Journal of Natural Sciences Research
ISSN 2224-3186 (Paper) ISSN 2225-0921 (Online)
Vol.3, No.7, 2013

Strong Convergence of an Algorithm about Strongly Quasi-Nonexpansive Mappings for the Split Common Fixed-Point Problem in Hilbert Space

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Abstract

Based on the recent work by Censor and Segal (2009 J. Convex Anal.16), and inspired by Moudafi (2010 Inverse Problem 26), in this paper, we study the modified algorithm of Yu and Sheng [29] for the strongly quasi - nonexpansive operators to solve the split common fixed-point problem (SCFP) in the framework of Hilbert space. Furthermore we proved the strong convergence for the (SCFPP) by imposing some conditions. Our results extend and improved/developed some recent result announced.

Keywords: Convex Feasibility, Split Feasibility, Split Common Fixed Point, Strongly Quasi-Nonexpansive Operator, Iterative Algorithm and Strong Convergence.

1. Introduction

Let H_1 *and* H_2 be a Hilbert spaces, $A: H_1 \to H_2$ be a bounded linear operator and A^* be an ad joint of A. Given integer's p, r ≥ 1 and also given sequence of nonempty, closed, convex subsets $\{C_i\}_i^p$ *and* $\{Q_j\}_j^r$ of H_1 *and* H_2 respectively, the convex feasibility problem (CFP) is formulated as finding a point $x^* \in H_1$ satisfying the property:

$$x^* \in \bigcap_i^p C_i \,. \tag{1.1}$$

Note that, CFP (1.1) has received a lot of attention due to its extensive applications in many applied displines, diverse as approximation theorem, image recovery, signal processing, control theory, biomedical engineering, communication and geophysics (see [1 - 3] and the reference therein).

The multiple set split feasibility problem (MSSFP) was recently introduced and studied by Censo, Elfving, Kopf and Bortfeld, see [4] and is formulated as finding a point $x^* \in H_1$ with the property:

$$x^* \in \bigcap_i^p C_i \text{ and } Ax^* \in \bigcap_j^r Q_j \tag{1.2}$$

If in a MSSFP (1.2) p = r = 1, we get what is called the split feasibility problem (SFP) see [5], which is formulated as finding a point $x^* \in H_1$ with the property:

$$x^* \in C \text{ and } Ax^* \in Q \tag{1.3}$$

where C and Q are nonempty, closed and convex subsets of H_1 *and* H_2 respectively.

Note that, SFP (1.3) and MSSFP (1.2) model image retrieval (see [5]) and intensity modulated radiation therapy (see [15, 16]) and have recently been studied by many Researchers [6, 7 and 17-25] and references therein.

The MSSFP (1.2) can be viewed as a special case of the CFP (1.1) since (1.2) can be rewriting as

$$x^* \in \bigcap_i^{p+r} C_i\,, \qquad C_{p+j} = \{x^* \in H: x^* \in A^{-1}(Q_j),\ 1 \le j \le r\}.$$

However, the methodologies for studying the MSSFP (1.2) are actually different from those for the CFP (1.1) in order to avoid usage of the inverse of A. In other word, the method for solving CFP (1.1) may not apply to solve

MSSFP (1.2) straight forwardly without involving the inverse of A. The CQ algorithm of Byne [6; 7] is such an example where only the operator of A is used without involving the inverse.
Since every closed convex subset of Hilbert space is the fixed point set of its associating projection, the CFP (1.1) becomes a special case of the common fixed-point problem (CFPP) of finding a point $x^* \in H_1$ with property:

$$x^* \in \bigcap_{i}^{p} Fix(T_i). \tag{1.4}$$

where each, $T_i: H_1 \to H_2$ are some (nonlinear) mapping. Similarly the MSSFP (1.2) becomes a special case of the split common fixed point problem (SCFPP) [8] of finding a point $x^* \in H_1$ with the property:

$$x^* \in \bigcap_{i}^{p} Fix(U_i) \; and \; Ax^* \in \bigcap_{j}^{r} Fix(T_j), \tag{1.5}$$

where each, $U_i: H_1 \to H_1$ $(i = 1, 2, 3 \dots p)$ and $T_j: H_2 \to H_2$ $(j = 1, 2, 3, \dots, r)$ are some nonlinear operators.
If p = r = 1, problem (1.5) is reduces to find a point $x^* \in H_1$ with property:

