Use of Counters for Simplifying The Teaching Of Number Bases In Secondary Schools

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Abstract
The notion of the pedagogical sequence of "concrete to abstract" is supported by educational theories, research, and practice, especially in mathematics and science education. But in the teaching of number bases, many teachers violate this pedagogical sequence and teach the topic abstractly. This has created difficulties for students and teachers in trying to study and understand the topic. Many teachers teach number bases abstractly by just stating the algorithms for the conversions from one base to the other without concrete materials to practicalise the concepts. In this paper, focus is on how to simplify the teaching and learning of number bases in schools using counters. Also, model strategies for carrying out basic operations in various number bases are discussed. The paper recommends that teachers should use counters to simplify the teaching of number bases in schools

Key words: Counters, Number Bases, Teaching

1. Introduction

Of all subjects in the school curriculum, mathematics is the most international. The understanding of mathematics influences decision making in all aspects of life including private, social and civil. Mathematics education is a key to increasing all opportunities in life after school, but unfortunately many students now struggle with its study and become disaffected as the continually encounter problems in the study of the subject. Apparently, it becomes imperative that teachers understand what effective mathematics teaching is expected to be and what teachers can do to key into pattern of effective teaching. One of the strategies for effective teaching is the use of teaching aids which helps to concretize mathematical concepts. The focus of this paper is to examine the strategies for effective mathematics teaching and practices that teachers can apply in the teaching of number bases.

In this era of educational development, there is the agreement that effective mathematics instruction requires the use of concrete materials. Concrete materials are objects that students can grasp with their hands (Douglas,1999). Counters are forms of concrete material. This sensory nature ostensibly makes manipulative "real", connected with one's intuitively meaningful personal self, and therefore helpful. Supporting this, Confucius in Thompson (1994) said that Learning without thought is labor lost. Also, Dewey in Thompson asserted that an experience is not a true experience until it is reflective. The use of concrete materials in teaching has been supported by Bruner's theorem. Concrete materials are used for two purposes. First, they enable the teacher and the students to have grounded conversations. Their use provides something "concrete" about which teacher and students can talk. Second, concrete materials provide something on which students can act (Thompson,1994).

The use of manipulative improves the academic achievement of students in mathematics topic because it "makes sense" for the topic. Students who use manipulative materials in their mathematics classes usually outer perform those who do not, though the benefits may be slight. These benefits hold across grade level, ability level, and topic. Manipulative use also increases scores on retention and problem solving tests. Attitudes towards mathematics are improved when students have instruction with concrete materials provided by teachers knowledgeable about their use (Douglas, 1999). Suydam and Higgins in ERIC Education Resources Information Centre (2012) studied the use of manipulative material in grade K-8 pupils and found that lessons using manipulative materials have higher probability of producing greater mathematical achievement than do non manipulative lessons. Use of manipulative materials and pictorial representations is highly effective, symbolic treatments alone are less effective. Also, Moyer-Packenham and Suh in ERIC Education Resources Information Centre (2012) examined the influence of virtual manipulative materials on different achievement groups during a teaching experiment in fifth grade classroom. Results of the study revealed that there was over all gain following the treatment. Follow –up paired samples individual t-test on the low, average, and high achieving groups indicated a statistically significant gain for students in the low achieving group, but only numerical gains for students in the average achieving and high achieving groups.

2. Number Bases.

A number is simply an idea or a concept of quantity. For instance, the number two(2) represents the concept of quantities that are two e.g. two boys, two cars, two legs, etc. But different cultures have different ways of representing two. Most Nigerian traditional communities used strokes (//) to represent two. Some even used objects such as stones. The Arabs represent two with the symbol 2 while the Romans represent two with ii. These symbols used to represent numbers are called numerals.

As there are various ways of representing numbers numerically, different cultures have various ways of grouping objects or quantities for the purpose of counting. For instance, many tribes in Nigeria count in fours, fives, sevens, tens, twelves, twenties and sixties. In most communities, there usually four market days and seven days in a week. In local Ibo markets in Nigeria, yams are sold in fives, tens and twenties. These systems of grouping numbers or systems of counting numbers are called number bases (Azuka, 2012). Students are very familiar with base 10 counting where we group objects in tens. For instance:

12 = 1 ten + 2 =10+ 2 = ////////// //
25 = 2 tens + 5 = 20 + 5 = //////////, //////////, /////
46 = 4 tens + 6 = 40 +6 = //////////, //////////, //////////, //////////, //////
127 = 1 ten square, 2 tens and 7 = 100+ 20 +7
1425 = 1 ten cube, 4 ten square, 2 tens and 5
$= 1 \times 10^3 + 4 \times 10^2 + 2 \times 10 + 5$
$= 1000 + 400 + 20 + 5$

