 - y_0\| + a_3\frac{\|x0-F(y0)\|\|y0-F(x0)\|}{\|x0-y0\|}$
$= (a_2+a_3)\|x_0 - y_0\|$
Since $a_2 + a_3 < 1$, therefore
$\|x_0 - y_0\| = 0$
Hence $x_0 = y_0$.

Case 2: when max [P, Q] = Q

Then $\|z - x\|$

$\leq a_1\,[\|y - F(y)\| + \|F(x) - F^2(x)\|] + a_2\,\|y - F(x)\| + a_3\left[\frac{\|y-F(y)\|\|y-F^2(x)\|+\|F(x)-F^2(x)\|\|F(x)-F(y)\|}{\|y-F(x)\|}\right]$

$= a_1\,[\|y - F(y)\| + \|F(x) - x\|] + a_2\,\|y - F(x)\| + a_3\left[\frac{\|y-F(y)\|\|y-x\|+\|F(x)-x\|\|F(x)-F(y)\|}{\|y-F(x)\|}\right]$

$= a_1 [\|y - F(y)\| + \|F(x) - x\|] + a_2 \left\|\frac{1}{2}(F + I)(x) - F(x)\right\| + a_3\left[\frac{\|y-F(y)\|\left\|\frac{1}{2}(F+I)(x)-x\right\|+\|F(x)-x\|\|F(x)-F(y)\|}{\left\|\frac{1}{2}(F+I)(x)-F(x)\right\|}\right]$

$= a_1 [\|y - F(y)\| + \|F(x) - x\|] + \frac{a_2}{2}\|x - F(x)\| + a_3 \|y - F(y)\| + 2 a_3 \|F(x) - F(y)\|$

$= a_1 [\|x - F(x)\| + \|y - F(y)\|] + \frac{a_2}{2}\|x - F(x)\| + a_3 \|y - F(y)\| + 2a_3 [\ \|F(x) - y\| + \|y - F(y)\|]$

$= a_1 [\|x - F(x)\| + \|y - F(y)\|] + \frac{a2}{2}\|x - F(x)\| + 3a_3\|y - F(y)\| + 2a_3 1/2\|x - F(x)\|$

$= (a_1 + 3a_3) [\ \|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + a_3\right) \|x - F(x)\|$

Therefore,

$\|z - x\| \leq (a_1 + 3a_3) [\ \|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + a_3\right) \|x - F(x)\|$

Also

$\|u - x\| = \|2y - z - x\| = \|(F + I)(x) - F(y) - x\| = \|F(x) - F(y)\|$

$\leq a_1 [\|x - F(x)\| + \|y - F(y)\|] + a_2 \|x - y\| + a_3\left[\frac{\|x-F(x)\|\|x-F(y)\| + \|y-F(y)\|\|y-F(x)\|}{\|x-y\|}\right]$

$= a_1 [\|x - F(x)\| + \|y - F(y)\|] + a_2 \left\|x - \frac{1}{2}(F + I)(x)\right\|$

$+ a_3\left[\frac{\|x-F(x)\|\|y-F(y)\| + \|x-F(y)\|\left\|\frac{1}{2}(F+I)(x)-F(x)\right\|}{\left\|x-\frac{1}{2}(F+I)(x)\right\|}\right]$

$= a_1 [\|x - F(x)\| + \|y - F(y)\|] + \frac{a_2}{2}\|x - F(x)\| + 2a_3\|x - y\| + a_3 \|y - F(y)\|$

$+ a_3 \|y - F(y)\|$

$= (a_1 + 3a_3) [\ \|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + a_3\right) \|x - F(x)\|$.

Therefore,

$\|u - x\| \leq (a_1 + 3a_3) [\ \|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + a_3\right) \|x - F(x)\|$. ……. (1.5)

Now

$\|z - u\| \leq \|z - x\| + \|x - u\|$

$= (a_1 + 3a_3) [\ \|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + a_3\right) \|x - F(x)\| + (a_1 + 3a_3) [\ \|y - F(y)\|] + \left(a_1 + \frac{a_2}{2} + a_3\right)$
$\|x - F(y)\|$

$= 2(a_1 + 3a_3) [\ \|y - F(y)\|] + 2\left(a_1 + \frac{a_2}{2} + a_3\right) \|x - F(x)\|$

Thus, $\|z - u\| \leq 2(a_1 + 3a_3) [\ \|y - F(y)\|] + 2\left(a_1 + \frac{a_2}{2} + a_3\right) \|x - F(x)\|$ ……. (1.6)

Also, $\|z - u\| = \|F(y) - (2y - z)\|$

$= \|F(y) - 2y + z\|$

$= 2\|F(y) - y\|$

Combining (1.5) and (1.6), we have

$\|Y - F(y)\| \leq 2[(a_1+3a_3)\ \|y - F(y)\| + (a_1 + a_2/2 + a_3)\ \|x - F(x)\|\ ]$

Therefore $\|y - F(y)\| \leq q\ \|x - F(x)\|$

Where

$q=\frac{(2a1+a2+2a3)}{(1-3a3-a1)} < 1,$

since $4a_1 + a_2 + 8a_3 < 2$

on taking

$G = \frac{1}{2}(F + I)$ then for every $x \in X$

$$\left\|G^2\ (x) - G(x)\right\| = \|G(y) - y\|$$

$$=\left\|\frac{1}{2}(F + I)(y) - y\right\|$$

$$=\frac{1}{2}\|y - F(y)\|$$

$$< {}^{q}/_{2}\ \|x - F(x)\|$$

By the definition of q, we claim that $\{G^n(x)\}$ is a Cauchy sequence in X. Therefore, by the property of completeness, $\{G^n(x)\}$ converges to some element x_0 in X.

i.e.$\lim_{n\to\infty} G^n\,(x) = x_0$
Which implies $G(\ x_0\ ) = x_0$
Hence $F(x_0) = x_0$
i.e. x_0 is a fixed point of F.
For the uniqueness, if possible let $y_0\ (\neq x_0)$ be another fixed point of F. Then
$\|x_0 - y_0\| = \|F(x_0) - F(y_0)\|$
$\leq a_1\ [\ \|x_0 - F(x_0)\| + \|\ y_0 - F(y_0)\ \|\ ] + a_2\ \|x_0 - y_0\|$
$+\ a_3\ \frac{\|x0-F(x0)\|\ \|x0-F(y0)\|+\|y0-F(y0)\|\|y0-F(x0)\|}{\|x0-yo\|}$
$=a_2\|x_0 - y_0\|$
Since $a_2 < 1$, therefore
$\|x_0 - y_0\| = 0$
Hence $x_0 = y_0$

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