Comparative Analysis of Static Differential GPS/GNSS Positioning Using Two and Three or More Receivers

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Abstract
Differential GPS/GNSS positioning gives accurate coordinates as well as positions of observed points on the earth surface. To obtain the accuracy as well as the reliability and the most probable positions of observed points using the post processing software that accompanies the receivers, least squares adjustment technique is applied. Applying least squares adjustment technique requires three or more receivers, that is two or more rovers and one base receiver to be used. Using three or more receivers makes the observation a closed loop observation. But in most cases where the number of points whose positions are to be determined is small, the observations are carried out with only two (rover and base) receivers. Using only two receivers (rover and base) implies that each of the points will be occupied one after the other with respect to the base receiver. In this, least squares adjustment cannot be applied as the observation is not a closed loop one, hence, the accuracy of the observations as well as those of the determined positions and the most probable positions of the observed points cannot be determined. Consequently, this study comparatively analyses the positions determined from static Differential GPS/GNSS observations using two and three or more receivers with a view to obtaining significant differences in the positions of observed points. A control station (ASPXW42A), two new points (ESO 1 and ESO 2), a base receiver and two rover receivers were used in the study. The positions of the new points were first determined with closed loop observation and then with opened loop observation. The two sets of observations were processed with compass post processing software. From the processing results, it was discovered that the reliability of the observations and those of the determined positions of the opened loop observation were not computed. The two sets of coordinates of the observed points were compared and their differences were used to compute the difference magnitudes. The computed difference magnitude of each point was compared with its corresponding 95% confidence level to determine if the difference between the two positions of each point is significant. The comparison results showed that there were significant differences in the positions. It is recommended that differential GPS/GNSS observation should be carried out with three or more receivers.

Keywords: Least Squares Technique, Differential GPS/GNSS, Analysis, Two or More Receivers, Positioning, Significant

1. Introduction
The Global Positioning System (GPS) provides a cost-effective capability that overcomes nearly all the limitations of existing Time and Space Position Information, TSPI sources. GPS is a passive system using satellites, which provides universal and accurate source of real-time position, and timing data to correlate mission events. The coverage area is unbounded and the number of users is unlimited (Sabatini and Palmerini, 2008).

The Global Positioning System (GPS) is a satellite-based navigation system which enables positions and times to be determined with high precision on the earth surface. Position determination using DGPS/GNSS observations involves measurements of ranges using GPS/GNSS receivers from the point of observation to not less than four different satellites in space. GPS receivers can be categorized into two types: single frequency receivers which access the L1 frequency only, while dual frequency receivers which access both L1 and L2 frequencies. The single frequency receivers output the raw C/A code pseudoranges, the L1 carrier phase measurements, and the navigation message, and gives accuracy of about 1cm±2ppm. The dual frequency receivers are the most sophisticated and most expensive receiver type. This type of receiver is capable of measuring all the GPS observables, which are: L1, L2 carriers, C/A code, and P-code, and gives accuracy 5mm±1ppm (El-Rabbany, 2002 and Abdel-Mageed, 2014). The GPS measurements are subjected to some errors, which are affecting the accuracy of the final results. To increase the accuracy as well as the reliability and integrity of the systems, differential methods are applied.

Using the DGPS methods, two or more receivers (base and rover receivers) are used such that the base or reference receiver is fixed at a station whose coordinates are known while the rover receiver(s) is/are moved from one of the unknown/new points to another. In this, the base receiver calculates its position and compares it with the known position to obtain the errors in the measurements. These errors are used to correct the positions/coordinates of the new points as it is assumed that during observation, both, rover and base receivers receive or acquire signals from the same set of satellites (see figure 1). According to Abdalla (2016) Differential
GPS is a system that was designed to eliminate ionosphere, troposphere, satellite clock, and satellite ephemerides introduced to GPS signal for eliminating or drastically reducing errors due to selective availability.