In general whole numbers in base 10 system are represented in the following way(Ward,2007):

$$a = a_n \cdot 10^n + a_{n-1} \cdot 10^{n-1} + a_{n-2} \cdot 10^{n-2} + \ldots\ldots + a_2 \cdot 10^2 + a_1 \cdot 10^1 + a_0 \cdot 10^0 \quad \ldots\ldots(1)$$

where "a" is a number, such as 1425, and the a_n to a_0 are numbers between 0 and 9

While this base ten looks trivial for teachers and students, they experience difficulties in the teaching and learning of other number bases. Many teachers usually resort to teaching number bases by talk and chalk method and the students only try to memorizing the algorithms for conversion to base ten and conversion from base ten to other number bases. As a matter of fact, students see number bases as being abstract and therefore fail to understand the concepts. Their lack of understanding usually manifests in trying to carry out the basic operations such as addition, subtraction, multiplication and division in other number bases apart from base ten. For instance, just ask the students to convert 45 in base ten to a number in base eight without the usual pen and paper algorithms. Many students cannot do this due to the method used in teaching them. These result in poor performance of students in questions involving number bases in school and public examinations.

Concrete materials such as counters can be used to practicalise the teaching of number bases for proper understanding by students. The author has tried this in the retraining of teachers in Nigeria and teachers have described it as a veritable strategy for effective teaching of number bases in schools. See picture below. With counters in teaching number bases in workshops, many teachers were excited and also confessed that they never understood number base before the workshop. The teachers expressed deeper understanding and comprehension of number bases and the basic operations using number bases after the use of counters to learn number bases. According to Bruner in Nnachi (2008) learning is carried out if the learner is given the opportunity to discover the fact by himself. This implies that cognition is achieved as individuals interacts with the environment and explores the materials in the environment. In teaching number bases students should be given opportunity to use concrete materials such as counters to develop the concept of number bases.

Fig. 1. Participants at a workshop in Nigeria being shown how to use counters to convert 14_{10} to 32_4 (3 fours and 2) using counters.

Fig.2. $8_{10} = 22_{3.}$ (2 threes and 2)

Fig.3. $14_{10} = 32_4$ (Three fours and two)

To teach this in the class every student is expected to have at least twenty counters each. Then the teacher should lead the students to group the counters into various number bases. For instance, ask every student to bring out five counters and arrange them in groups of (i) base 3 (ii) base 4 (iii) base 5 (iv) base 6.

Below are the arrangements using counters. Here the strokes represent the counters:

(i) 5_{10} = /// // =1 three and 2 = 12_3
(ii) 5_{10} = //// / = 1 four and 1 = 11_4
(iii) 5_{10} = ///// = 1 five and 0 = 10_5
(iv) 5_{10} = ///// = zero six and 5 units = 5_6

Next ask students to take 12 counters and arrange them in groups of (i) base 4 , (ii) 5, (iii) 6, (iv) base 8 (v) base 3

Below are the arrangements using strokes to represent the counters:

(i) 12_{10} = //// //// //// = 3fours and 0 = 30_4
(ii) 12_{10} = ///// ///// // = 2 fives and 2 = 22_5
(iii) 12_{10} = ////// ////// = 2 sixes and 0 = 20_6
(iv) 12_{10} = //////// //// = 1 eight and 4 =14_8
(v) 12_{10} = ///, ///, /// /// = 4 threes and 0 = 40_3 = 110_3 since 4 = 11_3

But this is better illustrated as:

Fig. 4. 12_{10} = 1 three square, 1 three and 0 = 110_3

After this stage lead the students use counters to convert the following numbers in base 10 to other number bases as shown in the table below:

