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Bayes Estimators for the Parameter of the Inverted Exponential Distribution under Symmetric and Asymmetric Loss Functions

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Abstract
This paper is devoted to discuss Bayes method to estimate the unknown scale parameter of the inverted exponential distribution along with the maximum likelihood method. Bayes estimators are obtained under symmetric "squared error" and asymmetric "precautionary" loss functions corresponding to informative "inverted gamma and Gumbel type II" and non-informative "Jeffrey and extension of Jeffrey" priors. The obtained Bayes estimators along with the maximum likelihood estimator are compared empirically for different cases and sample sizes using Monte-Carlo simulation method in terms of two statistical criteria which are mean squared error (MSE) and mean absolute percentage error (MAPE). Among the set of conclusions that have been reached, it is observed that, conjugate inverted gamma prior with hyper-parameters $\alpha = \beta$ and $\alpha < \beta$ record full appearance as best prior depending on the value of the parameter of inverted exponential distribution.
Keywords: Inverted exponential distribution; maximum likelihood estimator; Bayes estimator; informative prior; non-informative prior; squared error loss function; precautionary loss function; mean squared error; mean absolute percentage error.

1. Introduction
The inverted exponential distribution is a member of continuous probability distributions. It has been introduced by Keller and Kamath [4] in (1982). The probability density function (pdf) and distribution function (cdf) of inverted exponential distribution (IED) are defined as [8]:

$$f(t;\theta) = \frac{1}{\theta t^2} e^{-1/\theta t} \;; t > 0\,, \theta > 0 \qquad (1)$$

$$F(t;\theta) = e^{-1/\theta t} \quad ; \theta > 0 \qquad (2)$$

The IED has no finite moments where the r^{th} moment of the IED is given as [7]:

$$E(T^r) = \frac{1}{\theta^r}\Gamma(1-r)\,;\, r < 1 \qquad (3)$$

Thus the expectation and the variance of the IED do not exist.

2. Maximum Likelihood Estimation
The maximum likelihood estimation method is one of the most widely used as classical estimation. Classical view is that there is some fixed (unknown) value of the parameter that is driving a process and, hence, its value is reflected in the data we see [3]. Assume that $(t_1, t_2, \dots, t_n)$ are the n independent random sample drawn from the IED defined by (1), then the likelihood function is obtained as:

$$L(\theta|\underline{t}) = \prod_{i=1}^{n} f(t_i, \theta) = \prod_{i=1}^{n} \frac{1}{\theta t^2} e^{-1/\theta t} \Rightarrow L(\theta|\underline{t}) = \frac{1}{\theta^n}\prod_{i=1}^{n}\frac{1}{t_i^2}\, e^{-S/\theta} \qquad (4)$$

Where: $S = \sum_{i=1}^{n} \frac{1}{t_i}$

The maximum likelihood estimator of θ, denoted by $\hat{\theta}_{ML}$, which yields by taking the derivative of the natural log-likelihood function with respect to θ and setting it equal to zero is :

$$\hat{\theta}_{ML} = \frac{S}{n} \quad ; S = \sum_{i=1}^{n}\frac{1}{t_i} \qquad (5)$$

3. Bayes Estimation
In Bayesian analysis, the parameter of interest is considered to be a random variable. In this section, in order to obtain the Bayes estimation of the parameter θ, we consider different types of prior distribution: informative and non-informative priors.

◙ ***Informative Prior:*** The informative prior applied here are:

① ***Inverted Gamma Prior:*** The probability density function of inverted gamma prior is defined as [1]:

$$\pi_1(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)\theta^{\alpha+1}} e^{-\beta/\theta} \quad ; \theta > 0 \qquad (6)$$

Where $\alpha > 0$ and $\beta > 0$ are the shape and scale parameters respectively of the prior distribution.

② ***Gumbel Type II Prior:*** The probability density function of Gumbel type II prior defined as [1]:

$$\pi_2(\theta) = b\left(\frac{1}{\theta}\right)^2 e^{-b/\theta} \quad ;\theta > 0, b > 0 \qquad (7)$$

◙ **Non-Informative Prior:** The non-informative prior applied here are:

① ***Jeffrey's Prior:*** Jeffrey's prior is proposed by Harold Jeffrey in (1946). It is based on Fisher information [18], such that:

$\pi_3(\theta) \propto \sqrt{I(\theta)}$; Where $I(\theta) = -nE\left[\frac{\partial^2 \ln f(t,\theta)}{\partial^2\theta}\right]$ is the Fisher's information matrix.