Fig. 1: Differential GPS/GNSS Observation

DGPS/GNSS observation for position determination involves range measurements from the receivers to the satellites in their respective orbits. The processing of these observations involves the application of least squares adjustment technique which in turn requires observation equation to be written for each of the range measurements. This is because the position is to be fixed by resection method which involves the determination of the position of a point by taking observations from the point to not less than two known points. But in the case of satellite observation, a minimum of four satellites are required. Two range measurements from the unknown point are enough to obtain its position but to obtain the most probable values of the coordinates, not less than four satellites are observed and least squares method is applied. Using the relative/differential method, the rover and the base positions are determine with the same method. But as the coordinates of the reference receiver are known, the errors in the observations are determined by comparing the computed and the known coordinates of the base station.

In most cases, when the number of points to observe is small, a rover and a base receivers are used. When only two receivers are used, the accuracy of the observations as well as that of the determined position cannot be evaluated and outputted. This is because the observation is not a closed loop observation. The interaction/communication is between the two receivers only. When three or more receivers, that is, a base and two or more rovers are used, the observation is always a closed loop observation because there are communications between the base receiver and the rover receivers, and between any two rover receivers (see figure 2). In this, least squares adjustment technique is applied to determine the: most probable values of the coordinates/positions and heights of the observed points; standard errors/accuracy of the adjusted coordinates and heights; accuracy of the adjusted observations; relative accuracy of the adjusted observations and present the positional accuracy graphically by computing and plotting error ellipses of the adjusted positions using the post processing software that accompanied the receivers.

Fig. 2: DGPS/GNSS Closed Loop Observation

Here, only a base receiver is mentioned because during the processing of the DGPS/GNSS observations using the accompanied post processing software, the positions of the new points are constrained by fixing the
base station only. This is done to adjust the positions of the new points with respect to that of the base station and to fit the new points to existing network since the base station/control is a part of an existing network. Since the most probable positions of observed points are obtained from the post processing of DGPS/GNSS observations when three or more receivers are employed using the post processing software that accompanied the receivers, there is need to compare the positions of these points when they are occupied one after the other using a base and a rover receivers, and a base and two or more rover receivers. Therefore, this study comparatively analyses the positions determined from static Differential GPS/GNSS observations using two and three or more receivers with a view to obtaining significant differences in the positions of observed points.

1.1 Methods of Differential Positioning
There are two primary variations of the differential measurements and equations. One is based on ranging code measurements and the other is based on carrier-phase measurements (Sabatini and Palmerini, 2008).

1.2 GPS/GNSS Code Measurement
The pseudorange equation corresponds to the geometric distance that would be travelled by the signal in the propagation medium, that is, in a vacuum where there are no clock errors or other biases. Taking these errors and biases into account, the complete expression for the pseudorange takes the form (Kleusberg and Teunissen, 1996, and Bakula, 2010):

\[ P_k^i(t) = \rho_k^i + c[\delta_i(t) - \delta_k^i(t)] + I_k^i + T_k^i + d_{\text{eph}}^i + d_i^k(t) + d_k^i(t) + m_i(t) + \epsilon_i^k \]  

where: \( k \) (\( \rho_k^i \)) - geometric range between the satellite \( k \) (at transmit time) and the receiver \( i \) (at receiver time), computed from ephemeris data and station coordinates; \( \delta_i^k(t) \) - receiver clock error; \( \delta_k^i(t) \) - satellite clock error; \( I_k^i \) - measurement delay due to ionosphere; \( T_k^i \) - measurement delay due to troposphere; \( d_{\text{eph}}^i \) - effect of ephemeris error; \( d_i^k(t) \) - receiver hardware delay; \( d_k^i(t) \) - satellite hardware delay; \( m_i(t) \) - multipath; \( \epsilon_i^k \) - pseudorange measurement error.

1.3 GPS/GNSS Carrier-Phase Measurement
The range between the receiver and satellite can also be obtained through carrier phase measurement. The range would simply be the sum of the total number of full carrier cycles plus fractional cycle at the receiver and the satellite, multiplied by the carrier wave length. The ranges determined with the carriers are more accurate than those obtained by the codes (Kaplan, 1996 and Abdel-Mageed, 2014). This is due to the fact that, the wavelength of the carrier signal (19cm in case of L1) is smaller than the codes. The observation equation of the phase pseudorange as given by Abdel-Mageed (2014) is

\[ \Phi = \rho + c(dt - dT) + \lambda N - d_{\text{ion}} + d_{\text{prop}} + d_{\text{arb}} + \epsilon_p \]  

