On The Component Analysis and Transformation of An Explicit Fourth – Stage Fourth – Order Runge – Kutta Methods

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Abstract

This work is designed to transform the fourth stage – fourth order explicit Runge-Kutta method with the aim of projecting a new method of implementing it through tree diagram analysis. Efforts will be made to represent the equations derived from the y derivatives and x,y derivatives separately on Butcher's rooted trees. This idea is derivable from general graphs and combinatorics.

Key words: Rooted tree diagram, Transformation, Vertex, explicit, y partial derivatives, x,y partial derivatives, Runge-Kutta Methods, Linear and non- linear equations, Taylor series, Graphs.
Keywords: key words, Runge - Kutta, Combinatorics, Derivatives, Transformation, Analysis

1. Introduction

The object of this paper is totransform an explicit fourth- stage fourth- order Runge – Kutta methodto rooted tree diagram by separating the y derivatives from the (x, y) derivatives to get two different four-stage fourth – order Runge-kutta formulas after using Taylor's series expansion to expand the order. It also involves comparing the equations, rooted trees and partial derivatives gotten from the two methods to see if they have similarities or differences. Scientific implementation of the two formulason initial-value problems in ordinary differential equations of the form:

$y^1 = f(x, y), y(x_0) = y_0,\ a \leq x \leq b,$ h given, with a view to finding out their consistency and accuracy, is also carried out.

Explicit Runge – kutta (ERK) formulas are among the oldest and best – understood schemes in the numerical analysis tool kit. However, according to Byrne and Hindmarsh (1990); "despite the evolutions of a vast and comprehensive body of knowledge, ERK algorithms continue to be sources of active research". The history of ERK methods began almost a century ago. Classic references are Heun(1900), kutta (1901) andRunge (1985). According to Lambert(1991), the Runge –Kutta methods represent an important family of implicit and explicit iterative methods for approximation of ordinary differential equations in numerical analysis.

Because of their elegance and simplicity, ERK methods are usually among the first to be taught in the ODE section of a numerical methods course. Thankfully, good quality introductory texts no longer dismiss "the Runge – kutta method" as a fixed step size implementation of the classic 4^{th} order ERK formula. However, significant advancements in the state – of – the – art which post – date the work of Fehlberg (1970), even in the fundamental area of deriving ERK formulas, tend to be ignored. According to Kahanar(1989), another side effect of the simple nature of the ERK formula is that a generation of non – experts have been tempted to write their own "quick and dirty" codes. It is widely acknowledged that "squeaky – clean" codes require a great deal of expertise and programming effort. A high – level discussion of some of the issues involved in ERK is found in the work ofShampine and Gladwell(1992).Recent work on Runge-Kutta analysis include Agbeboh (2006,2010), Agbeboh, Ukpebor and Esekhaigbe (2009), Esekhaigbe (2007) and Butcher(2003). More recent works are that of Van Der Houwen and Sommeijer (2013, 2014, 2015).

The work of Butcher (1963,1966, 1987, 2003, 2010) revealed much successes in the analysis of explicit Runge-Kutta methods and their transformation to rooted tree diagrams. This was because the continuation of the process of Taylor Series gives rise to very complicated formulas. It was therefore, advantages to use a graphical representation for a convenient analysis of the order of a Runge – kutta method; hence, the basic tree theory was introduced. A tree is a rooted graph which contains no circuits. The symbol T is used to represent the tree with only one vertex. All rooted trees can be represented using T and the operation [$t_1, t_2,\ldots,t_m$].

Hence, it is the differentials and equations derived that are represented on trees so as to enable us compare the order condition with their differentials for varyingparameters.

Connes & Kreimer (1999,2007) pointed out that the Butcher group is the group of characters that had risen independently in their own work on rooted trees analysis in Runge – kutta methods for solving initial-value problems in ordinary differential equations.

Recent works on rooted tree analysis include Butcher (2008,2009,2010), Brouder (2000, 2004) e.t.c.

Conclusively, despite the fact that good, reliable explicit Runge-Kutta formulas exist, there is still need for their transformation to rooted tree diagrams. Traditionally, Runge – kutta methods are all explicit, although recently, implicit Runge – kutta methods, which have improved weak stability characteristics have been considered. However, the transformation of implicit Runge-Kutta methods to rooted tree diagrams can also be explored.

2. METHODS OF DERIVATION

i. From the general Runge-Kutta method, get a Fourth Stage-Fourth order method
ii. Obtain the Taylor series expansion of $k_{i's}^{;}$about the point (x_n, y_n), i=2,3,4,
iii. Carry out substitution to ensure that all the $k_{i's}^{;}$ are in terms of k_1 only.
iv. Insert the $k_{i's}^{;}$ in terms of k_1 only into $b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4$
v. Separate all partial derivatives involving only y with their coefficient from all partial derivatives involving x, y and their coefficients.
vi. Compare the coefficients of all partial derivatives involving only y with Taylor series expansion involving only partial derivatives with respect to y of the form:

$$\emptyset_T(x,y,h) = f + \frac{h}{2!}ff_y + \frac{h^2}{3!}\left(ff_y^2 + f^2f_{yy}\right) + \frac{h^3}{4!}(4f^2f_yf_{yy} + ff_y^3 + f^3f_{yyy}) + \frac{h^4}{5!}(7f^3f_yf_{yyy} + 4f^3f_{yy}^2 + 11f^2f_y^2f_{yy} + ff_y^4 + f^4f_{yyyy})$$

vii. As a result, a set of linear/non linear equations will be generated. Represent these equations and their partial derivatives on Butcher's rooted tree diagram.
viii. Compare the coefficient of all partial derivatives involving x, y only with Taylor series expansion involving partial derivative of x, y only of the form:

$$\phi(x,y,h) = f + \frac{h}{2!}f_x + \frac{h^2}{3!}\left(f_{xx} + 2ff_{xy} + f_xf_y\right) + \frac{h^3}{4!}(f_{xxx} + 3ff_{xxy} + 3f^2f_{xyy} + 3f_xf_{xy} + 5ff_yf_{xy} + 3ff_xf_{yy} + f_{xx}f_y + f_xf_y^2)$$

ix. As a result also, a set of linear/non-linear equations will be generated. Represent those equations and their x, y partial derivatives on Butcher's rooted tree diagram.
x. Vary the parameters from the two different sets of equations generated above. Two new fourth-stage fourth-order explicit Runge-Kutta formulas will be birthed with their various Butcher's rooted tree diagram.

2.1 DERIVATION OF THE FOURTH – ORDER FOURTH STAGE ERK METHOD

According to Lambert (1991), the general R – Stage Runge – Kutta method is:

$$y_{n+1} = y_n + h\,\phi(x_n, y_n, h)$$

$$\phi(x_n, y_n, h) = \textstyle\sum_{r=1}^{R} b_r\, k_r$$

$$k_1 = f(x,y)$$

$$k_r = f\left(x + hc_r, y + h\sum_{s=1}^{r-1} a_{rs}k_s\right), r = 2,3,\dots R$$

The formula is defined by the number of stages s, the nodes$[c_r]^s{}_{r=1}$, the internal weights $[a_{rs}]^{r-1,s}{}_{s=1,r=2}$ and the external weights$[b_r]^s{}_{r=1}$.