$$x^* \in Fix(U) \; and \; Ax^* \in Fix(T) \tag{1.6}$$

This is usually called the two-set SCFPP.
The concept of SCFPP in finite dimensional Hilbert space was first introduce by Censor and Segal (see [8]) who invented an algorithm of the two-set SCFPP which generate a sequence $\{x_n\}$ according to the following iterative procedure:

$$x_{n+1} = U(x_n + \gamma A^*(T - I)Ax_n), n \geq 0, \tag{1.7}$$

where the initial guess $x_0 \in H$ is choosing arbitrarily and $0 < \gamma \leq \frac{1}{\|A\|^2}$.
Inspired by the work of Censor and Segal [8], Moudafi [27] introduced the following algorithm for μ −demicontractive operator in Hilbert space:

$$\begin{cases} u_n = x_n + \gamma A^*(T - I)Ax_n \\ x_{n+1} = (1 - t_n)u_n + t_n U(u_n), \; n \geq 0 \end{cases} \tag{1.8}$$

where $\gamma \in \left(0, \frac{(1-\mu)}{\lambda}\right)$ with λ being the spectral radius of the operator A^*A and $t_n \in (0,1)$ and $x_0 \in H$ is choosing arbitrarily. Using fejer-monotone and the demiclosed properties of $(I - U)$ and $(I - T)$ at origin, in 2010, Moudafi (see [27]) proved convergence theorem based on the work of Censor and Segal [8]. And also in 2011, Moudafi, Sheng and Chen (see [28]) gave their result of pseudo-demicontractive operators for the split common fixed-point problems. In 2012, Yu and Sheng [29] that modified the algorithm proposed by Moudafi [27] and they extend the operator to the class of firmly pseudo-demicontractive operator. In this paper, we study the modified algorithm of Yu and Sheng [29] and we used the strongly quasi nonexpansive operator to obtain the strong convergence of SCFPP (1.5).

2. Preliminaries

Throughout this paper, we adopt the notation:

- I: the identity operator on Hilbert space H.
- $Fix\,(T)$: the set of fixed point of an operator $T: H \to H$
- Ω: The solution set of SCFPP (1.5).
- $\omega_\omega(x_n)$: The set of the cluster point of x_n in the weak topology i.e. $\left\{\exists\, x_{n_j} \text{ of } x_n \ni x_{n_j} \rightharpoonup x\right\}$
- $x_n \to x$: $\{x_n\}$ Converge in norm to x
- $x_n \rightharpoonup x$: $\{x_n\}$ Converge weakly to x

Definition 2.1 Assume that C is a closed convex nonempty subset of a real Hilbert space H. A sequence $\{x_n\}$ in H is said to be Fejer monotone with respect to C if and only if

$$\|x_{n+1} - z\| \leq \|x_n - z\|, \qquad for\ all\ n \geq 1\ and\ z \in C$$

Definition 2.2 let $T: H \to H$ be an operator. We say that $(I - T)$ is demi closed at zero, if for any sequence x_n in H, there holds the following implication:

$x_n \rightharpoonup x$ and $(I - T)x_n \to 0\ as\ n \to \infty$, then $(I - T)x = 0$.

Definition 2.3 A Banach space E has Kadec-Klee property, if for every sequence $x_n \in E$ such that $x_n \rightharpoonup x$

and $\|x_n\| \to \|x\|$, then $x_n \to x\ n \to \infty$.

Definition 2.4 An operator $T: H \to H$ is said to be

(a) nonexpansive if $\|Tx - Ty\| \le \|x - y\|, for\ all\ x, y \in H$
(b) quasi-nonexpansive if $Fix(T) \ne \emptyset$ and $\|Tx - z\| \le \|x - z\|$, for all $x \in H\ and\ z \in Fix(T)$
(c) strictly quasi-nonexpansive if $Fix(T) \ne \emptyset$ and $\|Tx - z\| < \|x - z\|$, $\forall\ x \in H/Fix(T)\ and\ z \in Fix(T)$
(d) α -strongly quasi-nonexpansive if there exist $\alpha > 0$ with the property: $\|Tx - z\|^2 \le \|x - z\|^2 - \alpha\|x - Tx\|^2$, for all $x \in H\ and\ z \in Fix(T)$.

This is an equivalent to

$\langle x - z, Tx - x\rangle \le \frac{-1-\alpha}{2}\|x - Tx\|^2$, for all $x \in H\ and\ z \in Fix(T)$.