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Base 10	**Base 4**	**Base 6**	**Base 8**
3	ΔΔΔ 3_4	ΔΔΔ 3_6	ΔΔΔ 3_8
4	ΔΔΔΔ 10_4 1 four and 0	ΔΔΔΔ 4_6	ΔΔΔΔ 4_8
5	ΔΔΔΔ 11_4 Δ 1 four and 1	ΔΔΔΔΔ 5_6	ΔΔΔΔΔ 5_8
6	ΔΔΔΔ 12_4 ΔΔ 1 four and 2	ΔΔΔΔΔΔ 10_6 1 six and 0	ΔΔΔΔΔΔ 6_8
8	ΔΔΔΔ 20_4 ΔΔΔΔ 2 fours and 0	ΔΔΔΔΔΔ 12_6 ΔΔ 1 six and 2	ΔΔΔΔΔΔΔΔ 10_8 1 eight and 0
9	ΔΔΔΔ 21_4 ΔΔΔΔ Δ 2 fours and 1	ΔΔΔΔΔΔ 13_6 ΔΔΔ 1 six and 3	ΔΔΔΔΔΔΔΔ Δ 11_8 1 eight and 1
12	ΔΔΔΔ ΔΔΔΔ 30_4 ΔΔΔΔ 3 fours and 0	ΔΔΔΔΔΔ 20_6 ΔΔΔΔΔΔ 2 six and 0	ΔΔΔΔΔΔΔΔ ΔΔΔΔ 14_8 1 eight and 4

Fig .1: *Conversion to other Number Bases using Counters*

From the table, students can discover that number bases are simply grouping in various numbers or quantities. For example,

$14_{10} = 16_8$, (1 eight and 6)

$20_{10} = 32_6$, (3 six and2)

Students can now complete the following orally:

(a) $14_{10} = (\quad)_6$

(b) $25_{10} = (\quad)_7$

(c) $30_{10} = (\quad)_8$

(d) $35_{10} = (\quad)_9$

(e) $18_{10} = (\quad)_5$

This helps to explain the conversion of numbers from base ten to other number bases and also conversions to base ten. It enhances the understanding of the operations in number bases such as addition, subtraction, multiplication and division

Conversion of Number Bases to Base 10

Place values in base ten (H. T U) are simply powers of ten. For instance 1000(TH) $=10^3$, 100(H) $=10^2$, 10(T) = 10^1 , 1(U) = 10^0 .Numbers in various number bases can be written in the expanded form using the powers of the number base. For example,

(a) 25_6 = 2 six and 5 $= 2 \times 6 + 5 = 17_{10}$.

(b) $213_4 = \dfrac{4^2\ 4^1\ 4^0}{2\ \ 1\ \ 3}$ = 2 four square, 1 four and 3

$= 2 \times 4^2 + 1 \times 4 + 3$

$= 32 + 4 + 3$

$= 39_{10}$.

(c) $132_5 = \dfrac{5^2\ 5^1\ 5^0}{1\ \ 3\ \ 2}$ = 1 five square, 3 fives and 2

$= 1 \times 5^2 + 3 \times 5 + 2$

$= 25 + 15 + 2$

$= 42_{10}$.

(d) $213_x = \dfrac{x^2\ x^1\ x^0}{2\ \ 1\ \ 3} = 2 \times x^2 + 1 \times x^1 + 3 \times x^0$

$= (2x^2 + x + 3)_{10}$.

This explains that *the place values of digits in a number base are the powers of the base as shown in (c) and (d) above. This is like TH H T U in base ten.*

Since any number raised to power zero is one, we can rewrite the expansions above as:

(a) $14_5 = 1\times5^1 + 4\times5^0 = 9_{10}$.

(b) $25_6 = 2\times6^1 + 5\times6^0 = 17_{10}$

(c) $213_4 = 2\times4^2 + 1\times4^1 + 3\times4^0 = 39_{10}$.

(d) $132_5 = 1\times5^2 + 3\times5^1 + 2\times5^0 = 42_{10}$.

Generalizing, we can represent numbers in the base, b, as follows (Ward, 2007):

$a = a_n . b^n + a_{n-1} . b^{n-1} + a_{n-2} . b^{n-2} + \ldots\ldots + a_2 . b^2 + a_1 .b^1 + a_0 .b^0$ …………(2)

where **a** is a number, such as 132, and the **a'**s with subscripts are digits in the given number. Also, b is the given number base system

If we divide a number a, by b continuously, we find that the remainders of the division are the digits of our new number base b.

Thus, a number in any number base can be converted to a number in base 10 by expansion as shown above. Generally, we put powers of the number base form zero starting from the unit digit on the right and increase the powers to the left.