From the above scheme, the fourth stage fourth – order method is:

$$y_{n+1} = y_n + h\,(b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4)$$

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$$k_1 = f\,(x_n, y_n)$$
$$k_2 = f\,(x_n + c_2, y_n + ha_{21}k_1)$$
$$k_3 = f\,(x_n + c_3h, y_n + h(a_{31}k_1 + a_{32}k_2))$$
$$k_4 = f\,(x_n + c_4h, y_n + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3))$$

Using Taylor's series expansion for $k_i's$, we have:

$$k_1 = f\,(x_n, y_n)$$
$$k_2 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_2h\,\frac{d}{dx} + ha_{21}k_1\frac{d}{dy})^r f(x_n, y_n)$$
$$k_3 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_3h\,\frac{d}{dx} + h(a_{31}k_1 + a_{32}k_2)\frac{d}{dy})^r f(x_n, y_n)$$
$$k_4 = \sum_{r=o}^{\infty}\frac{1}{r!}(c_4h\,\frac{d}{dx} + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\frac{d}{dy})^r f(x_n, y_n)$$

Hence, we have:

$$k_1 = f$$

$$k_2 = f + \left(c_2hf_x + ha_{21}k_1f_y\right) + \tfrac{1}{2!}\left(c_2hf_x + ha_{21}k_1f_y\right)^2 + \tfrac{1}{3!}\left(c_2hf_x + ha_{21}k_1f_y\right)^3 + \tfrac{1}{4!}\left(c_2hf_x + ha_{21}k_1f_y\right)^4 + 0(h^5)$$

$$k_3 = f + \left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right) + \tfrac{1}{2!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^2 + \tfrac{1}{3!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^3 + \tfrac{1}{4!}\left(c_3hf_x + h(a_{31}k_1 + a_{32}k_2)\,f_y\right)^4 + 0(h^5)$$

$$k_4 = f + \left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right) + \tfrac{1}{2!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^2 + \tfrac{1}{3!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^3 + \tfrac{1}{4!}\left(c_4hf_x + h(a_{41}k_1 + a_{42}k_2 + a_{43}k_3)\,f_y\right)^4 + 0(h^5)$$

Expanding fully and substituting the various k_i' s, i = 2, 3, 4 into their various positions in terms of k_1 only and collecting like terms, in terms of y derivatives and (x, y) derivatives separately, we have:

$$k_1 = f$$

$$k_2 = f + ha_{21}ff_y + \tfrac{h^2}{2!}a_{21}^2f^2f_{yy} + \tfrac{h^3}{3!}a_{21}^3f^3f_{yyy} + \tfrac{h^4}{4!}a_{21}^4f^4f_{yyyy} + hc_2f_x + \tfrac{h^2}{2!}c_2^2f_{xx} + h^2c_2a_{21}ff_{xy}\,\tfrac{h^3}{3!}c_2^3f_{xxx} + \tfrac{h^3}{2!}c_2^2a_{21}ff_{xxy} + \tfrac{h^3}{2!}c_2a_{21}^2f^2f_{xyy} + \tfrac{h^4}{4!}c_2^4f_{xxxx} + \tfrac{h^4}{3!}c_2^3a_{21}ff_{xxxy}\,\tfrac{h^4}{2!2!}c_2^2a_{21}^2f^2f_{xxyy} + \tfrac{h^4}{3!}c_2a_{21}^3f^3f_{xyyy} + 0\,(h^5)$$

$$k_3 = f + h\,(a_{31} + a_{32})ffy + h^2a_{21}a_{32}ff_y^2 + \tfrac{h^2}{2!}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{yy} + \tfrac{h^3}{3!}a_{21}a_{32}\big(a_{21} + 2(a_{31} + a_{32})\big)f^2f_yf_{yy} + \tfrac{h^3}{3!}(a_{31}^3 + 3a_{31}^2a_{32} + 3a_{31}a_{32}^2 + a_{32}^3)f^3f_{yyy} + \tfrac{h^4}{3!}(a_{32}a_{21}^3 + 3a_{31}^2a_{32}a_{21} + 3a_{32}^3a_{21} + 6a_{31}a_{32}^2a_{21})f^3f_yf_{yyy} + \tfrac{h^4}{2!}a_{21}^2a_{32}(a_{31} + a_{32})f^3f_{yy}^2 + \tfrac{h^4}{2!}a_{32}^2a_{21}^2f^2f_y^2f_{yy} + \tfrac{h^4}{4!}(a_{31}^4 + 4a_{31}^3a_{32} + 6a_{31}^2a_{32}^2 + 4a_{31}a_{32}^3 + a_{32}^4)f^4f_{yyyy} + hc_3f_x + \tfrac{h^2}{2!}c_3^2f_{xx} + h^2c_3(a_{31} + a_{32})ff_{xy} + h^2c_2a_{32}f_xf_y + \tfrac{h^3}{3!}c_3^3f_{xxx} + \tfrac{h^3}{2!}c_3^2(a_{31} + a_{32})ff_{xxy} + \tfrac{h^3}{2!}c_3(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{xyy} + h^3c_2a_{32}(a_{31} + a_{32})ff_xf_{yy} + h^3a_{21}a_{32}(c_2 + c_3)ff_yf_{xy} + \tfrac{h^3}{2!}c_2^2a_{32}f_yf_{xx} + h^3c_2c_3a_{32}f_xf_{xy} + \tfrac{h^4}{4!}c_3^4f_{xxxx} + \tfrac{h^4}{3!}c_2^3a_{32}f_{xxx}f_y + \tfrac{h^4}{2!}c_3^2c_2a_{32}f_xf_{xxy} + \tfrac{h^4}{2!}a_{21}a_{32}(c_2^2 + c_3^2)ff_yf_{xxy} + \tfrac{h^4}{3!}a_{21}a_{32}(2c_2a_{31} + 3c_2a_{21} + 6c_3a_{31})f^2f_yf_{xyy} + \tfrac{h^4}{2!}c_3a_{32}c_2^2f_{xx}f_{xy} + \tfrac{h^4}{2!}c_2^2a_{32}(a_{31} + a_{32})ff_{xx}f_{yy} + \tfrac{h^4}{2!}a_{21}a_{32}(2c_2a_{31} + 2c_2a_{32} + c_3a_{21})f^2f_{xy}f_{yy} + h^4c_3a_{32}c_2a_{21}ff_{xy}^2 + \tfrac{h^4}{2!}a_{32}^2c_2^2f_x^2f_{yy} + h^4a_{32}^2a_{21}c_2ff_xf_yf_{yy} + \tfrac{h^4}{2!}c_3c_2a_{32}(6a_{31} + 2a_{32})ff_xf_{xyy} + \tfrac{h^4}{2!}c_2a_{32}(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_xf_{yyy} + \tfrac{h^4}{3!}c_3^3(a_{31} + a_{32})ff_{xxxy} + \tfrac{h^4}{2!2!}c_3^2(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2)f^2f_{xxyy} + \tfrac{h^4}{3!}c_3(a_{31}^3 + 3a_{31}^2a_{32} + 3a_{31}a_{32}^2 + a_{32}^3)f^3f_{xyyy} + 0\,(h^5)$$

$$k_4 = f + h\,(a_{41} + a_{42} + a_{43})ff_y + h^2(a_{21}a_{42} + a_{31}a_{43} + a_{32}a_{43})ff_y^2 + \tfrac{h^2}{2!}(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} +$$