Definition 2.5 an operator $T: H \to H$ is said to be:

- Demicontractive [27]; if there exist a constant $\beta < 1$ such that $\|Tx - z\|^2 \le \|x - z\|^2 + \beta\|x - Tx\|^2$, for all $x \in H\ and\ z \in Fix(T)$.
- Pseudo-demicontractive [28]; if there exist a constant $\alpha > 1$ such that $\|Tx - z\|^2 \le \alpha\|x - z\|^2 + \|x - Tx\|^2$, for all $x \in H\ and\ z \in Fix(T)$.
- Firmly pseudo-demicontractive; if there exist a constants $\alpha > 1$ and $\beta > 1$ such that $\|Tx - z\|^2 \le \alpha\|x - z\|^2 + \beta\|x - Tx\|^2$, for all $x \in H\ and\ z \in Fix(T)$.

Lemma 2.6 [9] Let $T: H \to H$ be an operator. Then the following statements are equivalent

(a) T is class - τ operator ;
(b) $\|x - Tx\|^2 \le \langle x - z, x - Tx\rangle$, for all $x \in H\ and\ z \in Fix(T)$.
(c) there hold the relation: $\|Tx - z\|^2 \le \|x - z\|^2 - \|x - Tx\|^2$, for all $x \in H\ and\ z \in Fix(T)$.

Consequently a class - τ operator is 1 - strongly quasi - nonexpansive.

Lemma 2.7 [3] If a sequence $\{x_n\}$ is fejer monotone with respect to a closed convex nonempty subset C, then the following hold.

(i) $x_n \rightharpoonup x$ if and if $\omega_\omega(x_n) \subset C$;
(ii) The sequence $\{P_\Omega x_n\}$ converges strongly to some point in C;
(iii) $if\ x_n \rightharpoonup x \in C$, then, $x = \lim_{n\to\infty} P_\Omega x_n$.

Lemma 2.8 [30] Let H be a Hilbert space and let $\{x_n\}$ be a sequence in H such that there exist a nonempty set $S \subset H$ satisfying the following:

(a) For every sequence $x \in H, \lim_{n\to\infty}\|x_n - x\|$ exist;
(b) Any weak-cluster point of the sequence $\{x_n\}$ belongs in S. Then there exist x in S such that $\{x_n\}$ weakly converges to x.

3. Main Results

In what follows, we will focus our attention on the following general two-operator split common fixed-point problem: find

$$x^* \in C\ and\ Ax^* \in Q, \tag{3.1}$$

where $A: H_1 \to H_2$ is bounded and linear operator, U: $H_1 \to H_1$ and T: $H_2 \to H_2$ are two strongly quasi-nonexpansive operators with nonempty fixed-point set $Fix\ (U) = C\ and\ Fix\ (T) = Q$,

$$\|Ux - z\|^2 \le \|x - z\|^2 - \alpha\|x - Ux\|^2, \text{ for all } x \in H\ and\ z \in Fix(U). \tag{3.2}$$

$$\|Tx - z\|^2 \le \|x - z\|^2 - \beta\|x - Tx\|^2, \text{ for all } x \in H\ and\ z \in Fix(T). \tag{3.3}$$

And denote the solution set of the two-operator SCFPP by

$$\Omega = \{x^* \in C\ and\ Ax^* \in Q\}. \tag{3.4}$$

Based on the algorithm of [29], we have the following algorithm to solve (3.1)

$$\begin{cases} u_k = x_k + \gamma A^*(T - I)Ax_k \\ x_{k+1} = (1 - t_k)u_k + t_k U(u_k), \ \ k \ge 0 \end{cases} \tag{3.5}$$

where $0 < \gamma < \frac{1-\beta}{\lambda}$ with λ being the spectral radius of the operator A^*A ,$\propto > t_k > 0\ and\ x_0 \in H$ is choosing arbitrarily.

Theorem 3.1.Let $A: H_1 \to H_2$ be a bounded linear operator, U : $H_1 \to H_1$ and T : $H_2 \to H_2$ be two strongly quasi-nonexpansive operator with $Fix\ (U)\ =\ C\ and\ Fix(T)\ =\ Q$. Assume that $(U - I)\ and\ (T - I)$ are both demiclosed at zero and let P_Ω be a metric projection from H onto Ω satisfying $\langle x_k - x^*, x_k - P_\Omega x_k \rangle \leq 0$. If Ω is nonempty, then the sequence $\{x_k\}$ generated by algorithm (3.5) converges strongly to a split common fixed-point $x^* \in \Omega$.

Proof. To show that $x_k \to x^*$, it suffices to show that $x_k \rightharpoonup x^*$ and $\|x_k\| \to \|x^*\|$ as $k \to \infty$.

