Analysis and Reflections on a Comprehensive Function Problem

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Abstract

This article provides an analysis and reflection on a comprehensive function problem from the college entrance examination mathematics section. The problem involves the combination of logarithmic and polynomial functions, testing concepts such as derivatives, symmetry, and function value limits. Through an in-depth analysis of the problem, we can gain a better understanding of function properties and problem-solving approaches, while also exploring the characteristics of math questions and effective solving strategies. This article aims to offer detailed problem-solving methods for students and serve as a teaching reference for educators.

Keywords: Function, Derivative, Symmetry

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1. Introduction

Given the function $f(x) = \ln x - \ln(2-x) + ax + b(x-1)^3$ **.**

(1) If $b = 0$ and $f'(x) \geq 0$, find the minimum value of a ;

(2) Prove that the curve $y = f(x)$ is centrally symmetric;

(3) If $f(x) > -2$ if and only if $1 < x < 2$, determine the range of values for b .

The question above comes from question 18 of the 2024 National College Entrance Examination Mathematics Paper I.

Function problems are a crucial part of mathematics in college entrance exams, often involving derivatives, symmetry, and function construction. This article uses a comprehensive problem as an example to analyze the problem-solving process in detail and summarizes key knowledge points and techniques. Through analyzing and reflecting on the problem, we can not only master specific solution methods but also enhance our overall understanding and problem-solving skills for function-related questions.

This problem covers topics such as logarithmic functions, polynomial functions and their derivatives, symmetry, and the range of function values. Below is a detailed analysis of each sub-question.

2. Find the minimum value of a under the conditions $b = 0$ and $f'(x) \geq 0$

2.1 Function and Its Derivative

Let $b = 0$ **,** then the function simplifies to

$$f(x) = \ln\frac{x}{2-x} + ax.$$

First, calculate its derivative

$$f'(x)=\frac{d}{dx}\left(\ln\frac{x}{2-x}+ax\right)=\frac{d}{dx}\ln\frac{x}{2-x}+a.$$

By the derivative property of the logarithmic function

$$\frac{d}{dx}\ln\frac{x}{2-x}=\frac{1}{\frac{x}{2-x}}\cdot\frac{d}{dx}\left(\frac{x}{2-x}\right)=\frac{2-x}{x}\cdot\frac{(2-x)-x\cdot(-1)}{(2-x)^2}=\frac{2}{x(2-x)}.$$

Therefore, $f'(x)=\frac{2}{x(2-x)}+a$.

2.2 Analysis of $f'(x)\geq 0$

Assume $f'(x)\geq 0$, i.e., $f'(x)=\frac{2}{x(2-x)}+a\geq 0$ holds for $0<x<2$. We only need to ensure that the minimum value of $\frac{2}{x(2-x)}+a$ within this interval is greater than or equal to zero. Since

$$x(2-x)\leq\left(\frac{2-x+x}{2}\right)^2=1,$$

we have $\frac{2}{x(2-x)}\geq 2$, where equality holds if and only if $x=1$.

Therefore, the minimum value of $f'(x)$ is $2+a$. For $f'(x)\geq 0$ to hold, we require $a+2\geq 0$, then we have $a\geq -2$.

3. Proving that the curve $y=f(x)$ is centrally symmetric

A centrally symmetric figure with respect to a center point (x_0, y_0) satisfies $f(x_0+x)+f(x_0-x)=2y_0$.

The original function is $f(x)=\ln\frac{x}{2-x}+ax+b(x-1)^3$ with domain $(0,2)$. Consider the center point $(1, f(1))$, where $f(1)=a$. We examine $f(1+x)+f(1-x)$:

$$f(1+x)=\ln\frac{1+x}{2-(1+x)}+a(1+x)+b((1+x)-1)^3=\ln\frac{1+x}{1-x}+a(1+x)+bx^3,$$

$$f(1-x)=\ln\frac{1-x}{2-(1-x)}+a(1-x)+b((1-x)-1)^3=\ln\frac{1-x}{1+x}+a(1-x)-bx^3.$$

Simplifying the sum, we obtain $f(1+x)+f(1-x)=2a$. Thus, the curve satisfies the central symmetry condition about the point $(1,a)$.