For the model (1),

$$\pi_3(\theta) = \frac{w\sqrt{n}}{\theta} \; ; \theta > 0 \; , \; w\text{: constant} \qquad (8)$$

② ***Extension of Jeffrey's Prior:*** The extension of Jeffrey's prior is considered as [2]:

$\pi_4(\theta) \propto [I(\theta)]^k \, ; k \in R^+$

Where $I(\theta)$ is the Fisher's information matrix.

For the model (1),

$$\pi_4(\theta) = \frac{wn^k}{\theta^{2k}} \; ; \; \theta > 0 \; , \; w: constant \qquad (9)$$

Now, the posterior density of (θ) corresponding to the i^{th} prior, $\pi_i(\theta)$, is obtained as:

$$H_i(\theta|\underline{t}) = \frac{\prod_{i=1}^n f(t_i,\theta)\pi_{i(\theta)}}{\int_\theta \prod_{i=1}^n f(t_i,\theta)\pi_{i(\theta)} d\theta}$$

So, the posterior density of (θ) corresponding to the each prior is obtained as:

$$H_1(\theta|\underline{t}) = \frac{(S+\beta)^{n+\alpha}}{\Gamma(n+\alpha)\theta^{n+\alpha+1}} e^{-(S+\beta)/\theta} \qquad (10)$$

which implies that: $(\theta|\underline{t}) \backsim IG(n+\alpha\,, S+\beta)$

$$H_2(\theta|\underline{t}) = \frac{(S+b)^{n+1}}{\Gamma(n+1)\theta^{n+2}} e^{-(S+b)/\theta} \qquad (11)$$

Which implies that: $(\theta|\underline{t}) \backsim IG(n+1\,, S+b)$

$$H_3(\theta|\underline{t}) = \frac{S^n}{\Gamma(n)\theta^{n+1}} e^{-S/\theta} \qquad (12)$$

Which implies that: $(\theta|\underline{t}) \backsim IG(n\,, S)$

$$H_4(\theta|\underline{t}) = \frac{S^{n+2k-1}}{\Gamma(n+2k-1)\,\theta^{n+2k}} e^{-S/\theta} \qquad (13)$$

which implies that: $(\theta|\underline{t}) \backsim IG(n+2k-1, S)$

3.1 Bayes Estimation under Symmetric Loss Function: The symmetric loss function applied here is squared error. The squared error loss function was proposed by Legendre (1805) and Gauss (1810) to develop least square theory. The formula of this loss function is [1]:

$$L(\hat{\theta}_s, \theta) = (\hat{\theta}_s - \theta)^2 \qquad (14)$$

Where $\hat{\theta}_s$ is an estimate of θ under squared error loss function. The Bayes estimator of θ under squared error loss function is obtained as:

$$\hat{\theta}_s = E_H(\theta|\underline{t}) \qquad (15)$$

From (15), it is shown that the Bayes estimate under squared error loss function is the mean of the posterior probability density function. Bayes estimators of the parameter θ of IED corresponding to four posterior distributions are:

❖ ***Corresponding to $H_1(\theta|\underline{t})$:*** According to (10), posterior density function is $IG(n+\alpha, S+\beta)$. Hence, by using the properties of inverted gamma distribution, the Bayes estimator of θ under squared error loss function with first prior (inverted gamma prior distribution), denoted by $\hat{\theta}_{s_1}$, is:

$$\hat{\theta}_{s_1} = E_{H_1}(\theta|\underline{t}) = \frac{S+\beta}{n+\alpha-1} \qquad (16)$$

❖ ***Corresponding to $H_2(\theta|\underline{t})$:*** According to (11), posterior density function is $IG(n+1, S+b)$. Hence the Bayes estimator of θ under squared error loss function with second prior (Gumbel type II prior distribution), denoted by $\hat{\theta}_{s_2}$, is:

$$\hat{\theta}_{s_2} = E_{H_2}(\theta|\underline{t}) = \frac{S+b}{n} \qquad (17)$$

❖ ***Corresponding to*** $H_3(\theta|\underline{t})$**:** According to (12), posterior density function is $IG(n, S)$. Hence the Bayes estimator of θ under squared error loss function with third prior (Jeffrey's prior), denoted by $\hat{\theta}_{S_3}$, is:

$$\hat{\theta}_{S_3} = E_{H_3}(\theta|\underline{t}) = \frac{S}{n-1} \tag{18}$$

❖ ***Corresponding to*** $H_4(\theta|\underline{t})$**:** According to (13), posterior density function is $IG(n + 2k - 1, S)$, Hence the Bayes estimator of θ under squared error loss function with fourth prior (extension of Jeffrey's prior) , denoted by $\hat{\theta}_{S_4}$, is:

$$\hat{\theta}_{S_4} = E_{H_4}(\theta|\underline{t}) = \frac{S}{n+2k-2} \tag{19}$$