$2a_{42}a_{43} + a_{42}^2 + a_{43}^2)f^2 f_{yy} +$
$\frac{h^3}{2!}(a_{21}^2 a_{42} + a_{31}^2 a_{43} + 2a_{31}a_{32}a_{43} + a_{32}^2 a_{43} + 2a_{21}a_{41}a_{42} + 2a_{31}a_{41}a_{43} + 2a_{32}a_{41}a_{43} + 2a_{31}a_{42}a_{43} + 2a_{32}a_{42}a_{43} + 2a_{21}a_{42}a_{43} + 2a_{21}a_{42}^2 + 2a_{31}a_{43}^2 + 2a_{32}a_{43}^2)f^2 f_y f_{yy} + h^3 a_{21}a_{32}a_{43} f f_y^3 + \frac{h^3}{2!}(a_{41}^3 + 3a_{41}^2 a_{42} + 3a_{41}^2 a_{43} + 3a_{41}a_{42}^2 + 6a_{41}a_{42}a_{43} + 3a_{42}^2 a_{43} + 3a_{41}a_{43}^2 + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3)f^3 f_{yyy} + \frac{h^4}{3!}(a_{31}^3 a_{43} + 3a_{31}^2 a_{32}a_{43} + 3a_{31}a_{32}^2 a_{43} + 3a_{21}a_{41}^2 a_{42} + 3a_{31}a_{41}^2 a_{43} + 3a_{32}a_{41}^2 a_{43} + 6a_{21}a_{41}a_{42}^2 + 6a_{31}a_{41}a_{42}a_{43} + 6a_{32}a_{41}a_{42}a_{43} + 6a_{21}a_{41}a_{42}a_{43} + 3a_{31}a_{41}^2 a_{43} + a_{42}a_{21}^3 + 3a_{32}a_{42}^2 a_{43} + 6a_{21}a_{42}^2 a_{43} + 6a_{31}a_{41}a_{43}^2 + 6a_{32}a_{41}a_{43}^2 + 6a_{31}a_{42}a_{43}^2 + 6a_{32}a_{42}a_{43}^2 + 3a_{21}a_{42}a_{43}^2 + 3a_{21}a_{43}^3 + 3a_{31}a_{43}^3 + 3a_{32}a_{43}^3)f^3 f_y f_{yyy} + \frac{h^4}{2!}(a_{21}^2 a_{32}a_{43} + 2a_{21}a_{31}a_{32}a_{43} + 2a_{21}a_{32}^2 a_{43} + 2a_{21}a_{32}a_{41}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + 2a_{21}a_{31}a_{42}a_{43} + 2a_{21}a_{32}a_{42}a_{43} + a_{21}^2 a_{42}^2 + a_{21}a_{32}a_{43}^2 + a_{31}^2 a_{43}^2 + 2a_{31}a_{32}a_{43}^2 + a_{32}^2 a_{43}^2)f^2 f_y^2 f_{yy} +$
$\frac{h^4}{2!}\left(a_{21}^2 a_{41}a_{42} + a_{31}^2 a_{41}a_{42} + 2a_{31}a_{32}a_{41}a_{43} + a_{32}^2 a_{41}a_{43} + a_{31}^2 a_{41}a_{43} + 2a_{31}a_{32}a_{42}a_{43} + a_{32}^2 a_{42}a_{43} + \frac{a_{21}^2 a_{42}^2}{2!} + \frac{a_{31}^2 a_{43}^2}{2!} + a_{31}a_{32}a_{43}^2 + \frac{a_{31}^2 a_{43}^2}{2!}\right) f^3 f_{yy}^2 + \frac{h^4}{4!}(a_{41}^4 + 4_{41}^3 a_{42} + 4a_{41}^3 a_{43} + 6a_{41}^2 a_{41}^2 + 12a_{41}^2 a_{42}a_{43} + 6a_{42}^2 a_{43}^2 + 4a_{41}a_{43}^3 + 4a_{42}a_{43}^3 + 12a_{41}a_{42}^2 a_{43} + 2a_{41}a_{42}a_{43}^2 + 6a_{41}^2 a_{43}^2 + 4a_{42}^3 a_{43} + 4a_{41}a_{42}^3 + a_{42}^4 + a_{43}^4) f^4 f_{yyyy} + hc_4 f_x + h^2(c_4 a_{42} + c_3 a_{43})f_x f_y + \frac{h^2}{2!}c_4^2 f_{xx} + h^2 c_4(a_{41} + a_{42} + a_{43})f f_{xy} + \frac{h^3}{2!}(c_2^2 a_{42} + c_3^2 a_{43})f_{xx}f_y + h^2(c_4 a_{21}a_{42} + c_3 a_{31}a_{43} + c_3 a_{32}a_{43})f f_{xy} f_y + h^3 c_2 a_{32}a_{43} f_x f_y^2 + h^3(c_2 c_4 a_{42} + c_3 c_4 a_{43}) f_x f_{xy} + h^3 c_2 a_{32}a_{43} f_x f_y^2 + h^3(c_2 c_4 a_{42} + c_3 c_4 a_{43}) f_x f_{xy} + h^3(c_2 a_{21}a_{42} + c_4 a_{31}a_{43} + c_4 a_{32}a_{43})f f_y f_{xy} + h^3(c_2 a_{41}a_{42} + c_3 a_{41}a_{43} + c_3 a_{42}a_{43} + c_2 a_{42}a_{43} + c_2 a_{42}^2 + c_3 a_{43}^2)f f_x f_{yy} + \frac{h^3}{3!}c_4^3 f_{xxx} + \frac{h^3}{2!}(c_4^2 a_{41} + c_4^2 a_{42} + c_4^2 a_{43})f f_{xxy} + \frac{h^3}{2!}c_4(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + a_{42}^2 + 2a_{42}a_{43} + a_{43}^2)f^2 f_{xyy} + \frac{h^4}{3!}(c_2^3 a_{42} + c_3^3 a_{43})f_{xxx}f_y + \frac{4}{3!}(3c_2^2 a_{21}a_{42} + c_3^2 a_{31}a_{43} + 3c_3^2 a_{32}a_{43} + 3c_4^2 a_{21}a_{42} + 3c_4^2 a_{31}a_{43} + 3c_4^2 a_{32}a_{43})f f_{xxy} f_y + \frac{h^4}{2!}(c_2 a_{21}^2 a_{42} + 2c_3 a_{31}a_{32}a_{43} + c_3 a_{31}^2 a_{43} + c_3 a_{32}^2 a_{43} + 2c_4 a_{21}a_{41}a_{42} + 2c_4 a_{31}a_{41}a_{43} + 2c_4 a_{32}a_{41}a_{43} + 2c_4 a_{21}a_{42}^2 + 2c_4 a_{31}a_{42}a_{43} + 2c_4 a_{32}a_{42}a_{43} + 2c_4 a_{21}a_{42}a_{43} + 2c_4 a_{31}a_{43}^2 + 2c_4 a_{32}a_{43}^2)f^2 f_y f_{xyy} + \frac{h^4}{3!}(c_2^2 a_{32}a_{43})f_{xx}f_y^2 + h^4(c_2 a_{21}a_{32}a_{43} + c_3 a_{21}a_{32}a_{43} + c_4 a_{21}a_{32}a_{43})f f_{xy} f_y^2 + \frac{h^4}{2!}(2c_2 a_{31}a_{32}a_{43} + 2c_2 a_{32}^2 a_{43} + 2c_2 a_{32}a_{41}a_{43} + 2c_2 a_{32}a_{42}a_{43} + 2c_2 a_{31}a_{42}a_{43} + 2c_2 a_{32}a_{42}a_{43} + 2c_3 a_{21}a_{42}a_{43} + 2c_2 a_{21}a_{42}^2 + c_2 a_{32}a_{43}^2 + c_3 a_{31}a_{43}^2 + c_3 a_{32}a_{43}^2 + c_3 a_{31}a_{43}^2 + c_3 a_{32}a_{43}^2)f f_x f_y f_{yy} + h^4(c_2 c_3 a_{32}a_{43} + c_2 c_4 a_{32}a_{43})f_x f_y f_{xy} + \frac{h^4}{2!}(c_2^2 c_4 a_{42} + c_3^2 c_4 a_{43})f_{xx}f_{xy} + h^4(c_2 c_4 a_{21}a_{42} + c_3 c_4 a_{31}a_{43} + c_3 c_4 a_{32}a_{43})$
$f f_{xy}^2 + \frac{h^4}{2!}(c_4 a_{21}^2 a_{42} + c_4 a_{31}^2 a_{43} + 2c_4 a_{31}a_{32}a_{43} + c_4 a_{32}^2 a_{43} + 2c_2 a_{21}a_{41}a_{42} + 2c_3 a_{31}a_{41}a_{43} + 2c_3 a_{32}a_{41}a_{43} + 2c_3 a_{31}a_{42}a_{43} + 2c_3 a_{32}a_{42}a_{43} + c_2 a_{21}a_{42}^2 + c_3 a_{31}a_{43}^2 + c_3 a_{32}a_{43}^2) f^2 f_{xy} f_{yy} + \frac{h^4}{2!}(2c_2 c_3 a_{42}a_{43} + c_2^2 a_{42}^2 + c_3^2 a_{43}^2)f_x^2 f_{yy} + \frac{h^4}{2!}(c_2 c_4^2 a_{42} + c_3 c_4^2 a_{43})f_x f_{xxy} + h^4(c_2 c_4 a_{41}a_{42} + c_3 c_4 a_{41}a_{43} + c_2 c_4 a_{42}^2 + c_3 c_4 a_{42}a_{43}) + c_2 c_4 a_{42}a_{43} + c_3 c_4 a_{43}^2)f f_x f_{xyy} + \frac{h^4}{2!}(c_2 c_{41}^2 a_{42} + c_3 c_{41}^2 a_{43} + 2c_2 a_{41}a_{42}^2 + 2c_3 a_{41}a_{42}a_{43} + 2c_2 a_{41}a_{42}a_{43} + c_3 c_{42}^2 a_{43} + 2c_2 c_{42}^2 a_{43} + 2c_3 a_{41}a_{43}^2 + 2c_3 a_{42}a_{43}^2 + c_2 a_{42}a_{43}^2 + c_2 c_{42}^3 + c_3 a_{43}^3)f^2 f_x f_{yyy} + \frac{h^4}{4!}c_4^4 f_{xxxx} + \frac{h^4}{3!}(c_4^3 a_{41} + c_4^3 a_{42} + c_4^3 a_{43})f f_{xxxy} + \frac{h^4}{2!2!}c_4^2(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + a_{42}^2 + 2a_{42}a_{43} + a_{43}^2)f^2 f_{xxyy} + \frac{h^4}{3!}c_4(a_{41}^3 + 3a_{41}^2 a_{42} + 3a_{41}^2 a_{43} + 3a_{41}a_{42}^2 + 6a_{41}a_{42}a_{43} + 3a_{42}^2 a_{43} + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3) f^3 f_{xyyy} + \frac{h^4}{2!2!}(2c_2^2 a_{41}a_{42} + 2c_3^2 a_{41}a_{43} + 2c_3^2 a_{42}a_{43} + c_2^2 a_{42}^2 + c_3^2 a_{43}^2)f f_{xx} f_{yy.} + 0(h^5).$