Where, the measured phase is indicated in meters by \( \Phi \), \( \rho \) is the unknown geometric satellite to receiver range, \( c \) is speed of light which is approximately equal to 300,000 km/s, \( dt \) and \( dT \) are respectively satellite and receiver clock errors, \( \gamma \) is the carrier wavelength, \( N \) is the phase ambiguity, \( d_{\text{ion}} \), \( d_{\text{prop}} \), are the errors due to ionospheric, tropospheric refraction respectively, \( d_{\text{arb}} \) is the orbital error and \( \epsilon_p \) is the combined receiver and multipath noise. Gurtner et al (1989) explained that the bias terms associated with both cases of code pseudorange and carrier phase pseudorange can be eliminated, or minimized by following a certain technique for collecting GPS measurements, such as single, double, and triple differences; and/or using mathematical model; and/or using linear combination.

The model for the computation of the geometric range between the satellite position \((X_s, Y_s, Z_s)\) at the time of signal transmission and the receiver position \((x, y, z)\) at the time signal is received, is given by Ferrao (2013) as:

\[ \rho = \sqrt{(X_s - x)^2 + (Y_s - y)^2 + (Z_s - z)^2} \]  

1.4 Least Squares Adjustment of Differential GPS/GNSS Observation Loop
In order to perform least squares adjustment of a group of observations, the number of observations must be more than the number of the unknowns. Thus, there must be redundant observations. Least squares adjustment technique can be applied if a base receiver and two rover receivers are used simultaneously to determine the positions of two new points with respect to the base receiver. In this case, there are six unknowns and nine observations, hence, the redundant observation is three since at each rover station, there are X, Y and Z, and
there are three baselines each consists of $\Delta X, \Delta Y$ and $\Delta Z$.

The closure errors of the observation loop in X, Y and Z are respectively $cx, cy$ and $cz$. In this case, the algebraic sum of $\Delta X, \Delta Y$ and $\Delta Z$ is equal zero. The closure errors are obtained by (Akwasi, 2008):

$$
\begin{align*}
\Delta X & = \Delta X_{12} + \Delta X_{23} + \Delta X_{31} \\
\Delta Y & = \Delta Y_{12} + \Delta Y_{23} + \Delta Y_{31} \\
\Delta Z & = \Delta Z_{12} + \Delta Z_{23} + \Delta Z_{31}
\end{align*}
$$

(4)

Akwasi (2008) also stated that, the resultant loop closure error is determined by finding the square root of the sum of the squares of the closure errors, $cx, cy$ and $cz$ and it is given by

$$
\text{Resultant Loop Closure Error} = \left[ (cx)^2 + (cy)^2 + (cz)^2 \right]^{1/2}
$$

(5)

To carry out least squares adjustment of a closed loop DGPS/GNSS observations, the design matrix ($A$), residual matrix ($V$), observation matrix ($L$) and matrix of unknown parameters ($X$) are deduced. The above matrices are obtained from the written observation equations. The number of observation equations must equal the number of baselines. The general matrix notation for least squares adjustment using observation equation method is given by Eteje et al (2018) as:

$$
V = AX - L
$$

(6)

According to Eteje et al (2018), the estimated parameter, X is obtained using:

$$
X = \left( A^T W A \right)^{-1} A^T W L
$$

(7)

Where, W is weighted matrix and $\left( A^T W A \right)^{-1}$ is the inverse of the normal matrix, $N^{-1}$. The model for the computation of the a posteriori standard error as given by Xu (2016) is

$$
\hat{\sigma}_{o} = \sqrt{\frac{V^T W V}{n - m}} (n > m)
$$

(8)

Where, $n$ is the number of observation and $m$ is the number of unknown. The standard error of the adjusted parameter is computed as (Akwasi, 2008):

$$
\hat{\sigma}_{xi} = \hat{\sigma}_{o} \sqrt{Q_{nn}} = \sqrt{\hat{\sigma}_{o}^2 Q_{nn}}
$$

(9)

Where, $Q_{nn}$ is a diagonal element of the inverse of the normal matrix ($N^{-1}$).