3.2 Bayes Estimation under Asymmetric Loss Function: The asymmetric loss function applied here is precautionary loss function which introduced by Norstrom (1996) [6]. A very useful and simple asymmetric precautionary loss function is [5]:

$$L(\hat{\theta}_P , \theta) = \frac{(\hat{\theta}_P - \theta)^2}{\hat{\theta}_P \theta} \tag{20}$$

Where $\hat{\theta}_P$ is an estimate for θ under precautionary loss function. The Bayes estimator of θ under precautionary loss function is obtained as:

$$\hat{\theta}_P = \sqrt{\frac{E_H(\theta|\underline{t})}{E_H\left(\frac{1}{\theta}|\underline{t}\right)}} \tag{21}$$

Now, Bayes estimators for the parameter θ under precautionary loss function corresponding to four posterior distributions are:

❖ ***Corresponding to*** $H_1(\theta|\underline{t})$**:** From (10), $(\theta|\underline{t}) \backsim IG(n+\alpha , S+\beta) \Rightarrow \left(\frac{1}{\theta}|\underline{t}\right) \backsim G(n+\alpha , S+\beta)$. By using the properties of gamma distribution, we obtained:

$$E_{H_1}\left(\frac{1}{\theta}|\underline{t}\right) = \frac{n+\alpha}{S+\beta} \tag{22}$$

After substituting (16) and (22) in (21), Bayes estimator of parameter θ under precautionary loss function with first prior, denoted by $\hat{\theta}_{P_1}$, is:

$$\hat{\theta}_{P_1} = \frac{S+\beta}{\sqrt{(n+\alpha)(n+\alpha-1)}} \tag{23}$$

❖ ***Corresponding to*** $H_2(\theta|\underline{t})$**:** From$(11)$, $(\theta|\underline{t}) \backsim IG(n+1 , S+b) \Rightarrow \left(\frac{1}{\theta}|\underline{t}\right) \backsim G(n+1 , S+b)$. By using the properties of gamma distribution, we obtained:

$$E_{H_2}\left(\frac{1}{\theta}|\underline{t}\right) = \frac{n+1}{S+b} \tag{24}$$

After substituting (17) and (24) in (21), Bayes estimator of parameter θ under precautionary loss function with second prior , denoted by $\hat{\theta}_{P_2}$, is:

$$\hat{\theta}_{P_2} = \frac{S+b}{\sqrt{n(n+1)}} \tag{25}$$

❖ ***Corresponding to*** $H_3(\theta|\underline{t})$**:** From (12), $(\theta|\underline{t}) \backsim IG(n, S) \Rightarrow \left(\frac{1}{\theta}|\underline{t}\right) \backsim G(n, S)$. By using the properties of gamma distribution, we obtained:

$$E_{H_3}\left(\frac{1}{\theta}|\underline{t}\right) = \frac{n}{S} \tag{26}$$

After substituting (18) and (26) in (21), Bayes estimator of parameter θ under precautionary loss function with third prior, denoted by $\hat{\theta}_{P_3}$, is:

$$\hat{\theta}_{P_3} = \frac{S}{\sqrt{n(n-1)}} \tag{27}$$

❖ ***Corresponding to*** $H_4(\theta|\underline{t})$**:** From (13), $(\theta|\underline{t}) \backsim IG(n+2k-1, S) \Rightarrow \left(\frac{1}{\theta}|\underline{t}\right) \backsim G(n+2k-1, S)$**.**
By using the properties of gamma distribution, we obtained:

$$E_{H_4}\left(\frac{1}{\theta}|\underline{t}\right) = \frac{n+2k-1}{S} \tag{28}$$

After substituting (19) and (28) in (21), Bayes estimator of parameter θ under precautionary loss function with fourth prior, denoted by $\hat{\theta}_{P_4}$, is:

$$\hat{\theta}_{P_4} = \frac{S}{\sqrt{(n+2k-1)(n+2k-2)}} \quad (29)$$

4. Simulation Experiment Design and Results

In simulation experiment, two statistical criteria "Mean Squared Error (MSE) and Mean Absolute Percentage Error (MAPE) of the estimator" have been considered to compare the performance of estimators $\hat{\theta}_{ML}$, $\hat{\theta}_{S1}$, $\hat{\theta}_{S2}$, $\hat{\theta}_{S3}$, $\hat{\theta}_{S4}$, $\hat{\theta}_{P1}$, $\hat{\theta}_{P2}$, $\hat{\theta}_{P3}$ and $\hat{\theta}_{P4}$. The MSE and MAPE of an estimator are defined as:

$$MSE(\hat{\theta}) = \frac{\sum_{j=1}^{L}(\hat{\theta}_j - \theta)^2}{L} \quad (30)$$

$$MAPE(\hat{\theta}) = \frac{\sum_{j=1}^{L}\frac{|\hat{\theta}_j - \theta|}{\theta}}{L} \quad (31)$$