Putting the $k_{i's}^{;}$ *(y derivatives only) into* $y_{n+1} = y_n + h(b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4$ *where* $\phi(x, y, h) = b_1 k_1 + b_2 k_2 + b_3 k_3 + b_4 k_4$ *and equating coefficients with the Taylor series expansion:*

$\emptyset_T(x, y, h) = f + \frac{h}{2!}f f_y + \frac{h^2}{3!}(f f_y^2 + f^2 f_{yy}) + \frac{h^3}{4!}(4f_y^2 f_y f_{yy} + f f_y^3 + f^3 f_{yyy}) + \frac{h^4}{5!}(4f^3 f_y f_{yyy} + 4f^3 f_{yy}^2 + 11f^2 f^2 f_{yy} + f f_y^4 + f^4 f_{yyyy})$, we have the following equations:

$b_1 + b_2 + b_3 + b_4 = 1$ (1)

$b_2 a_{21} + b_3(a_{31} + a_{32}) + b_4(a_{41} + a_{42} + a_{43}) = ½$ (2)

$b_3a_{21} + b_4(a_{21}a_{41} + a_{43}(a_{31} + a_{32}) = 1/6$ (3)

$b_2a_{21}^2 + b_3(a_{31}^2 + 2a_{31}a_{32} + a_{32}^2) + b_4(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + 2a_{42}a_{43} + a_{42}^2 + a_{43}^2) = \frac{1}{3}$ (4)

$b_2a_{21}^3 + b_3(a_{31}^3 + 3a_{31}^2a_{32} + 3a_{31}a_{32}^2 + a_{32}^3) + b_4(a_{41}^3 + 3a_{41}^2a_{42} + 3a_{42}^2a_{43} + 3a_{41}a_{42}^2 + 6a_{41}a_{42}a_{43} + 3a_{42}^2a_{43} + 3a_{41}a_{43}^2 + 3a_{42}a_{43}^2 + a_{42}^3 + a_{43}^3) = 1/4.$ (5)

$b_3a_{21}a_{32}\big(a_{21} + 2(a_{31} + a_{32})\big) + b_4(a_{41}^2a_{42} + a_{43}(a_{31} + a_{32})^2 + 2a_{21}a_{42}(a_{41} + a_{42} + a_{43}) + 2a_{31}4_3(a_{41} + a_{42} + a_{43}) + 2a_{32}a_{43}(a_{41} + a_{42} + a_{43})) = 1/3$ (6)

$b_4a_{21}a_{32}a_{43} = 1/24$ (7)

From the above equations: let A = a_{21}, B = $a_{31} + a_{32}$, P = $a_{41} + a_{42} + a_{43}$
From equations (2), (3), (4), (5), (6) and (7) we have

$Ab_2 + Bb_3 + Pb_4 = 1/2$ (8)

$Aa_{32}b_3 + Aa_{42}b_4 + Ba_{43}b_4 = 1/6$ (9)

$A^2b_2 + B^2b_3 + P^2b_4 = 1/3$ (10)

$A^3b_2 + B^3b_3 + P^3b_4 = 1/4$ (11)

$A^2a_{32} + 2BAa_{32}b_3 + A^2a_{42}b_4 + B^2a_{43}b_4 + 2APa_{42}b_4 + 2Pa_{31}a_{43}b_4 + 2Pa_{32}a_{43}b_4 = 1/3$ (12)

$Aa_{32}a_{43}b_4 = 1/24$ (13)

Now from (1), setting $b_1 = b_4 = 1/6$, $b_2 = b_3 = 2/6$
(8) becomes: $2A + 2B + P = 3$ (14)
(10) becomes: $2A^2 + 2B^2 + P^2 = 2$ (15)
(11) becomes: $2A^3 + 2B^3 + P^3 = 3/2$ (16)

From (14), (15), (16)
A = ½, B = ½, P = 1,
Hence (9) becomes: $2a_{32} + a_{42} + a_{43} = 2$ (17)
(13) becomes: $a_{32}a_{43} = 1/2$ (18)
(12) becomes: $6a_{32} + 5a_{42} + a_{43} + 8a_{43}a_{43} + 8a_{42}a_{43} = 8$ (19)

From (18), let $a_{43} = \frac{1}{2}$, then $a_{32} = 1$, From (17), $a_{42} = -\frac{1}{2}$, But A = ½, B = ½, P = 1
Therefore, $a_{21} = \frac{1}{2}, a_{31} + a_{32} = \frac{1}{2}, \rightarrow a_{31} = -\frac{1}{2}, a_{41} + a_{42} + a_{43} = 1, a_{41} = 1$
In conclusion, $b_1 = 1/6$, $b_2 = 2/6$, $b_3 = 2/6$, $b_4 = 1/6$

$$a_{21} = \frac{1}{2}, \quad a_{31} = -\frac{1}{2}, \quad a_{32} = 1, \quad a_{41} = 1, \quad a_{42} = -\frac{1}{2}, \quad a_{43} = \frac{1}{2}.$$

