

One Step Continuous Hybrid Block Method for the Solution of $y''' = f(x, y, y', y'')$.

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Abstract

In this paper, we present a block method for the direct solution of third order initial value problems of ordinary differential equations. Collocation and interpolation approach was adopted to generate a continuous linear multistep method which was then solved for the independent solution to give a continuous block method. We evaluated the result at selected grid points to give a discrete block which eventually gave simultaneous solutions at both grid and off grid points. The one-step block method is consistent and A -stable, with good region of absolute stability. Experimental results confirmed the superiority of the new scheme over an existing method.

Keywords: consistent, convergent, collocation, hybrid points, independent solution, interpolation, zero stable

1. Introduction

We consider the solution to general third order initial value problem of the form

$$y''' = f(x, y, y', y''), \quad y^k(x_0) = y_0^k, \quad k = 0,1,2 \quad y \varepsilon \mathrm{R} \tag{1}$$

where x_0 is the initial point and f is continuous within the interval of integration and satisfies the existence and uniqueness condition.

Many real life problems in sciences, engineering, biology and social sciences are model of third order ordinary differential equations. Some of these models do not always have theoretical solutions, thus numerical methods are often employed to solve them. Researchers in most cases always use method of reduction of higher order ODEs into system of first order ODEs to solve (1). This technique though quite good, is bedeviled with many problems such tediousness, complexity of the method, waste of time, and the need for large computer storage memories because of too many auxiliary functions, etc. This approach has been extensively discussed in the literature (see Awoyemi 2001, Awoyemi & Kayode 2005). To address the challenges associated with method of reduction to system of first order, scholars developed direct methods for higher order ODEs among whom are Awoyemi *et al.* (2011), Kayode & Obaruah (2013), Adesanya *et al.* (2013), Jator (2007) and Anake *et al.* (2012). When the problem to be solved is stiff and oscillatory, Yakubu *et al.* (2013) proposed hybrid methods. A three-step p-stable collocation method for direct third order ODEs was also developed by Yahaya (2007). Some of the schemes can only solve special third order differential equations of the form $y''' = f(x), \quad y'' = f(y)$ and the level of accuracy is not high enough.

In order to avoid the limitation of the methods mentioned above, scholars adopted predictor -corrector approach, which has also been reported to be inefficient because the cost of developing separate predictors, human and computer time involved in the execution of the method are too costly (James *et al.* 2013, Adesanya *et al.* 2013). Scholars later developed block methods in which approximations are simultaneously generated at different grid points in the interval of integration and is less expensive in terms of the number of function evaluations compared to linear multistep methods. This assertion has been reported by Jator (2010), Adesanya *et al.* (2012), Anake *et al.* (2012) and Awoyemi *et al.* (2011).

In this paper, we developed a one step with four off-grid points implemented in block method. The grid points were carefully selected to ensure zero stability of the new method.

2. Derivation of the Method

We define the general power series approximate solution in the form:

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \tag{2}$$

where r and s are the numbers of interpolation and collocation points respectively. The third derivative of (2) gives

$$y'''(x)=\sum_{j=2}^{r+s-1} j(j-1)(j-2)a_j x^{j-3} \tag{3}$$

Substituting (3) into (1) gives

$$f(x,y,y',y'')=\sum_{j=2}^{r+s-1} j(j-1)(j-2)a_j x^{j-3} \tag{4}$$

Interpolating (2) at $x_{n+r}, r=0,\frac{1}{5},\frac{2}{5}$ and collocating (3) at $x_{n+s}, s=0(\frac{1}{5})1$ results into a system of nonlinear equation

$$AX=U \tag{5}$$

where

$A=\begin{bmatrix}a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8\end{bmatrix}^T$

$U=\begin{bmatrix}y_n & y_{n+\frac{1}{5}} & y_{n+\frac{2}{5}} & f_n & f_{n+\frac{1}{5}} & f_{n+\frac{2}{5}} & f_{n+\frac{3}{5}} & f_{n+\frac{4}{5}} & f_{n+1}\end{bmatrix}^T,$