3. Conversion of Fractions in other Number Bases to Base 10

We can also **c**onvert decimals to a numbers in base 10. A decimal fraction such as 0.201 can be written in terms of negative powers of 10:

$0.201 = \frac{2}{10} + \frac{0}{10^2} + \frac{1}{10^3}$

Alternatively, we can write it as:

$0.201 = 2.\ 10^{-1} + 0.\ 10^{-2} + 10^{-3}$

In general a fractional number, a, in base b can be written (Ward, 2007):

$a = a_{-1} .b^{-1} + a_{-2} . b^{-2} + a_{-3} .b^{-3} + \ldots$ where b = the given number base system…….(3)

Alternatively, we can write it as:

$a = \frac{a_{-1}}{b^1} + \frac{a_{-2}}{b^2} + \frac{a_{-3}}{b^3} + \ldots\ldots\ldots\ldots\ldots$ ……………………….(4)

For example

(a) $21.12_3 = \frac{3^1\ 3^0\ 3^{-1}\ 3^{-2}}{2\ \ 1\ \ .1\ \ 2} = 2 \times 3^1 + 1 \times 3^0 + 1 \times 3^{-1} + 2 \times 3^{-2}$

$$= 6 + 1 + \frac{1}{3} + \frac{2}{9}$$

$$= 7 + \frac{3+2}{9}$$

$$= 7\frac{5}{9} = 7.555_{10} \ldots$$

(b) $0.121_4 = \frac{4^{-1}\ 4^{-2}\ 4^{-3}}{0.1\ \ 2\ \ 1} = 1 \times 4^{-1} + 2 \times 4^{-2} + 1 \times 4^{-3}$

$$= \frac{1}{4} + \frac{2}{16} + \frac{1}{64}$$

$$0.121_4 = \frac{1}{4} + \frac{2}{16} + \frac{1}{64}$$

$$= \frac{16+8+1}{64} = \frac{25}{64}$$

$$= 0.3900625_{10}.$$

4. Conversion of Numbers in Base 10 to other Numbers Bases

From the arrangements of objects above we can see that:

(i) $9_{10} = 12_7$

(ii) $12_{10} = 14_8$

(iii) $15_{10} = 33_4$

(iv) $20_{10} = 24_8$ and so on.

You notice that we are dividing the numbers in base 10 by the number bases and writing out the remainders. For example, $20 \div 8 = 2$ remainder 4. That is, 2 eight and 4 and written as 24_8.

This could be generalized to convert numbers in base 10 to any other number base. Study the examples below to convert a 142 in base 10 to a number in base 6

Convert 142, base 10 to base 6		
	Number of 6's	Remainder
142	6 x 23 +	4
23	6 x 3 +	5
3	6 x 0 +	3

We read off the new number, base 6 from the remainders starting from below: 354_6. So 142 in base 10 is 354 in base 6.

This is usually arranged in the classroom as shown below:

(a) 142_{10} = ($)_6$.

6	142		
6	23	R	4
6	3	R	5
	0	R	3

The remainders are written from the bottom upwards to make up the answer.

$\therefore$ 142 = $(354)_6$.

(b) 184_{10} = ($)_{9?}$

$\therefore$ 184_{10} = $(224)_9$.

9	184		
9	20	R	4
9 2	R	2	
	0	R	2

5. Conversion of Fractions in other Number Bases to Base 10

We seen that in general a fractional number, a, in base b can be written:

$a = a_{-1} .b^{-1} + a_{-2} . b^{-2} + a_{-3} .b^{-3} + \ldots\ldots$, ………(3, repeated)

where b is any number base.

Alternatively, we can write it as:

$a = \frac{a_{-1}}{b^1} + \frac{a_{-2}}{b^2} + \frac{a_{-3}}{b^3} + \ldots\ldots$ ……………………….(4, repeated) where b is any number base (Ward, 2007).

Therefore, in converting a decimal in base 10 to a fraction in base b, we do the opposite of converting the whole numbers: we multiply the number in base 10 by b repeatedly, and our remainder is the ***integer*** part of the resulting number starting from the top. We continue to multiply each decimal part by the number base being corrected to until the decimals are no more or zeros.

For example

(a) $(0.25)_{10}$ = ($)_{2?}$

Convert 0.25 Base 10 to Base 2		
	Integer, Number of 2's	
0.25 x 2	0	.5
.5 x 2	1	0

Therefore, $(0.25)_{10} = 0.\ 05_2$

This can be arranged as shown below:

$$2 \times 0.25 = \downarrow 0.50$$
$$2 \times 0.50 = \quad 1.00$$

Write the answer as indicated by the arrow from up to down formed by the digits before the decimal points.

$\therefore$ 0.25_{10} = 0.01_2

(b) $(0.375)_{10}$ = ($)_4$

$$4 \times 0.375 = \downarrow 1.500$$
$$4 \times 0.500 = \quad 2.000$$
$$\therefore 0.375_{10} = 0.12_4.$$

(c) $(20.025)_{10}$ = ($)_5$. First change 20_{10} to base 5

You know that 20_{10} = 40_5.