THE FOURTH ORDER BECOMES:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$k_1 = f(y_n)$$
$$k_2 = f(y_n + \frac{h}{2}k_1)$$
$$k_3 = f(y_n + \frac{h}{2}(-k_1 + 2k_2))$$
$$k_4 = f(y_n + \frac{h}{2}(2k_1 - k_2 + k_3))$$

Below is Table 1 Showing the equations, partial derivatives of $f(x_n,y_n)$ with respect to y only been transformed into rooted trees:
Note $c_2 = a_{21}$, $c_3 = a_{31} + a_{32}$, $c_4 = a_{41} + a_{42} + a_{43}$

The Butchers Tableau for the above fourth Order is:

0				
0	½			
0	- ½	1		
0	1	-½	½	
	1/6	1/3	1/3	1/6

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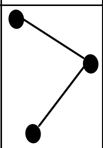
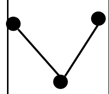
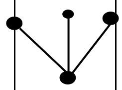
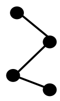
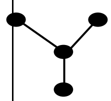
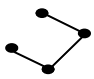
TABLE 1

EQUATIONS	ERIVATIVES	R(T)	TREE	T	$\emptyset(t) = \frac{1}{r(t)}$
$b_1 + b_2 + b_3 + b_4 = 1$	f	1		t	$\sum_{i=1}^{4} b_i = 1$
$b_2c_2 + b_3c_3 + b_4c_4 = 1/2$	ff_y	2		[t]	$\sum_{i=2}^{4} b_i\, c_i = 1/2$
$b_3c_2a_{32} + b_4c_2a_{42} + b_4c_3a_{43} = 1/6$	ff_y^2	3		$[_2t_2]$	$\sum_{i=3,j=2}^{4,3} b_i\, a_{ij}c_j = 1/6$
$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = 1/3$	f^2f_{yy}	3		$[t^2]$	$\sum_{i=2}^{4} b_i\, c_i^2 = 1/3$
$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 = 1/4$	f^3f_{yyy}	4		$[t^3]$	$\sum_{i=2}^{4} b_i\, c_i^3 = 1/4$
$b_4a_{43}a_{32}c_2 = 1/24$	ff_y^3	4		$[_3t_3]$	$\sum_{i=4,j=3,k=2}^{4,3,2} b_i\, a_{ij}a_{jk}c_k = 1/24$
$b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{43}c_3^2 = 1/12$	$f^2f_yf_{yy}$	4		$[_2t^2]_2$	$\sum_{i=3,j=2}^{4,3} b_i\, a_{ij}c_j^2 = 1/12$
$b_3a_{32}c_2c_3 + b_4a_{42}c_2c_4 + b_4a_{43}c_3c_4 = 1/8$	$f^2f_yf_{yy}$	4		[t[t]]	$\sum_{i=3,j=2}^{4,3} b_i\, c_ia_{ij}c_j = 1/8$

Also Putting the $k_{i's}^{'}$ (x, y derivatives only) into $y_{n+1} = y_n + \mathrm{h}(\mathrm{b}_1\mathrm{k}_1 + \mathrm{b}_2\mathrm{k}_2 + \mathrm{b}_3\mathrm{k}_3 + \mathrm{b}_4\mathrm{k}_4$ where $\phi(x,y,h) = b_1k_1 + b_2k_2 + b_3k_3 + b_4k_4$ and equating coefficients with the Taylor series expansion:

$$\Phi_T(x,y,h) = f + \frac{h}{2!}f_x + \frac{h^2}{3!}\left(f_{xx} + 2ff_{xy} + f_xf_y\right) + \frac{h^3}{4!}(f_{xxx} + 3ff_{xxy} + 3f^2f_{xyy} + 3f_xf_{xy} + 5ff_yf_{xy} + 3ff_xf_{yy} + f_{xx}f_y + f_xf_y^2)$$

The Equations become:

$b_1 + b_2 + b_3 + b_4 = 1$ (1)

$b_2c_2 + b_3c_3 + b_4c_4 = ½$ (2)

$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = 1/3$ (3)

$b_2c_2a_{21} + b_3c_3(a_{31} + a_{32}) + b_4c_4(a_{41} + a_{42} + a_{43}) = \frac{1}{3}$ (4)

$b_3c_2a_{32} + b_4(c_2a_{42} + c_3a_{43}) = \frac{1}{6}$ (5)

$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 = \frac{1}{4}$ (6)

$b_2c_2^3a_{21} + b_3c_3^2(a_{31} + a_{32}) + b_4c_4^2(a_{41} + a_{42} + a_{43}) = \frac{1}{4}$ (7)

$b_2c_2a_{21}^2 + b_3c_3\,(a_{31}^2 + 2a_{21}a_{32} + a_{32}^2) + b_4c_4(a_{41}^2 + 2a_{41}a_{42} + 2a_{41}a_{43} + a_{42}^2 + 2a_{41}a_{43} + a_{43}^2) = \frac{1}{4}$

(8)

$b_3c_2a_{32}(a_{31} + a_{32}) + b_4(c_2a_{42}(a_{41} + a_{42} + a_{43}) + c_3a_{43}(a_{41} + a_{42} + a_{43})) = 1/8$

(9)

$b_3c_{21}a_{32}(c_2 + c_3) + b_4(c_2a_{21}a_{42} + c_3a_{43}(a_{31} + a_{43}) + c_4a_{21}a_{42} + c_4a_{43}(a_{31} + a_{32}) = \frac{5}{24}$

(10)

$b_3c_2^2a_{32} + b_4(c_2^2a_{42} + c_3^2a_{43}) = \frac{1}{12}$ (11)

$b_3c_2c_3a_{32} + b_4(c_2c_4a_{42} + c_3c_4a_{43}) = \frac{1}{8}$ (12)

$b_4c_2a_{32}a_{43} = \frac{1}{24}$ (13)

Set $c_1 = 0$, $c_4 = 1$, $c_2 = 1/4$, $c_3 = ¾$

(2) becomes $b_2 + 3b_3 + 4b_4 = 2$ (14)

(3) becomes $3b_2 + 27b_3 + 48b_4 = 16$ (15)

(6) becomes $b_2 + 27b_3 + 48b_4 = 16$ (16)

Solving (1), (14), (15) and (16), we have:

$b_1 = 1/18$, $b_2 = 4/9$, $b_3 = 4/9$, $b_4 = 1/18$

From (4), we have: $2a_{21} + 6(a_{31} + a_{32}) + (a_{41} + a_{42} + a_{43}) = 6$ (17)

From (5), we have: $8a_{32} + a_{42} + 3a_{43} = 12$ (18)

From (7), we have: $a_{21} + 9(a_{31} + a_{32}) + 2(a_{41} + a_{42} + a_{43}) = 9$ (19)

From (8), we have: $4a_{21}^2 + 12(a_{31} + a_{32})^2 + 2(a_{41} + a_{42} + a_{43})^2 = 9$ (20)

From (9), we have: $8a_{21}(a_{31} + a_{32}) + a_{42}(a_{41} + a_{42} + a_{43}) + 3a_{42}(a_{41} + a_{42} + a_{43}) = 9$

(21)

From (10), we have: $32a_{31}a_{32} + 5a_{21}a_{42} + 7a_{43}(a_{31} + a_{32}) = 15$ (22)

From (11), we have: $8a_{32} + a_{42} + 9a_{43} = 24$ (23)

From (12), we have: $6a_{32} + a_{42} + 3a_{43} = 9$ (24)