Where $\hat{\theta}_j$ is the estimate of θ at the j^{th} replicate. The number of sample replicated used was L = 3000 samples from the inverted exponential distribution of sizes n = 10, 15, 25, 30, 50, 100 to represent small, moderate, and large dataset. The set of default (true) values for the parameters and constant upon which build the simulation experiment are shown below:

Parameter of the inverted exponential distribution	θ	1, 1.5, 3
Hyper-parameters of inverted gamma distribution	(α, β)	(4, 4), (3, 2), (6, 10)
Hyper-parameter of Gumbel type II distribution	b	3, 5
Constant of the extension of Jeffrey	k	1, 3

The simulation experiment program has been written by using MATLAB (R2011b) program. The results of simulation experiment have been summarized in the tables (1)…(5).

5. Conclusions and Recommendation

The most important conclusions of simulation experiment are:

1. When $\theta = 1,1.5$ and 3, the maximum likelihood method gives estimate values greater than default value (overestimate values) for $n \geq 50$, $n \geq 15$ and $n \leq 30$ respectively.
2. Corresponding to inverted gamma prior, when $\theta = 1$, Bayes method under squared error loss function gives overestimate values with hyper-parameters $(\alpha = \beta = 4)$ and $(\alpha = 6, \beta = 10)$ for all sample sizes as well as with $(\alpha = 3, \beta = 2)$ for $(n \geq 50)$. Under precautionary loss function, Bayes method gives overestimate values for all sample sizes with $(\alpha = \beta = 4)$ and $(\alpha = 6, \beta = 10)$. When $\theta = 1.5$ with $(\alpha = 6, \beta = 10)$, Bayes method under squared error and precautionary loss functions gives overestimate values for all sample sizes. Bayes method gives underestimate values for all sample sizes when $\theta = 3$.
3. Corresponding to Gumbel type II prior, Bayes method with different values of hyper-parameter $b = 3,5$ gives overestimate values for all sample sizes.
4. Corresponding to Jeffrey's prior, Bayes method under squared error and precautionary loss functions gives overestimate values for all sample sizes.
5. Corresponding to extension of Jeffrey's prior with $(k = 1)$, Bayes method under squared error loss function gives overestimate values for $n \geq 50, n \geq 15$ and $n \leq 30$ with $\theta = 1,1.5$ and 3 respectively.
6. Depending on the MSE and MAPE values when $\theta = 1$, the performance of Bayes estimator corresponding to inverted gamma prior under the assumption of precautionary loss function is better "record the minimum values of MSE and MAPE" than that under the assumption of squared error loss function while the reverse is true when $\theta = 3$ for all sample sizes and different values of hyper-parameters (α, β). When $\theta = 1.5$, the values of the hyper-parameters, sample size and statistical criteria have had a significant impact on the performance of Bayes estimator corresponding to inverted gamma prior such that, according to MSE values, the performance with hyper-parameters $(\alpha = 4, \beta = 4)$ under the assumption of precautionary loss function is better than that under squared error loss function for all sample sizes. Also this true, according to MAPE values for large sample sizes $(n \geq 50)$ while the reverse is true for small and moderate sample sizes $(n \leq 30)$. With hyper-parameters $(\alpha = 3, \beta = 2)$, squared error loss function record the minimum values of MSE only for $n = 10$ and precautionary record the minimum values of MSE for $n \geq 15$. While according to MAPE values, squared error loss function record the minimum values of MAPE for all sample sizes. With hyper-parameters $(\alpha = 6, \beta = 10)$ precautionary loss function record the minimum MSE and MAPE values for all sample sizes.

