Valuations, Valued Field and Construction of Valuation over the Rings

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Abstract
The definition of the valuation is studied on the finite structures. Since the ideals can be classified in particular, the finite chain rings are dealt with as a definition set. With these ideals, the maximal ideals of the valuation rings are matched. In addition, the unit groups of the writing valuations and the unit elements of the ring were compared.

Keywords: Valuations, Valuation Theory, Valued Fields, Valuation Ring.

1.Introduction

The arithmetic solutions of Weber and Dedekind are considered to be the beginning of valuation theory. Later, the early years of the 20th century led the way in the valuation area. In (Özkan M., 1), The definition set of the valuation is explained by me over the finite chain rings. The structure, properties and application of these rings are presented in (Özkan M., Öke F., 2), (Jian-Fa Q., Zhang L. N. and Zhu S. X., 3) and (Özkan M., Öke F., 4). The basics of the theory of valuation have been described in Endler's Valuation Theory book (Endler O., 5).

2. Preliminaries

In the case of $u^2=0$, the ring $\mathrm{F}_2[u] \big/ {<u^2>} = \{\, a_0 + a_1.u + {<u^2>} \mid a_0, a_1 \in \mathrm{F}_2 \,\}$ is isomorphic to the ring $\mathrm{F}_2 + u\mathrm{F}_2$.

$\mathrm{F}_2 + u\mathrm{F}_2 = \{0,1,u,1+u\}$ is a ring with the $+$ and $\bullet$ operations defined below;

$+$	0	1	u	$1+u$
0	0	1	u	$1+u$
1	1	0	$1+u$	u
u	u	$1+u$	0	1
$1+u$	$1+u$	u	1	0

$\bullet$	0	1	u	$1+u$
0	0	0	0	0
1	0	1	u	$1+u$
u	0	u	0	u
$1+u$	0	$1+u$	u	1

The ring $\mathrm{F}_2 + u\mathrm{F}_2$ has three different ideals. These ideals are $\langle 0\rangle$, $\langle 1\rangle$ ve $\langle u\rangle$, and are ensured $\langle 0\rangle \subseteq \langle u\rangle \subseteq \langle 1\rangle = \mathrm{F}_2 + u\mathrm{F}_2$.

In the case of $v^2=1$, the ring $\mathrm{F}_2[v] \big/ \langle v^2-1\rangle$ is isomorphic to the ring $\mathrm{F}_2 + v\mathrm{F}_2$.

$\mathrm{F}_2 + v\mathrm{F}_2$ where $v^2=1$ is a ring with the $+$ and $\bullet$ operations defined below;

$+$	0	1	v	$1+v$
0	0	1	v	$1+v$
1	1	0	$1+v$	v
v	v	$1+v$	0	1
$1+v$	$1+v$	v	1	0

$\bullet$	0	1	v	$1+v$
0	0	0	0	0
1	0	1	v	$1+v$
v	0	v	1	$1+v$
$1+v$	0	$1+v$	$1+v$	0

The ideals of the ring F_2+vF_2 in the case $v^2=1$ are $\langle 0\rangle=\{0\}$, $\langle 1+v\rangle=\{0,1+v\}$ and $\langle v\rangle=\langle 1\rangle=F_2+vF_2$. Among ideals of this ring is $\langle 0\rangle\subseteq\langle 1+v\rangle\subseteq\langle v\rangle\subseteq\langle 1\rangle=F_2+vF_2$ a relationship and a local ring.

$(G,+)$ is an commutative group. $>$ the relation is a ordered relation on group G; For every $x,y,z\in G$,

i) if there are $x>y$ and $y>z$ then $x>z$.
ii) only one conditions $x>y$, $x=y$, $y>x$ is provided.
iii) for $m\in G$, if $x>y$ then $x+m>y+m$.

If the conditions are true, with $>$ the relation $(G,+)$ is called a total order group.

Here we take G as a multiplicative group instead of an additive group;

iv) for $m\in G$, if $x>y$ then $x.m>y.m$.

If the conditions i , ii and iv are true, with $>$ the relation $(G,.)$ is called a total order group.

3. Structure of Valuations

3.1. Definition Let R be a finite ring and $(G,+)$ a total order group.

$$v\ :\ R\rightarrow G\cup\{\infty\}$$

$$x\mapsto v(x)$$

with map, for each $x,y\in R$, $x\neq y$;

i) $x=0$ if and only if $v(x)=\infty$.
ii) $v(x.y)=v(x)+v(y)$.
iii) $v(x+y)\geq min\{v(x),v(y)\}$.

If the conditions are satisfied, v are called additive valuation of R.

3.2. Definition Let R be a finite ring and $(G,.)$ a total order group.

$$w:R\rightarrow G\cup\{0\}$$

$$x\mapsto w(x)$$

with map, for each $x,y\in R$, $x\neq y$;

i) $x=0$ if and only if $w(x)=0$ olmasıdır.
ii) $w(x.y)=w(x).w(y)$.
iii) $w(x+y)\leq max\{w(x),w(y)\}$.

If the conditions are satisfied, w are called multiplicative valuation of R.

The group used in the definition of valuations is called the value group.