From (13), we have: $a_{32}a_{43} = 3$ (25)

Solving (18), (23) and (24), we have:

$a_{32} = \frac{3}{2}$, $a_{42} = -6$, $a_{43} = 2$, Let $A = a_{21}$, $B = a_{31} + a_{32}$, $D = a_{41} + a_{42} + a_{43}$

Hence, (17) becomes: 2A + 6B + D = 6 (26)

(19) becomes: A + 9B + 2D = 9 (27)

(20) becomes: $4A^2 + 12B^2 + 2D^2 = 9$ (28)

(21) becomes: 12B = 9 (29)

(22) becomes: 18A + 14B = 15 (30)

From (29) B = ¾
Hence, (26) becomes: 2A + D = 3/2 (31)
(27) becomes: A + 2D = 9/4 (32)
(28) becomes: $4A^2 + 2D^2 = 9/4$ (33)
From (30), A = ¼
Putting A into (31), (32) and (33), we get:
D = 1, A = $a_{21} = \frac{1}{4}$, B = $a_{31} + a_{32} = \frac{3}{4}, a_{31} = -\frac{3}{4}, \quad a_{32} = \frac{3}{2}$,
Also, D = $a_{41} + a_{42} + a_{43} = 1$, but $a_{42} = -6, \; a_{43} = 2, a_{41} = 5$,
The parameters put together are:
$C_1 = 0$, $c_2 = \frac{1}{4}$ $c_3 = \frac{3}{4}$, $c_4 = 1$, $b_1 = \frac{1}{18}$, $b_2 = \frac{4}{9}$, $b_3 = \frac{4}{9}$, $b_4 = \frac{4}{9}$
$a_{21} = \frac{1}{4}, \; a_{31} = -\frac{3}{4}, \; a_{32} = \frac{3}{2}, \; a_{41} = 5, \quad a_{42} = -6, \; a_{43} = 2,$
The fourth order becomes:

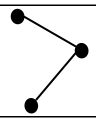
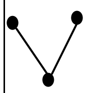
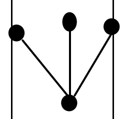
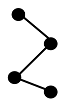
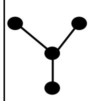
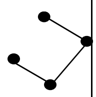
$$y_{n+1} = y_n + \frac{h}{18}(k_1 + 8k_2 + 8k_3 + k_4)$$
$$k_1 = f(x_n, y_n)$$
$$k_2 = f(x_n + \frac{h}{4}, y_n + \frac{h}{4}k_1)$$
$$k_3 = f(x_n + \frac{3h}{4}, y_n + \frac{h}{4}(-3k_1 + 6k_2))$$
$$k_4 = f(x_n + h, y_n + h(5k_1 - 6k_2 + 2k_3))$$

Below is Table 2 Showing the above thirteen (13) equations, partial derivatives of $f(x_n, y_n)$ with respect to x,y only and their rooted trees:

Butcher's tableau for (x, y) derivatives parameters

0				
1/4	1/4			
3/4	- 3/4	3/2		
1	5	- 6	2	
	1/18	4/9	4/9	1/18

TABLE

EQUATIONS	**DERIVATIVES**	**R(T)**	**TREE**	**T**	$\emptyset(t) = \frac{1}{r(t)}$
$b_1 + b_2 + b_3 + b_4 = 1$	f	1		t	$\sum_{i=1}^{4} b_i = 1$
$b_2c_2 + b_3c_3 + b_4c_4 = {}^1/_2$	f_x	2		[t]	$\sum_{i=2}^{4} b_i\, c_i = {}^1/_2$
$b_3a_{32}c_2 + b_4a_{42}c_2 + b_4a_{43}c_3 = {}^1/_6$	f_xf_y	3		$[_2t]_2$	$\sum_{i=3,j=2}^{4,3} b_i\, a_{ij}c_j = {}^1/_6$
$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = {}^1/_3$	f_{xx}	3		$[t^2]$	$\sum_{i=2}^{4} b_i\, c_i^2 = {}^1/_3$
$b_2c_2^2 + b_3c_3^2 + b_4c_4^2 = {}^1/_3$	ff_{xy}				
$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 = {}^1/_4$	f_{xxx}	4		$[t^3]$	$\sum_{i=2}^{4} b_i\, c_i^3 = {}^1/_4$
$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 = {}^1/_4$	ff_{xxy}				
$b_2c_2^3 + b_3c_3^3 + b_4c_4^3 = {}^1/_4$	f^2f_{xyy}				
$b_4a_{43}a_{32}c_2 = {}^1/_{24}$	$f_xf_y^2$	4		$[3t]_3$	$\sum_{i=4,j=3,k=2}^{4,3,2} b_i\, a_{ij}a_{jk}c_k = {}^1/_{24}$
$b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{43}c_3^2 = {}^1/_{12}$	$f_{xx}f_y$	4		$[_2t^2]_2$	$\sum_{i=3,j=2}^{4,3} b_i\, a_{ij}c_j^2 = {}^1/_{12}$
$b_3a_{32}c_2^2 + b_4a_{42}c_2^2 + b_4a_{43}c_3^2 = {}^1/_{12}$	ff_yf_{xy}				
$b_3a_{32}c_2c_3 + b_4a_{42}c_2c_4 + b_4a_{43}c_3c_4 = {}^1/_8$	ff_xf_{yy}	4		{t[t]}	$\sum_{i=3,j=2}^{4,3} b_i\, c_ia_{ij}c_j = {}^1/_8$
$b_3a_{32}c_2c_3 + b_4a_{42}c_2c_4 + b_4a_{43}c_3c_4 = {}^1/_8$	f_xf_{xy}				
$b_3a_{32}c_2c_3 + b_4a_{42}c_2c_4 + b_4a_{43}c_3c_4 = {}^1/_8$	ff_yf_{xy}				

3. IMPLEMENTATION OF THE FORMULAS

The two formulas are implemented on the initial – value problems below with the aid of FORTRAN programming language:

(i) $y^1 = -y, \quad y(0) = 1, 0 \le x \le 1, \quad y(x_n) = \frac{1}{e^{xn}}$

(ii) $y^1 = y, \quad y(0) = 1, 0 \le x \le 1, \quad y(x_n) = e^{xn}$

(iii) $y^1 = 1 + y^2, \quad y(0) = 1, 0 \le x \le 1, \quad y(x_n) = \tan\left(x_n + {}^{\pi}/_{4}\right), h = 0.1$

4. RESULTS

Below Are Tables of Results for the Above Initial - Value Problems

TABLE 3

PROBLEM 1(y Derivatives)

XN	YN	TSOL	ERROR
.1D+00	0.9048375000000D+00	0.9048374180360D+00	-.8196404044369D-07
.2D+00	0.8187309014063D+00	0.8187307530780D+00	-.1483282683346D-06
.3D+00	0.7408184220012D+00	0.7408182206817D+00	-.2013194597694D-06
.4D+00	0.6703202889175D+00	0.6703200460356D+00	-.2428818514089D-06
.5D+00	0.6065309344234D+00	0.6065306597126D+00	-.2747107467060D-06
.6D+00	0.5488119343763D+00	0.5488116360940D+00	-.2982822888686D-06
.7D+00	0.4965856186712D+00	0.4965853037914D+00	-.3148798197183D-06
.8D+00	0.4493292897344D+00	0.4493289641172D+00	-.3256172068089D-06
.9D+00	0.4065699912001D+00	0.4065696597406D+00	-.3314594766990D-06
.1D+01	0.3678797744125D+00	0.3678794411714D+00	-.3332410563051D-06

PROBLEM 1 (x,y Derivatives)