4. Determining the range of b such that $f(x)>-2$ if and only if $1<x<2$

4.1 Analysis of Function Conditions

From the symmetry of the problem, $f(x)>-2$ if and only if $1<x<2$. This implies that the function equals -2 at $x=1$ or $x=2-\varepsilon$ (where ε is an infinitesimal positive value). By evaluating the limit $\lim_{x\to 2^-}\ln\frac{x}{2-x}=+\infty$, we find $\lim_{x\to 2^-}f(x)=+\infty$. Thus, $f(1)=-2$, which gives

$$\ln\frac{1}{2-1}+a\cdot 1+b(1-1)^2=-2.$$

Thus we have $a = -2$.

4.2 Constructing an auxiliary function

For $1 < x < 2$, the inequality becomes

$$\ln\left(\frac{x}{2-x}\right) - 2(x-1) + b(x-1)^3 > 0.$$

Let $t = x-1$, so $t \in (0,1)$. The inequality transforms to $\ln\left(\frac{t+1}{1-t}\right) - 2t + bt^3 > 0$. Define the auxiliary function $g(t) = \ln\left(\frac{t+1}{1-t}\right) - 2t + bt^3$, $t \in (0,1)$. Compute the derivative of $g(t)$:

$$g'(t) = \frac{2}{1-t^2} - 2 + 3bt^2.$$

Rewrite it with a common denominator: $g'(t) = \frac{t^2(2+3b-3bt^2)}{1-t^2}$.

4.3 Determine the range of values for b

It is required that $g(t) > 0$ for $t \in (0,1)$. Find the range of values for b. Given $K(t) = -3bt^2 + 3b + 2$, it is derived that $\Delta = 36b(b+\frac{2}{3})$. Based on the range of values for b, the situation is analyzed as follows:

(1) $b \geq 0$

At this point, we have $\Delta \geq 0$, $K(0) = 3b + 2 > 0$ and $K(1) = 2 > 0$. Therefore, $K(t) > 0$.

Thus $g'(t) > 0$, the $g(t)$ is an increasing function on $(0,1)$. Therefore, $g(t) > g(0) = 0$ on $(0,1)$, and $g(t)$ is increasing.

(2) $\frac{-2}{3} \leq b < 0$

At this point, $K(t) = -3bt^2 + 3b + 2 \geq 2 + 3b \geq 0$, so $g'(t) \geq 0$ is established, and thus $g(t)$ is monotonically increasing. Therefore, $g(t) > g(0) = 0$ holds for $t \in (0,1)$.

(3) $b < -\frac{2}{3}$

Since $K(0) = 3b + 2 < 0$, $K(1) = 2 > 0$, it follows that there exists a point t_0 in $(0,1)$ such that $K(t_0) = 0$. From $K(t_0) = 0$, we solve for t_0, which gives

$$t_0 = \sqrt{1 + \frac{2}{3b}}.$$

At this point, for $t \in (0, t_0)$, $K(t) < 0$, and for $t \in (t_0, 1)$, $K(t) > 0$. Therefore, $g(t)$ decreases for $t \in (0, t_0)$ and increases for $t \in (t_0, 1)$, so $g(t)$ at $t = t_0$ reaches its minimum value. Note that $g(0) = 0$, so $g(t_0) < g(0) = 0$. Therefore, it does not align with the intended meaning of the question.

In conclusion, $f(x) > -2$ is established in $(1,2)$, and $b \geq -\frac{2}{3}$.

5. Conclusion

For students, solving such problems helps apply mathematical concepts to real-world situations, understanding the role of function changes and dynamic factors. The strict analysis range cultivates rigor and prevents logical errors, while the integration of multiple knowledge points improves technical proficiency and thinking ability. Repeated practice lays the foundation for future study of advanced mathematics or interdisciplinary research.

For teachers, these problems serve as a bridge between basic and advanced thinking. Teachers stimulate interest through diverse teaching strategies, emphasize the rigor of the analysis process, and help students broaden their thinking perspectives, fostering critical thinking. Encouraging students to combine mathematical tools with real-world problems enhances the practicality and purpose of learning, promoting the development of problem-solving skills and innovative thinking.

References

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