7. Depending on the MSE and MAPE values with different values of θ and all sample sizes, the performance of Bayes estimates under precautionary loss function are better than those estimates under squared error loss function corresponding to Gumbel type II prior "with different values of hyper-parameter b" as well as corresponding to Jeffrey's prior.
8. With different values of θ, the MSE and MAPE values associated with Bayes estimates corresponding to Gumbel type II prior are increase as hyper-parameter value (b) increase.
9. With different values of θ, the MSE and MAPE values associated with Bayes estimator corresponding to extension of Jeffrey prior are increases as extension constant (k) increases.
10. Depending on the MSE and MAPE values with different values of θ, the performance of Bayes estimator corresponding to extension of Jeffrey's prior with extension constant $(k = 1)$ under the assumption of precautionary loss function is better than that under the assumption of squared error loss function while the reverse is true with extension constant $(k = 3)$ for all sample sizes.
11. The simulation experiment results show a convergence between the estimates to true values of the parameter (θ) with increasing the sample size.
12. The conjugate inverted gamma prior record full appearance as best prior depending on the values of hyper-parameter. According to MSE and MAPE values, inverted gamma prior with hyper-parameters $\alpha = \beta = 4$ record full appearance as best prior under squared error and precautionary loss functions for all sample sizes when $\theta = 1$ as well as with hyper-parameters $\alpha = 6, \beta = 10$ for all sample sizes when $\theta = 1.5\ ,3$.
13. The MSE and MAPE values associated with maximum likelihood estimate, as well as with each Bayes estimate under each prior and every loss function, reduce with the increase in the sample size and this conforms to the statistical theory.
14. There is not always a consistency between the error measurement given by the MSE and the MAPE criteria.
15. The maximum likelihood estimates are equivalent to Bayes estimates under squared error loss function corresponding to extension of Jeffrey's prior with $(k = 1)$.
16. For some situations, the maximum likelihood method may give estimate values better than Bayes method. Such that, depending on MSE and MAPE values, the maximum likelihood method gives estimate values better than Bayes method for all sample sizes corresponding to Jeffrey's prior with different values of θ and corresponding to Gumbel type II prior with $\theta = 1, 1.5$.

In the light of the conclusions that have been obtained for the estimation of the parameter of inverted exponential distribution (IED), we recommend to using inverted gamma as an appropriate prior distribution under squared error and precautionary loss functions with $(\alpha = \beta = 4)$ for the situation that $(\theta = 1)$ and with $(\alpha = 6 < \beta = 10)$ for $(\theta = 1.5, 3)$.

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Table (1): Estimated, MSE and MAPE Values for Maximum Likelihood Estimator of θ

θ	Criteria	n=10	n=15	n=25	n=30	n=50	n=100
1	Est.	0.9967291	0.9958662	0.9980574	0.9965871	1.0013716	1.0006285
	MSE	0.0972233	0.0628100	0.0396510	0.0331380	0.0202021	0.0100923
	MAPE	0.2449796	0.2005705	0.1594937	0.1449568	0.1129120	0.0795727
1.5	Est.	1.4920928	1.5029435	1.5078325	1.5005268	1.5018286	1.5030866
	MSE	0.2217303	0.1591439	0.0904074	0.0771929	0.0449750	0.0218426
	MAPE	0.2502248	0.2100975	0.1598033	0.1483118	0.1132632	0.0785541
3	Est.	3.0077182	3.0258928	3.0061211	3.0038504	2.9958573	2.9974142
	MSE	0.9071278	0.5845579	0.3802260	0.2878400	0.1792408	0.0924489
	MAPE	0.2490088	0.1996128	0.1634844	0.1434657	0.1128616	0.0805002

Table (2): Estimated, MSE and MAPE Values for Bayes Estimator of θ with Inverted Gamma Prior