3.3. Definition Let v_0 and w_0 are respectively additive and multiplicative valuations on R ring; if for each $x\in R$, $x\neq 0$; $v_0(x)=0$, $w_0(x)=1$ is satisfied, then v_0 and w_0 valuations are called apparent valuation on the ring R.

3.4. Example For the ring R_1= $F_2+uF_2=\{0,1,u,1+u\}$ where $u^2=0$,

$$v_1\ :R_1\rightarrow F_2\cup\{\infty\}$$

$$0\mapsto\ v_1(0)=\infty$$

$$1\mapsto\ v_1(1)=0$$

$1+u \mapsto v_1\,(1+u)=0$

$u \mapsto v_1\,(u)=1$

For the ring R_2= $\mathrm{F}_2+v\mathrm{F}_2$ where $v^2=1$,

$v_2: R_2 \to \mathrm{F}_2 \cup \{\infty\}$

$0 \mapsto v_2(0)=\infty$

$1 \mapsto v_2(1)=0$

$1+v \mapsto v_2(1+v)=1$

$v \mapsto v_2(v)=0$

maps defined in the form of v_1 and v_2 are an additive valuation.

3.5. Proposition Let v_a and v_b are two valuation on the ring R.

Her $x \in R$ için $v_a(x)>0 \Rightarrow v_b(x)>0$ dir.

3.6. Proposition Let v is a valuation on the ring R.

For each $x \in R$, $v(x)=v(-x)$ is satisfied.

Proof: When we consider the R_1= $\mathrm{F}_2+u\mathrm{F}_2=\{0,1,u,1+u\}$ and R_2= $\mathrm{F}_2+v\mathrm{F}_2=\{0,1,u,1+u\}$ rings, we have $v(x)=v(-x)$ where the coefficients for all elements are in the field F_2 .

4.Valuation Rings and Ideals

4.1. Definition Let v and w are respectively additive and multiplicative valuations on the ring R.

$O_v=\{x \in R|v(x)\geq 0\}$ set is called valuation ring of the v additional valuation and

$O_w=\{x \in R|w(x)\leq 1\}$ set is called valuation ring of the w multiplicative valuation.

$M_v=\{x \in O_v|v(x)>0\}$ set is called maximal ideal corresponding to v additive valuation and

$M_w=\{x \in O_w|w(x)>0\}$ set is called maximal ideal corresponding to w multiplicative valuation. $U_v=\{x \in R|v(x)=0\}$ set is called unit group of the v additional valuation and

$U_w=\{x \in R|w(x)=1\}$ set is called unit group of the w multiplicative valuation.

4.2. Proposition Let v and w are additive and multiplicative valuations on the ring R,

i) The valuation rings O_v and O_w are sub ring of the ring.

ii) M_v is maximal ideal of valuation ring O_v and M_w is maximal ideal of valuation ring O_w.

4.3. Example According to the v_1 and v_2 valuations given in the 3.4. Example, the valuation ring, the maximal ideal corresponding to the valuation ring, and the unit group is in the form

$O_{v_1}=\{x \in R_1|v_1(x)\geq 0\}=\{x \in R_1|0,1,u,1+u\}=R_1$,

$O_{v_2}=\{x \in R_2|v_2(x)\geq 0\}=\{x \in R_2|0,1,v,1+v\}=R_2$,

$M_{v_1}=\{x \in O_{v_1}|v_1(x)>0\}$=$\{x \in R_1|0,u\}$=$\{0,u\}$,

$M_{v_2}=\{x \in O_{v_2}|v_2(x)>0\}$=$\{x \in R_2|0,1+v\}$=$\{0,1+v\}$,

$U_{v_1}=\{x \in R_1|v_1(x)=0\}=\{x \in R_1|1,1+u\}$=$\{1,1+u\}$ and

$U_{v_2}=\{x \in R_2|v_2(x)=0\}=\{x \in R_2|1,v\}$=$\{1,v\}$. Thus $O_{v_1}=M_{v_1}\cup U_{v_1}$ and

$O_{v_2} = M_{v_2} \cup U_{v_2}$ are found.

4.4. Example For v_0 additive apparent valuations on rings both R_1 and R_2,

the valuation ring is $O_{v_0} = R_1$ and $O_{v_0} = R_2$ respectively.

The maximal ideal corresponding to the valuation ring is $M_{v_0} = \{0\}$ (for both rings) and

Unit group is formed $U_{v_0} = R_1^*$ and $U_{v_0} = R_2^*$.

For w_0 multiplicative apparent valuations on rings both R_1 and R_2,

the valuation ring is $O_{w_0} = R_1$ and $O_{w_0} = R_2$ respectively.

The maximal ideal corresponding to the valuation ring is $M_{w_0} = \{0\}$ (for both rings) and

Unit group is formed $U_{w_0} = R_1^*$ and $U_{w_0} = R_2^*$. Hence on both R_1 and R_2, $O_{v_0} = M_{v_0} \cup U_0$ and $O_{w_0} = M_{w_0} \cup U_{w_0}$ are found.

5. Results

It has been shown that valuations other than apparent valuation are written on the finite structure.

It has been determined that the maximal ideals corresponding to the valuation rings formed by the valuations defined in this way are the same as the maximal ideals of the finite chain rings.It is also explained that the unit elements of the rings are elements of the unit groups corresponding to the valuations.

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