XN	YN	TSOL	ERROR
.1D+00	0.9048375000000D+00	0.9048374180360D+00	-.8196404044369D-07
.2D+00	0.8187309014063D+00	0.8187307530780D+00	-.1483282683346D-06
.3D+00	0.7408184220012D+00	0.7408182206817D+00	-.2013194597694D-06
.4D+00	0.6703202889175D+00	0.6703200460356D+00	-.2428818514089D-06
.5D+00	0.6065309344234D+00	0.6065306597126D+00	-.2747107467060D-06
.6D+00	0.5488119343763D+00	0.5488116360940D+00	-.2982822888686D-06
.7D+00	0.4965856186712D+00	0.4965853037914D+00	-.3148798197183D-06
.8D+00	0.4493292897344D+00	0.4493289641172D+00	-.3256172068089D-06
.9D+00	0.4065699912001D+00	0.4065696597406D+00	-.3314594766990D-06
.1D+01	0.3678797744125D+00	0.3678794411714D+00	-.3332410563051D-06

PROBLEM 2(y Derivatives)

XN	YN	TSOL	ERROR
.1D+00	0.1105170833333D+01	0.1105170918076D+01	0.8474231405486D-07

.2D+00	0.1221402570851D+01	0.1221402758160D+01	0.1873094752636D-06
.3D+00	0.1349858497063D+01	0.1349858807576D+01	0.3105134649406D-06
.4D+00	0.1491824240081D+01	0.1491824697641D+01	0.4575605843105D-06
.5D+00	0.1648720638597D+01	0.1648721270700D+01	0.6321032899326D-06
.6D+00	0.1822117962092D+01	0.1822118800391D+01	0.8382985758892D-06
.7D+00	0.2013751626597D+01	0.2013752707470D+01	0.1080873699877D-05
.8D+00	0.2225539563292D+01	0.2225540928492D+01	0.1365200152481D-05
.9D+00	0.2459601413780D+01	0.2459603111157D+01	0.1697376878607D-05
.1D+01	0.2718279744135D+01	0.2718281828459D+01	0.2084323879270D-05

PROBLEM 2(x,y Derivatives)

XN	YN	TSOL	ERROR
.1D+00	0.1105170833333D+01	0.1105170918076D+01	0.8474231405486D-07
.2D+00	0.1221402570851D+01	0.1221402758160D+01	0.1873094752636D-06
.3D+00	0.1349858497063D+01	0.1349858807576D+01	0.3105134649406D-06
.4D+00	0.1491824240081D+01	0.1491824697641D+01	0.4575605843105D-06
.5D+00	0.1648720638597D+01	0.1648721270700D+01	0.6321032899326D-06
.6D+00	0.1822117962092D+01	0.1822118800391D+01	0.8382985758892D-06
.7D+00	0.2013751626597D+01	0.2013752707470D+01	0.1080873699877D-05
.8D+00	0.2225539563292D+01	0.2225540928492D+01	0.1365200152481D-05
.9D+00	0.2459601413780D+01	0.2459603111157D+01	0.1697376878607D-05
.1D+01	0.2718279744135D+01	0.2718281828459D+01	0.2084323879270D-05

PROBLEM 3(y Derivatives)

XN	YN	TSOL	ERROR
.1D+00	0.1223051005569D+01	0.1223048880450D+01	-.2125119075158D-05
.2D+00	0.1508502732390D+01	0.1508497647121D+01	-.5085268468541D-05
.3D+00	0.1895771003842D+01	0.1895765122854D+01	-.5880987590245D-05
.4D+00	0.2464942965339D+01	0.2464962756723D+01	0.1979138375674D-04
.5D+00	0.3407951033347D+01	0.3408223442336D+01	0.2724089890727D-03
.6D+00	0.5328707710968D+01	0.5331855223459D+01	0.3147512490389D-02
.7D+00	0.1159500710295D+02	0.1168137380031D+02	0.8636669735614D-01
.8D+00	0.2841447010395D+03	-.6847966834558D+02	-.3526243693850D+03
.9D+00	0.8635045424394D+20	-.8687629546482D+01	-.8635045424394D+20
.1D+01	0.1640237043432+300	-.4588037824984D+01	-.1640237043432+300

PROBLEM 3 (x, y Derivatives)

XN	YN	TSOL	ERROR
.1D+00	0.1223051005569D+01	0.1223048880450D+01	-.2125119075158D-05
.2D+00	0.1508502732390D+01	0.1508497647121D+01	-.5085268468541D-05
.3D+00	0.1895771003842D+01	0.1895765122854D+01	-.5880987590245D-05
.4D+00	0.2464942965339D+01	0.2464962756723D+01	0.1979138375674D-04

.5D+00	0.3407951033347D+01	0.3408223442336D+01	0.2724089890727D-03
.6D+00	0.5328707710968D+01	0.5331855223459D+01	0.3147512490389D-02
.7D+00	0.1159500710295D+02	0.1168137380031D+02	0.8636669735614D-01
.8D+00	0.2841447010395D+03	-.6847966834558D+02	-.3526243693850D+03
.9D+00	0.8635045424394D+20	-.8687629546482D+01	-.8635045424394D+20
.1D+01	0.1640237043432+300	-.4588037824984D+01	-.1640237043432+300

5. Findings/Contribution to Knowledge and Conclusion

After our implementation, it shows from the tables of numerical results that the two methods compared favourably well. It was revealed that the two sets of equations derived from both derivatives are the same, hence generating the same rooted trees. This research has revealed the fact that researchers can take the y derivatives only or the x,y derivatives only and generate the same results and rooted trees even when parameters are varied. This can also be seen in our tables containing the derived equations, their various partial derivatives and rooted trees.

Hence, we conclude that this research has shown that more research can still be done in this area without any rigorous or stressful expansions that may give rise to complicated formulas.

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APPENDIX

FORTRAN PROGRAM THAT GENERATED THE RESULTS

PROBLEM 1 (y Derivatives)

```
C PRO
C FOURTH ORDER FOURTH STAGE RUNGE KUTTA METHOD
C OUR PROBLEM IS :Y'=-Y, Y(0)=1
C THEORETICAL SOLUTION:Y(XN)= 1/EXP(XN)
      DOUBLE PRECISION XN,YN,H,ONE,TWO
      DOUBLE PRECISION TSOL,ERROR,K1,K2,K3,K4,SIX
      OPEN(6,FILE='RUNG2.OUT')
      H=0.1D0
      YN=1.0D0
      XN=0.1D0
      TWO=2.0D0
      SIX=6.0D0
      ONE=1.0D0
      WRITE(6,101)
 3    K1=-YN
      K2=-(YN+ONE/TWO*(H*K1))
      K3=-(YN+H/TWO*(-K1+TWO*K2))
      K4=-(YN+H/TWO*(TWO*K1-K2+K3))
      YN=YN+H/SIX*(K1+TWO*K2+TWO*K3+K4)
      TSOL=ONE/EXP(XN)
      ERROR=TSOL-YN
      WRITE(6,100)XN,YN,TSOL,ERROR
      XN=XN+H
      IF(XN.LE.ONE) GOTO 3
100   FORMAT(D6.1,1X,3(3X,D19.13))
101   FORMAT(2X,'XN',13X,'YN',16X,'TSOL',16X,'ERROR')
      END
```