Then convert 0.025_{10} to base 5.

$$5 \times 0.025 = \downarrow 0.125$$

$$5 \times 0.125 = \quad 0.625$$

$5 \times 0.625 = \quad 3.125$

$5 \times 0.125 = \quad 0.625$

$5 \times 0.625 = \quad 3.125$

Notice that the decimals are recurring. So we can stop at any point. Therefore,

$0.025_{10} = (0.00303\ldots)_5$.

Therefore $(20.025)_{10} = (40.00303\ldots.)_5$.

6. Multiplication and Division of Numbers in Various Number Bases

To be able to multiply in the various number bases, you should be able to convert numbers from base 10 to other number bases properly. Also you should be able to add in the various number bases. So students need to revise these again before proceeding in the next unit.

To multiply numbers in other number bases, you should first multiply in base 10 and then correct immediately to the required number base you are working in. For example,

(i) $4_6 \times 5_6$

$4 \times 5 = 20_{10}$ and $20_{10} = 32_6$

$\therefore \quad 4_6 \times 5_6 = 32_6$

(ii) Simplify 23_4 x 32_4

```
      2  3
x     3  2
    1  1  2
  1 3  1
  2 0  2  2₄
```

The multiplication and addition should be done in the given number base.

Students and teachers usually have problems with division in other number bases apart from base ten. This is just because students and teachers find difficulties to comprehend multiplication tables in such other number base. Most teachers have the practice of converting the numbers to base ten first, divide out and then convert back to the required number base. This is really a long and cumbersome procedure. The easier one way out of this problem is to guide the students to work out the multiplication table using the divisor and the digits in the required number first before the division. In this case, the students can easily pick out the digits for the division from the multiplication table developed in the number base. Also, in division of numbers the digits selected to make up the quotient (answer) should be multiplied by the divisor in the given number base. For example, $32_6 \div 5_6 = 4_6$ because $4_6 \times 5_6 = 32_6$.

Let us look at these divisions in other number bases.

(a) $10213_6 \div 25_6$

(b) $13233_4 \div 23_4$

For division, it is faster to first multiply out the divisor by the digits in the number base (Azuka, 2012). For example, in $10213_6 \div 25_6$, multiply out using digits 1-5 since these are the digits in base six.

Digits	1	2	3	4	5
$\times 25_6$	25_6	54_6	123_6	152_6	221_6

Also, in $13233_4 \div 23_4$ multiply out using digits 1-3 since these are the digits in base four.

Digits	1	2	3
$\times 23_4$	23_4	112_4	201_4

These would save you the trouble of deciding the numbers to pick when dividing. Do same for other number bases. Below are the solutions to the above problems.

(*a*)

$$
\begin{array}{r l}
 & 213 \\
25_6 & \overline{)\,1\ 0\ 2\ 1\ 3} \\
 & -5\,4 \qquad \rightarrow (2_6 \times 25_6 = 54_6) \\
 & 4\ 1 \\
 & -2\,5 \qquad \rightarrow (1 \times 25_6 = 25_6) \\
 & 1\ 2\ 3 \\
 & -1\ 2\ 3 \qquad \rightarrow (3_6 \times 25_6 = 123_6) \\
 & 0
\end{array}
$$

$\therefore\ 10213_6 \div 25_6 = 213_6$

(*b*)

$$
\begin{array}{r l}
 & 2\ 3\ 1 \\
23_4 & \overline{)\,1\ 3\ 2\ 3\ 3} \\
 & -1\ 1\ 2 \qquad \rightarrow (2 \times 23_4 = 112_4) \\
 & 2\ 0\ 3 \\
 & -2\ 0\ 1 \qquad \rightarrow (3 \times 23_4 = 201_4) \\
 & 2\ 3 \\
 & -\ 2\ 3 \qquad \rightarrow (1 \times 23_4 = 23_4) \\
 & 0
\end{array}
$$

$\therefore\ 13233_4 \div 23_4 = 231_4$

7. Conclusion

For a long time now many teachers teach number bases abstractly by just stating the stating the algorithms for the conversions from one base to the other. This has created difficulties for both teachers and students in the study of this important topic. Teachers and students are expected to use concrete materials to simplify the study of the topic. The use of counters helps to concretize numbers in the various number bases other than base ten.

When students are exposed to number bases using counters they easily understand and comprehend the various number bases. In particular, this strategy would enable students understand the basis of the usual algorithms for conversion from one base to the other and the operations of addition, subtraction, multiplication and division in the various number bases other than ten. Teachers are encouraged use counters to introduce number bases to students in schools and use the method of division used in this paper.

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