As we are in Hilbert space, now, taking $x^* \in \Omega$ that is $x^* \in Fix(U)\ and\ Ax^* \in Fix(T)$ and by definition (2.4 (d)) we deduce that that

$$\begin{aligned}\|x_{k+1} - x^*\|^2 &= \|(1 - t_k)u_k + t_k U(u_k) - x^*\|^2 \\ &= \|u_k - x^*\|^2 + 2t_k\langle u_k - x^*, Uu_k - u_k\rangle + {t_k}^2\|Uu_k - u_k\|^2 \\ &\leq \|u_k - x^*\|^2 - t_k(1+\alpha)\|Uu_k - u_k\|^2 + {t_k}^2\|Uu_k - u_k\|^2 \\ &\leq \|u_k - x^*\|^2 - t_k(1 + \alpha - t_k)\|Uu_k - u_k\|^2\end{aligned}$$

$$\Longrightarrow \|x_{k+1} - x^*\|^2 \leq \|u_k - x^*\|^2 - t_k(1 + \alpha - t_k)\|Uu_k - u_k\|^2 \tag{3.6}$$

On the other hand, we have

$$\begin{aligned}\|u_k - x^*\|^2 &= \|x_k + \gamma A^*(T - I)Ax_k - x^*\|^2 \\ &= \|x_k - x^*\|^2 + 2\gamma\langle x_k - x^*, A^*(T - I)Ax_k\rangle + \gamma^2\|A^*(T - I)Ax_k\|^2 \\ &= \|x_k - x^*\|^2 + 2\gamma\langle A(x_k - x^*),\ (T - I)Ax_k\rangle + \gamma^2\langle A^*(T - I)Ax_k,\ A^*(T - I)Ax_k\rangle \\ &= \|x_k - x^*\|^2 + 2\gamma\langle Ax_k - Ax^*, (T - I)Ax_k\rangle + \gamma^2\langle (T - I)Ax_k,\ AA^*(T - I)Ax_k\rangle.\end{aligned} \tag{3.7}$$

From the definition of λ, it follows that

$$\gamma^2\langle (T - I)Ax_k,\ AA^*(T - I)Ax_k\rangle \leq \lambda\gamma^2\|(T - I)Ax_k\|^2. \tag{3.8}$$

Now, by setting $\theta := 2\gamma\langle Ax_k - Ax^*, (T - I)Ax_k\rangle$, and using the fact that (2.4(d)) and its equivalent form, we infer that

$$\begin{aligned}\theta := 2\gamma\langle Ax_k - Ax^*, (T - I)Ax_k\rangle &\leq 2\gamma\left(\frac{-1 - \beta}{2}\right)\|(T - I)Ax_k\|^2 \\ &= \gamma(-1 - \beta)\|(T - I)Ax_k\|^2\end{aligned} \tag{3.9}$$

Substituting (3.9), (3.8), into (3.7) we get the following inequality

$$\|u_k - x^*\|^2 \leq \|x_k - x^*\|^2 - \gamma(1 + \beta - \gamma\lambda)\|(T - I)Ax_k\|^2. \tag{3.10}$$

Also, substituting (3.10) into (3.6), we get the following:

$$\|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2 - \gamma(1 + \beta - \gamma\lambda)\|(T - I)Ax_k\|^2 - t_k(1 + \alpha - t_k)\|Uu_k - u_k\|^2 \tag{3.11}$$

Since γ > 0; $\beta > 0$; $\alpha > t_k > 0$; $\lambda > 0$ and $t_k > 0$, we obtain that $-\gamma(1 + \beta - \gamma\lambda) < 0\ and$ $-t_k(1 + \alpha - t_k) < 0$

then from equation (3.11), we deduce that $\{x_k\}$ is a fejer monotone and moreover $\{\|x_{k+1} - x^*\|\}_{k\in\mathbb{N}}$ is monotonically decreasing sequence, hence converges. Therefore we have

$$\lim_{k\to\infty}\|(T - I)Ax_k\| = 0. \tag{3.12}$$

From the fejer monotonicity of $\{x_k\}$, it follows that the sequence is bounded. Denoting by x^* a weak cluster point of $\{x_k\}$, let j = 0,1,2,... be the sequence of indices, such that