The semi-major axis $\sigma_{x}^2$, semi-minor axis $\sigma_{y}^2$ and the orientation of the error ellipse $\theta$ as given by Mikhail and Gracie (1981) and Ono et al (2018) are:

$$
\begin{align*}
\sigma_{x}^2 & = \frac{\sigma_{x}^2 + \sigma_{y}^2}{2} + \left[ \frac{(\sigma_{x}^2 - \sigma_{y}^2)^2}{4} \right] + \sigma_{xy}^2 \\
\sigma_{y}^2 & = \frac{\sigma_{x}^2 + \sigma_{y}^2}{2} - \left[ \frac{(\sigma_{x}^2 - \sigma_{y}^2)^2}{4} \right] + \sigma_{xy}^2 \\
\tan 2\theta & = \frac{2\sigma_{xy}}{\sigma_{x}^2 - \sigma_{y}^2}
\end{align*}
$$

(10)

1.5 Determination of Significant of the Difference between Two Positions of a Point

To compute the significant of the difference between two positions of a point, the magnitude of the difference is computed and compared with the 95% confidence level. The magnitude of the difference is computed as (Eteje et al, 2018):

$$
\text{Magnitude, } |D| = \sqrt{(\Delta N)^2 + (\Delta E)^2}
$$

(12)

Where, $(\Delta N)^2$ = change in northing squared and $(\Delta E)^2$ = change in easting squared.

The 95% confidence level $\epsilon$ is computed by multiplying the resultant error, $\sigma$ that is associated with the two positions with 95% confidence level expansion factor, 1.96. The 95% confidence level $\epsilon$ is computed as:
\[ e = 1.96\sqrt{\sigma^2} \]  

If the computed magnitude, \( |D| \) is less than the 95% confidence level, \( e \), that is, \( |D| < e \), it implies that the difference between the two positions of the point is not significant. But if on the other hand, the computed magnitude, \( |D| \) is greater than the 95% confidence level, \( e \), that is, \( |D| > e \), it implies that the difference between the two positions of the point is significant.

2. Methodology
The adopted methodology is divided into data acquisition, data processing, and results presentation and analysis. Figure 3 shows the flow chart of the adopted methodology.

2.1 Data Acquisition
The positions of two points, ESO1 and ESO2 were determined using three CHC900 dual frequency GNSS receivers. During the observation, two rover receivers simultaneously occupied the two points. The observations of the two points were carried out relative to a base receiver set at a control station, ASPXW42A (see figure 4). The period of occupation of the new/rover points was about 60 minutes. The data were collected at 1 second rate.

Fig. 4: Observation of the Two points Relative to Control Station ASPXW424
2.2 Data Processing
The GNSS observations of the two points were processed with Compass post processing software. The processing was done in two ways. Firstly, the observations were processed considering the three baselines, that is, the interactions between the base receiver and each of the rover receivers, and the interaction between the two rover receivers (see figure 5). Secondly, the interaction of the rover receivers was not considered. This was done to avoid the reoccupation of the points one after the other with respect to the base receiver. In other words, the processing of the GNSS observations of the points when each of them was occupied with respect to the base receiver.

Since the first set of observations was a closed loop observation, least squares adjustment and statistical evaluations were also carried out by the post processing software. The least squares adjustment of the observations was done using equation (6). The a posteriori standard error of the adjusted observations and the standard errors of the adjusted parameters were respectively obtained using equations (8) and (9). The semi-major axes and semi-minor axes of the adjusted positions were computed with equation (10) while their respective orientations were computed using equation (11) (see figure 5).

2.3 Results Presentation and Analysis
Table 1 presents the adjusted coordinates and standard errors of the observed points when one base and two rover receivers were used for the observation. The coordinates of the points were adjusted to obtain the most probable positions of the observed points while their respective standard errors were computed to determine the accuracy of the adjusted positions. It can be seen from table 1 that the standard errors in northing and easting of points ESO 1 and ESO 2 are respectively 0.0013m and 0.0013m, and 0.0018m and 0.0018m which show the high accuracy of the adjusted positions. This also implies that using two or more rover receivers together with a
base receiver for differential positioning, the reliability of the determined positions can be determined.