and

$$X=\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8\\
1 & x_{n+\frac{1}{5}} & x_{n+\frac{1}{5}}^2 & x_{n+\frac{1}{5}}^3 & x_{n+\frac{1}{5}}^4 & x_{n+\frac{1}{5}}^5 & x_{n+\frac{1}{5}}^6 & x_{n+\frac{1}{5}}^7 & x_{n+\frac{1}{5}}^8\\
1 & x_{n+\frac{2}{5}} & x_{n+\frac{2}{5}}^2 & x_{n+\frac{2}{5}}^3 & x_{n+\frac{2}{5}}^4 & x_{n+\frac{2}{5}}^5 & x_{n+\frac{2}{5}}^6 & x_{n+\frac{2}{5}}^7 & x_{n+\frac{2}{5}}^8\\
0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5\\
0 & 0 & 0 & 6 & 24x_{n+\frac{1}{5}} & 60x_{n+\frac{1}{5}}^2 & 120x_{n+\frac{1}{5}}^3 & 210x_{n+\frac{1}{5}}^4 & 336x_{n+\frac{1}{5}}^5\\
0 & 0 & 0 & 6 & 24x_{n+\frac{2}{5}} & 60x_{n+\frac{2}{5}}^2 & 120x_{n+\frac{2}{5}}^3 & 210x_{n+\frac{2}{5}}^4 & 336x_{n+\frac{2}{5}}^5\\
0 & 0 & 0 & 6 & 24x_{n+\frac{3}{5}} & 60x_{n+\frac{3}{5}}^2 & 120x_{n+\frac{3}{5}}^3 & 210x_{n+\frac{3}{5}}^4 & 336x_{n+\frac{3}{5}}^5\\
0 & 0 & 0 & 6 & 24x_{n+\frac{4}{5}} & 60x_{n+\frac{4}{5}}^2 & 120x_{n+\frac{4}{5}}^3 & 210x_{n+\frac{4}{5}}^4 & 336x_{n+\frac{4}{5}}^5\\
0 & 0 & 0 & 6 & 24x_{n+1} & 60x_{n+1}^2 & 120x_{n+1}^3 & 210x_{n+1}^4 & 336x_{n+1}^5
\end{bmatrix}$$

Solving (5) for the unknown constant $a_j^{'s}$ using Gaussian elimination method and substituting back into (2) gives a continuous hybrid linear multistep method in the form

$$y(x)=\alpha_0 y_n+\alpha_{\frac{1}{5}} y_{n+\frac{1}{5}}+\alpha_{\frac{2}{5}} y_{n+\frac{2}{5}}+h^3\left[\sum_{j=0}^{1}\beta_j f_{n+j}+\beta_v f_{n+v}\right],\ v=\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5} \tag{6}$$

where

$$\alpha_0=\frac{25}{2}t^2-\frac{15}{2}t+1$$

$$\alpha_{\frac{1}{5}}=10t-25t^2$$

$$\alpha_{\frac{2}{5}}=\frac{25}{2}t^2-\frac{5}{2}t$$

$$\beta_0 = -\frac{1}{1008000}(78125t^8 - 375000t^7 + 743750t^6 - 787500t^5 + 479500t^4 - 168000t^3 + 31075t^2 - 2286t)$$

$$\beta_{\frac{1}{5}} = \frac{1}{1008000}(390625t^8 - 1750000t^7 + 3106250t^6 - 2695000t^5 + 1050000t^4 - 92565t^2 + 13538t)$$

$$\beta_{\frac{1}{5}} = -\frac{1}{504000}(390625t^8 - 1625000t^7 + 2581250t^6 - 1872500t^5 + 525000t^4 - 20745t^2 + 2218t)$$

$$\beta_{\frac{3}{5}} = \frac{1}{504000}(390625t^8 - 1500000t^7 + 2143750t^6 - 1365000t^5 + 350000t^4 - 13465t^2 + 1482t)$$

$$\beta_{\frac{4}{5}} = -\frac{1}{1008000}(390625t^8 - 1375000t^7 + 1793750t^6 - 1067500t^5 + 262500t^4 - 9825t^2 + 1082t)$$

$$\beta_1 = \frac{1}{201600}(15625t^8 - 50000t^7 + 61250t^6 - 35000t^5 + 8400t^4 - 309t^2 + 34t)$$