θ	n	α	β	Est.		MSE		MAPE	
				Sq. Error	Prec.	Sq. Error	Prec.	Sq. Error	Prec.
1	10	4	4	1.0744070	1.0353245	0.0630586	0.0546613	0.1920065	0.1806594
		3	2	0.9972742	0.9581501	0.0675162	0.0640672	0.2041497	0.2021860
		6	10	1.3311527	1.2888831	0.1528677	0.1239587	0.3342249	0.2947493
	15	4	4	1.0521107	1.0240494	0.0463217	0.0418895	0.1691507	0.1620730
		3	2	0.9963525	0.9682806	0.0489005	0.0471774	0.1769739	0.1758494
		6	10	1.2468996	1.2168494	0.0962804	0.0806627	0.2565939	0.2309833
	25	4	4	1.0339798	1.0159962	0.0327611	0.0307725	0.1427983	0.1394416
		3	2	0.9982013	0.9802142	0.0339943	0.0331686	0.1476794	0.1468509
		6	10	1.1650478	1.1461027	0.0547736	0.0479906	0.1872655	0.1734556
	30	4	4	1.0272004	1.0119817	0.0281170	0.0267155	0.1322927	0.1295192
		3	2	0.9968004	0.9815811	0.0291252	0.0285720	0.1358970	0.1352774
		6	10	1.1399318	1.1239879	0.0439186	0.0390347	0.1665888	0.1559226
	50	4	4	1.0201619	1.0106718	0.0183846	0.0177591	0.1068964	0.1054629
		3	2	1.0013188	0.9918274	0.0186780	0.0183906	0.1085692	0.1081444
		6	10	1.0921560	1.0823607	0.0251871	0.0231796	0.1248066	0.1193418
	100	4	4	1.0103189	1.0054499	0.0096190	0.0094508	0.0774637	0.0768837
		3	2	1.0006161	0.9957469	0.0097004	0.0096239	0.0780125	0.0778333
		6	10	1.0482176	1.0432615	0.0114786	0.0109388	0.0845442	0.0824785
1.5	10	4	4	1.4554560	1.4025125	0.1331485	0.1312993	0.1959165	0.1970315
		3	2	1.4100773	1.3547585	0.1620220	0.1631898	0.2180110	0.2214860
		6	10	1.6613952	1.6086390	0.1245674	0.1041640	0.1799851	0.1656851
	15	4	4	1.4746751	1.4353433	0.1111520	0.1088747	0.1768115	0.1770340
		3	2	1.4437736	1.4030958	0.1270556	0.1264017	0.1905071	0.1922306
		6	10	1.6272076	1.5879920	0.1056954	0.0929936	0.1663020	0.1568105
	25	4	4	1.4891362	1.4632362	0.0721413	0.0708913	0.1431321	0.1435135
		3	2	1.4702153	1.4437228	0.0783443	0.0778580	0.1495756	0.1509428
		6	10	1.5898604	1.5640073	0.0708152	0.0648134	0.1389029	0.1335964
	30	4	4	1.4853274	1.4633213	0.0640108	0.0632645	0.1357387	0.1358748
		3	2	1.4692439	1.4468113	0.0687910	0.0686182	0.1413912	0.1420644
		6	10	1.5718801	1.5498947	0.0618797	0.0576270	0.1297706	0.1260191
	50	4	4	1.4922911	1.4784090	0.0400841	0.0397496	0.1071559	0.1070809
		3	2	1.4825275	1.4684747	0.0418841	0.0417882	0.1097912	0.1100731
		6	10	1.5471169	1.5332411	0.0393867	0.0376080	0.1049492	0.1028308
	100	4	4	1.4981423	1.4909223	0.0205832	0.0204643	0.0763265	0.0762171
		3	2	1.4932221	1.4859558	0.0210312	0.0209788	0.0772233	0.0772507
		6	10	1.5267491	1.5195304	0.0205188	0.0199979	0.0758570	0.0749515
3	10	4	4	2.6213217	2.5259688	0.6801240	0.7230947	0.2261398	0.2361123
		3	2	2.6730985	2.5682299	0.7367731	0.7678794	0.2334724	0.2412329
		6	10	2.6718121	2.5869710	0.5108487	0.5485381	0.1959878	0.2056542
	15	4	4	2.7437995	2.6706183	0.4711161	0.4926288	0.1848078	0.1909807
		3	2	2.7875524	2.7090140	0.4997176	0.5140019	0.1890740	0.1937068
		6	10	2.7694196	2.7026768	0.3816040	0.4011980	0.1663270	0.1723515
	25	4	4	2.8179653	2.7689534	0.3362385	0.3460326	0.1576348	0.1607414
		3	2	2.8482603	2.7969361	0.3489946	0.3555628	0.1598914	0.1623901
		6	10	2.8301009	2.7840800	0.2929011	0.3021396	0.1471258	0.1502015
	30	4	4	2.8519853	2.8097313	0.2597804	0.2670780	0.1378888	0.1401843
		3	2	2.8786098	2.8346590	0.2677069	0.2726432	0.1397502	0.1413889
		6	10	2.8604432	2.8204351	0.2309395	0.2378329	0.1300095	0.1322869
	50	4	4	2.9017522	2.8747585	0.1691610	0.1722400	0.1107360	0.1119844
		3	2	2.9190935	2.8914238	0.1722482	0.1743647	0.1115669	0.1125170
		6	10	2.9053248	2.8792676	0.1570821	0.1600501	0.1067093	0.1079490
	100	4	4	2.9489458	2.9347340	0.0897422	0.0905575	0.0799472	0.0804804
		3	2	2.9582492	2.9438538	0.0905957	0.0911423	0.0802014	0.0806259
		6	10	2.9499183	2.9359706	0.0863560	0.0871566	0.0784244	0.0789547

Table (3): Estimated, MSE and MAPE Values for Bayes Estimator of θ with Gumbel Type II Prior