PROBLEM 1(x,y Derivatives)

```
C PRO
C FOURTH ORDER FOURTH STAGE RUNGE KUTTA METHOD
C OUR PROBLEM IS :Y'=-Y, Y(0)=1
C THEORETICAL SOLUTION:Y(XN)= 1/EXP(XN)
      DOUBLE PRECISION XN,YN,H,ONE,TWO,EIGHT,EIT,FIV,THREE
      DOUBLE PRECISION TSOL,ERROR,K1,K2,K3,K4,SIX,FOUR
      OPEN(6,FILE='RUNG2.OUT')
```

```
      H=0.1D0
      YN=1.0D0
      XN=0.1D0
      TWO=2.0D0
      EIGHT=18.0D0
      EIT=8.0D0
      THREE=3.0D0
      FIV=5.0D0
      FOUR=4.0D0
      SIX=6.0D0
      ONE=1.0D0
      WRITE(6,101)
 3    K1=-YN
      K2=-(YN+ONE/FOUR*(H*K1))
      K3=-(YN+H/FOUR*(-THREE*K1+SIX*K2))
      K4=-(YN+H*(FIV*K1-SIX*K2+TWO*K3))
      YN=YN+H/EIGHT*(K1+EIT*K2+EIT*K3+K4)
      TSOL=ONE/EXP(XN)
      ERROR=TSOL-YN
      WRITE(6,100)XN,YN,TSOL,ERROR
      XN=XN+H
      IF(XN.LE.ONE) GOTO 3
100   FORMAT(D6.1,1X,3(3X,D19.13))
101   FORMAT(2X,'XN',13X,'YN',16X,'TSOL',16X,'ERROR')
      END
```

PROBLEM 2 (y Derivatives)

```
C PRO
  C FOURTH ORDER FOURTH STAGE RUNGE KUTTA METHOD
  C OUR PROBLEM IS :Y'=Y, Y(0)=1
  C THEORETICAL SOLUTION:Y(XN)=EXP(XN)
      DOUBLE PRECISION XN,YN,H,ONE,TWO
      DOUBLE PRECISION TSOL,ERROR,K1,K2,K3,K4,SIX
      OPEN(6,FILE='RUNG2.OUT')
      H=0.1D0
      YN=1.0D0
      XN=0.1D0
      TWO=2.0D0
      SIX=6.0D0
      ONE=1.0D0
      WRITE(6,101)
 3    K1=YN
      K2=YN+ONE/TWO*(H*K1)
      K3=YN+H/TWO*(-K1+TWO*K2)
```

```
          K4=YN+H/TWO*(TWO*K1-K2+K3)
          YN=YN+H/SIX*(K1+TWO*K2+TWO*K3+K4)
          TSOL=EXP(XN)
          ERROR=TSOL-YN
          WRITE(6,100)XN,YN,TSOL,ERROR
          XN=XN+H
          IF(XN.LE.ONE) GOTO 3
    100   FORMAT(D6.1,1X,3(3X,D19.13))
    101   FORMAT(2X,'XN',13X,'YN',16X,'TSOL',16X,'ERROR')
          END
```

PROBLEM 2 (x,y Derivatives)

```
C PRO
  C FOURTH ORDER FOURTH STAGE RUNGE KUTTA METHOD
  C OUR PROBLEM IS :Y'=Y, Y(0)=1
  C THEORETICAL SOLUTION:Y(XN)=EXP(XN)
          DOUBLE PRECISION XN,YN,H,ONE,THREE,TWO,EIGHT,EIT,FOUR
          DOUBLE PRECISION TSOL,ERROR,K1,K2,K3,K4,FIVE,SIX
          OPEN(6,FILE='RUNG2.OUT')
          H=0.1D0
          YN=1.0D0
          XN=0.1D0
          TWO=2.0D0
          THREE=3.0D0
          FOUR=4.0D0
          SIX=6.0D0
          ONE=1.0D0
          EIT=8.0D0
          EIGHT=18.0D0
          FIVE=5.0D0
          WRITE(6,101)
      3   K1=YN
          K2=YN+ONE/FOUR*(H*K1)
          K3=YN+H/FOUR*(-THREE*K1+SIX*K2)
          K4=YN+H*(FIVE*K1-SIX*K2+TWO*K3)
          YN=YN+H/EIGHT*(K1+EIT*K2+EIT*K3+K4)
          TSOL=EXP(XN)
          ERROR=TSOL-YN
          WRITE(6,100)XN,YN,TSOL,ERROR
          XN=XN+H
          IF(XN.LE.ONE) GOTO 3
    100   FORMAT(D6.1,1X,3(3X,D19.13))
    101   FORMAT(2X,'XN',13X,'YN',16X,'TSOL',16X,'ERROR')
```

```
          END
```

PROBLEM 3 (y Derivatives)

```
C PRO
C FOURTH ORDER FOURTH STAGE RUNGE KUTTA METHOD
C OUR PROBLEM IS :Y'=1+Y**2, Y(0)=1
C THEORETICAL SOLUTION:Y(XN)=TAN(XN+PI/4)
       DOUBLE PRECISION XN,YN,H,ONE,TWO,PI
       DOUBLE PRECISION TSOL,ERROR,K1,K2,K3,K4,SIX,FOUR
       OPEN(6,FILE='RUNG2.OUT')
       H=0.1D0
       YN=1.0D0
       ONE=1.0D0
       XN=0.1D0
       TWO=2.0D0
       FOUR=4.0D0
       SIX=6.0D0
       PI=FOUR*DATAN(ONE)
       WRITE(6,101)
  3    K1=ONE+YN*YN
       K2=ONE+(YN+(H/TWO)*K1)**TWO
       K3=ONE+(YN+(H/TWO)*(-K1+TWO*K2))**TWO
       K4=ONE+(YN+(H/TWO)*(TWO*K1-K2+K3))**TWO
       YN=YN+H/SIX*(K1+TWO*K2+TWO*K3+K4)
       TSOL=TAN(XN+(PI/FOUR))
       ERROR=TSOL-YN
       WRITE(6,100)XN,YN,TSOL,ERROR
       XN=XN+H
       IF(XN.LE.ONE) GOTO 3
 100   FORMAT(D6.1,1X,3(3X,D19.13))
 101   FORMAT(2X,'XN',13X,'YN',16X,'TSOL',16X,'ERROR')
       END
```

PROBLEM 3(x,y Derivatives)

```
C PRO
C FOURTH ORDER FOURTH STAGE RUNGE KUTTA METHOD
C OUR PROBLEM IS :Y'=1+Y**2, Y(0)=1
C THEORETICAL SOLUTION:Y(XN)=TAN(XN+PI/4)
       DOUBLE PRECISION XN,YN,H,ONE,TWO,EIGHT,PI,THREE,EIT
       DOUBLE PRECISION TSOL,ERROR,K1,K2,K3,K4,SIX,FOUR,FIV
       OPEN(6,FILE='RUNG2.OUT')
       H=0.1D0
       YN=1.0D0
       ONE=1.0D0
       FOUR=4.0D0
```

```
      XN=0.1D0
      TWO=2.0D0
      EIGHT=18.0D0
      EIT=8.0D0
      THREE=3.0D0
      FIV=5.0D0
      SIX=6.0D0
      PI=FOUR*DATAN(ONE)
      WRITE(6,101)
 3    K1=ONE+YN*YN
      K2=ONE+(YN+(H/FOUR)*K1)**TWO
      K3=ONE+(YN+(H/FOUR)*(-THREE*K1+SIX*K2))**TWO
      K4=ONE+(YN+H*(FIV*K1-SIX*K2+TWO*K3))**TWO
      YN=YN+H/EIGHT*(K1+EIT*K2+EIT*K3+K4)
      TSOL=TAN(XN+(PI/FOUR))
      ERROR=TSOL-YN
      WRITE(6,100)XN,YN,TSOL,ERROR
      XN=XN+H
      IF(XN.LE.ONE) GOTO 3
100   FORMAT(D6.1,1X,3(3X,D19.13))
101   FORMAT(2X,'XN',13X,'YN',16X,'TSOL',16X,'ERROR')
      END
```