$$w - \lim_{v\to\infty} x_{k_v} = x^*. \tag{3.13}$$

Then, from (3.12) and the demiclosedness of $(T - I)$ at zero, we obtain

$$T(Ax^*) = Ax^*. \tag{3.14}$$

From which it follows $Ax^* \in Q$. from (3.5), by considering $u_k = x_k + \gamma A^*(T - I)Ax_k$, it follows that

$$w - \lim_{v\to\infty} u_{k_v} = x^*. \tag{3.15}$$

Again from (3.11) and the convergence of the sequence $\{\|x_{k+1} - x^*\|\}_{k\in\mathbb{N}}$, we also have

$$\lim_{k\to\infty}\|(U - I)u_k\| = 0. \tag{3.16}$$

Which, combined with the demicloseness of $(U - I)$ at zero and weak convergence of $\{u_{k_v}\}\ to\ x^*$, yields

$$Ux^* = x^* \tag{3.17}$$

Hence $x^* \in C$, and therefore $x^* \in \Omega$. Since there is no morethan one weak-cluster point, the weak convergence of the whole sequence x_k follows by applying Lemma (2.8) with $S = \Omega$. i.e.

$$x_k \rightharpoonup x^* \tag{3.18}$$

Next, we show that
$\|x_k\| \to \|x^*\|$ as k→ ∞
To show this,
it suffices to show that
$\|x_{k+1}\| \to \|x^*\|$ as k → ∞.
Now, from (3.11) we deduce that

$$\big|\|x_{k+1}\| - \|x^*\|\big|^2 \leq \|x_{k+1} - x^*\|^2 \leq \|x_k - x^*\|^2.$$

Therefore, we have

$$\big|\|x_{k+1}\| - \|x^*\|\big|^2 \leq \|x_k - x^*\|^2$$

$$\Rightarrow \big|\|x_{k+1}\| - \|x^*\|\big| \leq \|x_k - x^*\| = \|x_k - P_\Omega x_k + P_\Omega x_k - x^*\|$$

$$\leq \|x_k - P_\Omega x_k\| + \|P_\Omega x_k - x^*\| \qquad (3.19)$$

Claim $\|x_k - P_\Omega x_k\| \leq \|P_\Omega x_k - x^*\|$
Proof of claim

$$\|x_k - P_\Omega x_k\|^2 = \|x_k - x^* + x^* - P_\Omega x_k\|^2$$

$$= \|x_k - x^*\|^2 + 2\langle x_k - x^*, x^* - P_\Omega x_k\rangle + \|x^* - P_\Omega x_k\|^2$$

$$= \|x_k - x^*\|^2 + 2\langle x_k - x^*, x^* - x_k + x_k - P_\Omega x_k\rangle + \|x^* - P_\Omega x_k\|^2$$

$$= \|x_k - x^*\|^2 + 2\langle x_k - x^*, x^* - x_k\rangle + 2\langle x_k - x^*, x_k - P_\Omega x_k\rangle + \|x^* - P_\Omega x_k\|^2$$

$$= -\|x_k - x^*\|^2 + 2\langle x_k - x^*, x_k - P_\Omega x_k\rangle + \|x^* - P_\Omega x_k\|^2$$

$$\leq \|x^* - P_\Omega x_k\|^2$$

$$\Rightarrow \|x_k - P_\Omega x_k\|^2 \leq \|x^* - P_\Omega x_k\|^2$$

$$\Rightarrow \|x_k - P_\Omega x_k\| \leq \|x^* - P_\Omega x_k\| \qquad (3.20)$$

Now, put (3.20) in (3.19), it follows that

$$\big|\|x_{k+1}\| - \|x^*\|\big| \leq 2\|x^* - P_\Omega x_k\|$$

$$\Rightarrow 0 \leq \text{limsup}_{k\to\infty}\big|\|x_{k+1}\| - \|x^*\|\big| \ \text{limsup}_{k\to\infty} 2\|x^* - P_\Omega x_k\| = 0$$

$$\Rightarrow \text{limsup}_{k\to\infty}\big|\|x_{k+1}\| - \|x^*\|\big| = 0.$$

Hence

$$\|x_{k+1}\| \to \|x^*\| \qquad (3.21)$$

By (3.15) and (3.18), we have that
$x_k \to x^*$as n → ∞ ■

Conclusion

In this paper, we study the modified algorithm of Yu and Sheng [29] for the strongly quasi nonexpansive operators to solve the split common fixed-point (1.5) and use some beautiful lemmas to prove the strong convergence of the modified algorithm. Our result extends and improved some recent result announced.

Acknowledgment

I would like to thanks Bauchi state University, Gadau, Nigeria for the financial support given to me.

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