Solving (6) for the independent solution at the selected grid points give a continuous block method of the form

$$y_{n+k}^{(m)} = \sum_{m=1}^{2} \frac{(kh)^m}{m!} y_n^m + h^3 \left[\sum_{j=0}^{1} \sigma_j \ f_{n+j} + \sigma_v \ f_{n+v} \right], v = \frac{1}{5}(\frac{1}{5})\frac{4}{5} \tag{7}$$

where

$$\sigma_0 = -\frac{1}{8064}(625t^8 - 3000t^7 + 5950t^6 - 6300t^5 + 3836t^4 - 1344t^3)$$

$$\sigma_{\frac{1}{5}} = \frac{1}{8064}(3125t^8 - 14000t^7 + 24850t^6 - 21560t^5 + 8400t^4)$$

$$\sigma_{\frac{2}{5}} = -\frac{1}{4032}(3125t^8 - 13000t^7 + 20650t^6 - 14980t^5 + 4200t^4)$$

$$\sigma_{\frac{3}{5}} = \frac{1}{4032}(3125t^8 - 12000t^7 + 17150t^6 - 10920t^5 + 2800t^4)$$

$$\sigma_{\frac{4}{5}} = -\frac{1}{8064}(3125t^8 - 11000t^7 + 14350t^6 - 8540t^5 + 2100t^4)$$

$$\sigma_1 = \frac{1}{8064}(625t^8 - 2000t^7 + 2450t^6 - 1400t^5 + 336t^4)$$

Evaluating (7) at $t = \frac{1}{5}(\frac{1}{5})1$ gives a discrete block method of the form

$$A^0 Y_m^{(i)} = \sum_{j=0}^{2} h^1 e_i y_n^{(i)} + h^{(3-i)} \left[d_i \ f(y_n) + b_i F(Y_m) \right] \tag{8}$$

where

$$Y_m^{(i)} = \left[y_{n+\frac{1}{5}}^{(i)}, y_{n+\frac{2}{5}}^{(i)}, y_{n+\frac{3}{5}}^{(i)}, y_{n+\frac{4}{5}}^{(i)}, y_{n+1}^{(i)} \right]^T,$$

$$F(Y_m) = \left[f_{n+\frac{1}{5}}, f_{n+\frac{2}{5}}, f_{n+\frac{3}{5}}, f_{n+\frac{4}{5}}, f_{n+1} \right]^T,$$

$$y_n^{(i)} = \left[y_{n-\frac{1}{5}}^{(i)}, y_{n-\frac{2}{5}}^{(i)}, y_{n-\frac{3}{5}}^{(i)}, y_{n-\frac{4}{5}}^{(i)}, y_n^{(i)} \right]^T$$

A^0 is a 5X5 identity matrix.

When $i = 0$,

$$e_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{50} \\ 0 & 0 & 0 & 0 & \frac{2}{25} \\ 0 & 0 & 0 & 0 & \frac{9}{50} \\ 0 & 0 & 0 & 0 & \frac{8}{25} \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

$$d_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{3929}{5040000} \\ 0 & 0 & 0 & 0 & \frac{317}{78750} \\ 0 & 0 & 0 & 0 & \frac{783}{80000} \\ 0 & 0 & 0 & 0 & \frac{712}{39375} \\ 0 & 0 & 0 & 0 & \frac{233}{8064} \end{bmatrix}, \quad b_0 = \begin{bmatrix} \frac{199}{201600} & \frac{-1931}{2520000} & \frac{173}{360000} & \frac{-883}{5040000} & \frac{139}{5040000} \\ \frac{367}{39375} & \frac{-38}{7875} & \frac{122}{39375} & \frac{-89}{78750} & \frac{1}{5625} \\ \frac{16119}{560000} & \frac{-2187}{280000} & \frac{423}{56000} & \frac{-1539}{560000} & \frac{243}{560000} \\ \frac{2336}{39375} & \frac{-32}{5625} & \frac{704}{39375} & \frac{-8}{1575} & \frac{32}{39375} \\ \frac{815}{8064} & \frac{5}{4032} & \frac{155}{4032} & \frac{-5}{1152} & \frac{11}{8064} \end{bmatrix},$$

When $i = 1$,

$$e_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \; d_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1231}{126000} \\ 0 & 0 & 0 & 0 & \frac{71}{3150} \\ 0 & 0 & 0 & 0 & \frac{123}{3500} \\ 0 & 0 & 0 & 0 & \frac{376}{7875} \\ 0 & 0 & 0 & 0 & \frac{61}{1008} \end{bmatrix}$$

$$b_1 = \begin{bmatrix} \frac{863}{50400} & \frac{-761}{63000} & \frac{941}{126000} & \frac{-341}{126000} & \frac{107}{252000} \\ \frac{544}{7875} & \frac{-37}{1575} & \frac{136}{7875} & \frac{-101}{15750} & \frac{8}{7875} \\ \frac{3501}{28000} & \frac{-9}{3500} & \frac{87}{2800} & \frac{-9}{875} & \frac{9}{5600} \\ \frac{1424}{7875} & \frac{176}{7875} & \frac{608}{7875} & \frac{-16}{1575} & \frac{16}{7875} \\ \frac{475}{2016} & \frac{25}{504} & \frac{125}{1008} & \frac{25}{1008} & \frac{11}{2016} \end{bmatrix}$$