θ	n	b	Est.		MSE		MAPE	
			Sq. Error	Prec.	Sq. Error	Prec.	Sq. Error	Prec.
1	10	3	1.2967291	1.2363826	0.1852607	0.1442518	0.3371100	0.2932024
		5	1.4967291	1.4270752	0.3439523	0.2707683	0.5013374	0.4356989
	15	3	1.1958662	1.1578924	0.1011565	0.0837984	0.2490768	0.2253779
		5	1.3291995	1.2869919	0.1711652	0.1412327	0.3421253	0.3056608
	25	3	1.1180574	1.0963455	0.0535847	0.0474047	0.1801743	0.1692578
		5	1.1980574	1.1747919	0.0788739	0.0686745	0.2247187	0.2074985
	30	3	1.0965871	1.0787552	0.0424554	0.0382602	0.1607711	0.1528273
		5	1.1632537	1.1443378	0.0597782	0.0528912	0.1943536	0.1815009
	50	3	1.0613716	1.0509145	0.0239667	0.0223964	0.1208827	0.1169694
		5	1.1013716	1.0905204	0.0304764	0.0279981	0.1372872	0.1311610
	100	3	1.0306285	1.0255137	0.0110300	0.0106429	0.0827263	0.0812884
		5	1.0506285	1.0454144	0.0126551	0.0120544	0.0887714	0.0865823
1.5	10	3	1.7920928	1.7086934	0.3069859	0.2450691	0.2827150	0.2527303
		5	1.9920928	1.8993859	0.4638230	0.3610253	0.3592729	0.3116558
	15	3	1.7029435	1.6488679	0.2003213	0.1713510	0.2286397	0.2119789
		5	1.8362768	1.7779674	0.2722173	0.2264552	0.2708195	0.2445856
	25	3	1.6278325	1.5962211	0.1066872	0.0961297	0.1701916	0.1620731
		5	1.7078325	1.6746676	0.1335404	0.1173799	0.1914299	0.1787416
	30	3	1.6005268	1.5745002	0.0872982	0.0802528	0.1536026	0.1483617
		5	1.6671934	1.6400828	0.1051462	0.0943257	0.1680934	0.1590391
	50	3	1.5618286	1.5464407	0.0487945	0.0462466	0.1166262	0.1138163
		5	1.6018286	1.5860466	0.0553407	0.0514939	0.1236530	0.1193977
	100	3	1.5330866	1.5254781	0.0229278	0.0222661	0.0801310	0.0790427
		5	1.5530866	1.5453789	0.0246513	0.0236762	0.0828712	0.0812787
3	10	3	3.3077182	3.1537855	1.0017587	0.8482574	0.2540611	0.2238300
		5	3.5077182	3.3444780	1.1648459	0.9432726	0.2724594	0.2107902
	15	3	3.2258928	3.1234572	0.6349150	0.5626362	0.2051684	0.1839184
		5	3.3592261	3.2525567	0.7129309	0.6111794	0.2168081	0.1773281
	25	3	3.1161211	3.0556081	0.3936951	0.3686798	0.1636887	0.1568511
		5	3.1961211	3.1340546	0.4186745	0.3835581	0.1673859	0.1525753
	30	3	3.1038504	3.0533780	0.2986101	0.2813897	0.1447907	0.1377458
		5	3.1705171	3.1189606	0.3169012	0.2926921	0.1480761	0.1350829
	50	3	3.0558573	3.0257496	0.1823437	0.1763725	0.1128111	0.1105019
		5	3.0958573	3.0653555	0.1884123	0.1799808	0.1141304	0.1089866
	100	3	3.0274142	3.0123897	0.0931938	0.0916805	0.0804370	0.0797486
		5	3.0474142	3.0322905	0.0946903	0.0925696	0.0808840	0.0791090

Table (4): Estimated, MSE and MAPE Values for Bayes Estimator of θ with Jeffrey's Prior

θ	n	Est.		MSE		MAPE	
		Sq. Error	Prec.	Sq. Error	Prec.	Sq. Error	Prec.
1	10	1.1074767	1.0506447	0.1315668	0.1105789	0.2773428	0.2569280
	15	1.0669995	1.0308194	0.0765726	0.0682280	0.2174794	0.2068347
	25	1.0396431	1.0186381	0.0445915	0.0416465	0.1665980	0.1622167
	30	1.0309521	1.0136240	0.0364084	0.0344543	0.1505400	0.1470861
	50	1.0218077	1.0115381	0.0215087	0.0207456	0.1156227	0.1139861
	100	1.0107358	1.0056695	0.0104120	0.0102259	0.0805936	0.0799746
1.5	10	1.6578809	1.5728039	0.2985902	0.2515979	0.2818273	0.2623490
	15	1.6102966	1.5556943	0.1948461	0.1736039	0.2276709	0.2170554
	25	1.5706589	1.5389251	0.1030244	0.0956256	0.1688014	0.1635494
	30	1.5522691	1.5261787	0.0853401	0.0805397	0.1537007	0.1504071
	50	1.5324781	1.5170759	0.0478808	0.0461811	0.1161544	0.1144385
	100	1.5182692	1.5106588	0.0226102	0.0221673	0.0797214	0.0790295
3	10	3.3419091	3.1704133	1.2367391	1.0368942	0.2822901	0.2617293
	15	3.2420280	3.1320966	0.7288565	0.6430432	0.2198233	0.2077062
	25	3.1209595	3.0579033	0.4271865	0.3994059	0.1705091	0.1661859
	30	3.1074315	3.0552020	0.3195589	0.3007974	0.1497835	0.1450104
	50	3.0569972	3.0262729	0.1898622	0.1835715	0.1151134	0.1136772
	100	3.0276911	3.0125146	0.0950860	0.0935326	0.0812495	0.0807596