Table 1: Adjusted Coordinates and Standard Errors of Observed Points

<table>
<thead>
<tr>
<th>STATION</th>
<th>NORTHING (m)</th>
<th>EASTING (m)</th>
<th>HEIGHT (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASPXW42A</td>
<td>203429.7650</td>
<td>388440.4400</td>
<td>30.9948</td>
</tr>
<tr>
<td>ESO 1</td>
<td>202188.0039</td>
<td>391567.2922</td>
<td>14.3770</td>
</tr>
<tr>
<td>ESO 2</td>
<td>202028.8359</td>
<td>391512.7008</td>
<td>13.6221</td>
</tr>
</tbody>
</table>

Table 2 also presents the unadjusted coordinates of the observed points when the points were occupied one after the other using one rover and base receivers. It can be seen from table 2 that the standard errors of the determined positions are all zero. Because their observations were carried out one after the other and not closed loop observations. Consequently, least squares adjustment was not applied during data processing. Thus, their standard errors were not evaluated by the post processing software. This shows that the accuracy as well as the reliability of the positions determined with Differential GPS/GNSS receivers using a rover receiver and one base receiver such that the points are occupied one after the other with respect to the control/base station cannot be determined.

Table 2: Unadjusted Coordinates of the Observed Points

<table>
<thead>
<tr>
<th>STATION</th>
<th>NORTHING (m)</th>
<th>EASTING (m)</th>
<th>HEIGHT (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASPXW42A</td>
<td>203429.7650</td>
<td>388440.4400</td>
<td>30.9948</td>
</tr>
<tr>
<td>ESO 1</td>
<td>202188.0006</td>
<td>391567.2989</td>
<td>14.3777</td>
</tr>
<tr>
<td>ESO 2</td>
<td>202028.8303</td>
<td>391512.7071</td>
<td>13.6036</td>
</tr>
</tbody>
</table>

Tables 3 and figure 6 show the computed semi-major axes, semi-minor axes and orientations of the error ellipses of the adjusted positions. This was done to present the positional accuracy of the adjusted parameters graphically. It can be seen from table 3 that the semi-major axes and semi-minor axes of points ESO 1 and ESO 2 are respectively 0.0018m and 0.0013m, and 0.0018m and 0.0013m which show the high accuracy of the adjusted positions. It can also be seen from table 3 and figure 6 that the orientations of the error ellipses of the adjusted positions of the points, ESO 1 and ESO are respectively 94° 43' 49" and 94° 46' 01" which implies that the positional accuracy of the adjusted positions is to the south east direction.

Table 3: Semi-major Axes, Semi-minor Axes and Orientations (Error Ellipses)

<table>
<thead>
<tr>
<th>STATION</th>
<th>SEMIMAJOR AXIS (m)</th>
<th>SEMIMINOR AXIS (m)</th>
<th>ORIENTATION (DMS)</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASPXW42A</td>
<td>0.0018</td>
<td>0.0013</td>
<td>94:43:49</td>
</tr>
<tr>
<td>ESO 1</td>
<td>0.0018</td>
<td>0.0013</td>
<td>94:46:01</td>
</tr>
</tbody>
</table>

Fig. 6: Plot of the Error Ellipses
Table 4 also presents the distances from the control/base station, ASPXW42A to the two observed points, ESO 1 and ESO 2, and from ESO 1 to ESO 2 and their respective relative errors. The distances and the errors associated with the observations were computed so that the relative errors could be computed. The relative error is the ratio of the observation error to the distance between the two observed points/stations. The relative errors were computed to determine the linear accuracy of the observations between the control station and the observed points and between the observed points. It can be seen from table 4 that the linear accuracy of stations ASPXW42A to ESO 1, ASPXW42A to ESO 2 and ESO 1 to ESO 2 are respectively 1: 1511302, 1: 1515574 and 1: 214415 which show the high accuracy of the adjusted observations.