When $i = 2$,

$$e_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, d_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{19}{288} \\ 0 & 0 & 0 & 0 & \frac{14}{225} \\ 0 & 0 & 0 & 0 & \frac{51}{800} \\ 0 & 0 & 0 & 0 & \frac{14}{225} \\ 0 & 0 & 0 & 0 & \frac{19}{288} \end{bmatrix} \quad b_2 = \begin{bmatrix} \frac{1427}{7200} & \frac{-133}{1200} & \frac{241}{3600} & \frac{-173}{7200} & \frac{3}{800} \\ \frac{43}{150} & \frac{7}{225} & \frac{7}{225} & \frac{-1}{75} & \frac{1}{450} \\ \frac{219}{800} & \frac{57}{400} & \frac{57}{400} & \frac{-21}{800} & \frac{3}{800} \\ \frac{64}{225} & \frac{8}{75} & \frac{64}{225} & \frac{14}{225} & 0 \\ \frac{25}{96} & \frac{25}{144} & \frac{25}{144} & \frac{25}{96} & \frac{19}{288} \end{bmatrix}$$

3. Implementation of the Method

Writing equation (8) in the generalized form

$$Y_m = Ey_n + h^{\mu} DF(y_n) + h^{\mu} BF(Y_m) \tag{9}$$

where $Y_m = \left[y_{n+1,} y_{n+2,...,} y_{n+k} \right]^T$, μ is the order of the differential equation, k is the step length, E, D and B are matrices. We then propose a prediction equation in the form

$$Y_m^{(0)} = Ey_n + \sum_{\lambda=0}^{2} h^{\mu+\lambda} F^{\lambda}(y_n) \tag{10}$$

where $F^{(\lambda)}(y_n) = \frac{\partial^{\lambda}}{\partial x^{\lambda}} f\left(x, y, y'\right)_{y_n}$. Substituting (9) into (8) gives

$$Y_m = Ey_n + h^{\mu} DF(y_n) + h^{\mu} BF(y_m^{(0)}) \tag{11}$$

Equation (11) is our non-self-starting block method since the prediction equation is not obtained directly from the block formula (Awoyemi *et al*. 2011).

4.0 Basic Properties of the Developed Method

4.1 Order of the block

Let the linear operator $L\{y(x):\ h\}$ on (7) be

$$L\{y(x):\ h\} = \mathbf{A}^0 y_m^{(i)} - \sum_{i=0}^{2} h^i e_i y_n^{(i)} - h^{3-i}\left[df(y_n) + bF(y_m)\right] \tag{12}$$

Expanding y_{n+j} and f_{n+j} in Taylor series and comparing the coefficients of h gives

$$L\{y(x):\ h\} = C_0 y(x) + C_1 y'(x) + ... + C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x) + C_{p+2} h^{p+2} y^{p+2}(x) + ...$$

Definition: *The linear operator* L *and associated block method are said to be of order* p *if* $C_0 = C_1 = ...$, $= C_p = C_{p+1} = 0, C_{p+2} \neq 0$. C_{p+2} *is called the error constant and implies that the truncation error is given by* $t_{n+k} = C_{p+2} h^{p+2} y^{p+2}(x) + O\left(h^{p+3}\right)$.

Comparing the coefficients of h, the order of the method is five with error constant

$$c_7 = \left[\tfrac{-9807}{7087500000000}, \tfrac{-491}{55371093750}, \tfrac{-1917}{87500000000}, \tfrac{-1136}{27685546875}, \tfrac{-149}{2268000000}\right]^T$$

4.2 Consistency

A method is said to be consistent, if it has order greater than one.

From the above analysis, it is obvious that our method is consistent.

4.3 Zero stability

A block method is said to be zero stable as $h \to 0$ if the roots of the first characteristics polynomial $\rho(r) = 0$ satisfy $\left\| \left[\sum A^0 R^{k-1} \right] \right\| \leq 1$, and those roots with $|R| = 1$ must be simple.

For our method,

$$\rho(r)=\left| r\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\right|=0$$

$r^4(r-1)=0 \Leftrightarrow r=0,0,0,0,1$ showing that our method is zero stable.