Table (5): Estimated, MSE and MAPE Values for Bayes Estimator of θ with Extension of Jeffrey Prior

θ	n	k	Est.		MSE		MAPE	
			Sq. Error	Prec.	Sq.Error	Prec.	Sq.Error	Prec.
1	10	1	0.9967291	0.9503439	0.0972233	0.0908408	0.2449796	0.2407388
		3	0.7119493	0.6878084	0.1325715	0.1437553	0.3190081	0.3352891
	15	1	0.9958662	0.9642433	0.0628100	0.0601469	0.2005705	0.1985509
		3	0.7862101	0.7663029	0.0848430	0.0917944	0.2479618	0.2603757
	25	1	0.9980574	0.9786758	0.0396510	0.0385770	0.1594937	0.1583709
		3	0.8603943	0.8459329	0.0489541	0.0522189	0.1850584	0.1923466
	30	1	0.9965871	0.9803813	0.0331380	0.0324427	0.1449568	0.1441493
		3	0.8793415	0.8666885	0.0403489	0.0428255	0.1668293	0.1728843
	50	1	1.0013716	0.9915056	0.0202021	0.0198763	0.1129120	0.1124276
		3	0.9271959	0.9187282	0.0226189	0.0236087	0.1229521	0.1261581
	100	1	1.0006285	0.9956625	0.0100923	0.0100108	0.0795727	0.0793823
		3	0.9621428	0.9575502	0.0107637	0.0110436	0.0834775	0.0847681
1.5	10	1	1.4920928	1.4226546	0.2217303	0.2074984	0.2502248	0.2455746
		3	1.0657806	1.0296418	0.3016423	0.3267929	0.3194136	0.3360311
	15	1	1.5029435	1.4552187	0.1591439	0.1511947	0.2100975	0.2069680
		3	1.1865343	1.1564906	0.1974448	0.2122235	0.2537522	0.2652806
	25	1	1.5078325	1.4785514	0.0904074	0.0873312	0.1598033	0.1576935
		3	1.2998556	1.2780077	0.1071996	0.1141843	0.1810182	0.1880813
	30	1	1.5005268	1.4761263	0.0771929	0.0752725	0.1483118	0.1475531
		3	1.3239942	1.3049429	0.0910761	0.0964282	0.1686016	0.1741877
	50	1	1.5018286	1.4870319	0.0449750	0.0442581	0.1132632	0.1127298
		3	1.3905820	1.3778823	0.0505283	0.0527677	0.1230282	0.1261092
	100	1	1.5030866	1.4956270	0.0218426	0.0216361	0.0785541	0.0782868
		3	1.4452755	1.4383768	0.0231807	0.0237911	0.0821241	0.0833443
3	10	1	3.0077182	2.8677468	0.9071278	0.8420984	0.2490088	0.2447265
		3	2.1483701	2.0755227	1.1880633	1.2865955	0.3165818	0.3333847
	15	1	3.0258928	2.9298081	0.5845579	0.5523214	0.1996128	0.1957630
		3	2.3888627	2.3283753	0.7374076	0.7968025	0.2430313	0.2554565
	25	1	3.0061211	2.9379385	0.3802260	0.3694391	0.1634844	0.1629612
		3	2.5828630	2.5394505	0.4565620	0.4852460	0.1882041	0.1951234
	30	1	3.0038504	2.9550041	0.2878400	0.2805651	0.1434657	0.1422221
		3	2.6504563	2.6123182	0.3462661	0.3679800	0.1619501	0.1679385
	50	1	2.9958573	2.9663407	0.1792408	0.1768424	0.1128616	0.1125327
		3	2.7739419	2.7486086	0.2047576	0.2140593	0.1228963	0.1260811
	100	1	2.9974142	2.9825386	0.0924489	0.0918319	0.0805002	0.0804162
		3	2.8821290	2.8683718	0.0993616	0.1019800	0.0850076	0.0863490