Table 4: Computed Distances, Positions Errors and Relative Errors

<table>
<thead>
<tr>
<th>START POINT</th>
<th>END POINT</th>
<th>DISTANCE</th>
<th>ERROR</th>
<th>RELATIVE ERROR</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASPXW42A</td>
<td>ESO 1</td>
<td>3364.3982</td>
<td>0.0022</td>
<td>1: 1511302</td>
</tr>
<tr>
<td>ASPXW42A</td>
<td>ESO 2</td>
<td>3376.5942</td>
<td>0.0022</td>
<td>1: 1515574</td>
</tr>
<tr>
<td>ESO 1</td>
<td>ESO 2</td>
<td>168.2696</td>
<td>0.0008</td>
<td>1: 214415</td>
</tr>
</tbody>
</table>

Table 4 and figures 7 and 8 respectively show the computed horizontal and vertical difference magnitudes and their respective 95% significant levels. These were computed and compared to determine if the difference between the two positions of each points, that is, the positions of the point obtained from closed and opened loop observations are significant or not. If the computed difference magnitude of the observed point is less than its corresponding confidence level, it implies that there is no significant difference between the two methods of observation. But if the computed difference magnitude is greater than its corresponding confidence level, it implies that there is significant difference between the two observation methods which in turn shows that differential GPS/GNSS observation/positioning should be carried out using not less than three receivers, that is, a base and two rover receivers. It can be seen from table 4 and figures 7 and 8 that the computed difference magnitudes, 0.0075m, 0.0084m and 0.0185m of the points except the vertical dimension of ESO 1 are greater than their respective 95% confidence ellipses, 0.0062m, 0.0062m and 0.0061m which shows that differential GPS/GNSS observation/positioning should be carried out using not less than three receivers as the most probable positions of points can only be determined if the observation is a closed loop observation. Point ESO 1 vertical difference is not significant because the difference in height of the point is very small, less than a millimetre.

Table 5: Computed Horizontal and Vertical Difference Magnitudes and their Respective 95% Confidence Levels

<table>
<thead>
<tr>
<th>STATION</th>
<th>DIFFERENCE IN NORTHING (m)</th>
<th>DIFFERENCE IN EASTING (m)</th>
<th>DIFFERENCE IN HEIGHT (m)</th>
<th>MAGNITUDE OF HORIZONTAL DIFFERENCE (m)</th>
<th>MAGNITUDE OF VERTICAL DIFFERENCE (m)</th>
<th>95% CONFIDENCE LEVEL (HORIZONTAL) (m)</th>
<th>95% CONFIDENCE LEVEL VERTICAL (m)</th>
<th>REMARK</th>
</tr>
</thead>
<tbody>
<tr>
<td>ESO 1</td>
<td>0.0033</td>
<td>-0.0067</td>
<td>-0.0007</td>
<td>0.0075</td>
<td>0.0007</td>
<td>0.0062</td>
<td>0.0061</td>
<td>SIGNIFICANT</td>
</tr>
<tr>
<td>ESO 2</td>
<td>0.0056</td>
<td>-0.0063</td>
<td>0.0185</td>
<td>0.0084</td>
<td>0.0185</td>
<td>0.0062</td>
<td>0.0061</td>
<td>SIGNIFICANT</td>
</tr>
</tbody>
</table>

Fig. 7: Comparison of the Positions Difference Magnitudes and Their Respective 95% Confidence Levels
Conclusion

The determination of the reliability of adjusted observations and those of adjusted positions is an important aspect of position determination. The determination of the reliability of a group of observations and those of the adjusted parameters respectively requires the computation of the mean accuracy, a posteriori standard error of the adjusted observations and the standard errors of the adjusted parameters. The reliability determination requires the application of least squares adjustment technique. The determination of the accurate positions of points using the differential GPS/GNSS positioning technique requires a base and rover receivers to be used. To obtain the reliability of the observed points using the differential GPS/GNSS technique, a minimum of a base and two rover receivers are required. Using a minimum of a base and two rover receivers enables interaction between the base receiver and each of the rover receivers. It also allows communication between any two rover receivers which makes the observation a closed loop observation. Thus, the possibility of least squares adjustment of the acquired observations. But as in most cases where the number of points to be observed is small, the observations are carried out with only two (rover and base) receivers. This study has comparatively analyzed the positions determined from static Differential GPS/GNSS observations using two and three or more receivers with a view to obtaining significant differences in the positions of observed points. The comparison and analysis results showed that there were significant differences which implies that differential GPS/GNSS observation should be carried out with not less than three GPS/GNSS receivers.

References