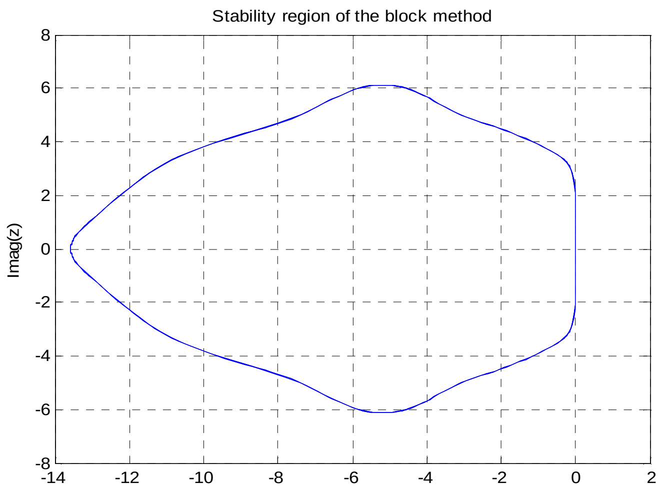
4.4 Convergence

Definition: *The necessary and sufficient conditions for a linear multistep method to be convergent are that it must be consistent and zero stable.*

Hence our method is convergent.

4.5 Stability region

The method (6) is said to be absolutely stable if for a given h, all roots z_s of the characteristics polynomial $\pi(z,h)=\rho(z)+h^3\sigma(z)=0$, satisfies $|z_s|<1, s=1,2,...,n,$ where $h=-\lambda^3h^3$ and $\lambda=\frac{\partial f}{\partial y}$. The boundary locus method is adopted to determine the region of absolute stability. Substituting the test equation $y'''=-\lambda^3 y,\ y''=-\lambda^2 y$ and $y'=\lambda y$ into the polynomial gives the stability region as shown in fig. 1



5.0 Numerical Experiments

5.1 Numerical Examples

Problem 1. Solve $y''' = 3\cos x$ such that $y(0)=1, y'(0)=0, y''(0)=2, 0\le x\le 1, h=0.05.$

Exact Solution: $y(x)=x^2+3x-3\sin x+1$.

Source: Agam & Irhebbhude (2011) (see the result is shown in Table 1).

Problem 2. Solve $y'''+y'=0$ such that $y(0)=0, y'(0)=1, y''(0)=2, x\in[0,1], h=0.05.$

Exact Solution: $y(x)=2(1-\cos x)+\sin x$.

Source: Agam & Irhebbhude (2011).

The result is shown in Table 2

$$\text{Error} = |\text{Exact result} - \text{Computed result}|$$

AI = Error in Agam & Irhebbhude (2011)
YB = Error in Yahaya and Badmus (2008)
NM= Error in New Method

Table 1: Comparison of absolute errors for Problem I

X	Error in NM	Error in AI	Error in YB
0.1	2.220(–16)	6.400(–08)	1.040(–06)
0.2	1.332(–15)	1.260(–07)	5.060(–06)
0.3	3.997(–15)	1.520(–07)	1.210(–05)
0.4	9.548(–15)	2.130(–07)	2.220(–05)
0.5	1.799(–14)	2.730(–07)	3.530(–05)

Table 2: Comparison of absolute errors for Problem 2

X	Error in NM	Error in AI	Error in YB
0.1	2.065(–14)	3.530(–07)	1.54055(–09)
0.2	1.950(–11)	5.170(–07)	9.84550(–09)
0.3	8.094(–11)	5.471(–07)	2.36528(–08)
0.4	1.964(–10)	4.860(–07)	4.32732(–08)
0.5	3.702(–10)	7.971(–07)	3.90181(–08)

Note $a(b) := a \times 10^{-b}$.

6.0 Discussion of Result

Two numerical examples were used to test the efficiency of our developed scheme. Agam & Irhebbhude (2011) solved Problems1 and 2 where they developed modified Runge-Kutta methods with step size $h = 0.1$. Our block method competes favourably with this method and with less computational cost. The accuracy of the method is demonstrated in Tables 1 and 2.

6.1 Conclusion

An improved block method for direct solution of third order ordinary differential equations has been developed and implemented in this paper. The good convergent and stability properties of our method make it more attractive for numerical integrator of linear initial value problems of third order ordinary differential equations. Its accuracy and effectiveness are shown clearly in tables 1 and 2. Our method proves to be a good estimate of the exact solution for the test examples.

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