The existence of common fixed point theorems of generalized contractive mappings in cone metric spaces

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Abstract

The purpose of this paper is to the study of the existence of common fixed point theorem for a sequence of self maps satisfying generalized contractive condition for a cone metrice space and obtains some new results in it. Also the paper contains generalized fixed point theorems of [10, 13, 19] and many others from the current literature.

1. Introduction and preliminaries

The well known Banach contraction principal and its several generalizations in the setting of metric spaces play a central role for solving many problems of non linear analysis. For example, see[2,5,6,15,16,17] Huang and Zang[8] generalized the concept of the metric spaces by introducing cone metric spaces and proved some fixed point theorems for mappings satisfying some contractive conditions, subsequentially, several other authors [1,9,16,19,21] studied the existence of fixed points and common fixed points of mappings satisfying contractive type condition on a normal cone metric space.

Recently Razapour and Hamilbarani[19] omitted the assumption of normality in cone metric space, which is milestone in developing fixed point theory in cone metric space. In[11] the authors introduced the concept of a compatible pair of self maps in a cone metric space and established a basic result for a non normal cone metric space with an example which [12] weakly compatible maps have been studied. In this paper we prove a common fixed point theorem for a sequence of self maps satisfying a generalized contractive condition for a non normal cone metric space.

Definition:-1.1 [sec [8]] :- Let E be a real Banach space a sub set of p of E is called a cone whenever the following condition holds.

(c1) P is closed, nonempty and P ≠ {0}
(c2) a,b∈R, a,b≥0 and x, y∈P imply ax+by∈P,
(c3)P∩(-P) = {0}

Given a cone P⊂E, we define a partial ordering ≤ with respect to P by x≤y if and only if y-x∈P. We shall write x<y to indicate that x≤y but x≠y while x≪y will stand for y-x∈p0 where P0 stands for the interior of P. If P0≠Ø.then P is called a solid cone[see[20]].

There exist two kinds of cones-normal (with the normal constant k) and non-normal cone [6]. Let E be a real Banach space,P⊂E a cone and ≤ partial ordering defined by P. Then P is called normal it there is a number k>0 such that for all x,y∈P.

0≤x≤y implies ∥x∥≤k∥y∥

(1.1)
or equivalently if (∀ n) x_n≤y_n≤z_n

and

lim_{n→∞} x_n = lim_{n→∞} z_n = x imply lim_{n→∞} y_n = x

(1.2)
The least positive number K satisfying (1.1) is called the normal constant of P.

Example 1.2 (see[20]) let E= C[0,1] with ∥x∥=∥x∥_∞+∥x∥_1 on p={x∈E:x(t)≥0}.This cone is not normal.

Consider for example, x_n(t)=n/n and y(t)=1/n, then 0≤x_n≤y_n and lim_{n→∞} y_n = 0 but ∥x_n∥=max_{t∈[0,1]} ∥x_n∥_∞ + ∥x_n∥_1

max_{t∈[0,1]} |x_n(t)-1| = \frac{1}{n} + 1 > 1 :hence x_n does not converge to zero. It follows by (1.2) that P is a non-normal cone.

Definition 1.3 (see [13]): Let P be a cone in a real Banach space E. If for a∈P and a≤ka for same K∈ [0,1],then a=0.
Definition 1.4 (see [10]): Let $P$ be a cone in a real Banach space $E$ with non-empty interior. If for $a \in E$ and $a \preceq c$ for all $c \in P$, then $a = 0$.

Remarks 1.5 (see [19]): $\lambda p \preceq p$ for $\lambda > 0$ and $p + p = p$.

Definition 1.6 (see [8.22]): Let $X$ be a nonempty set suppose that the mapping $d : X \times X \to E$ satisfies:

- $(d_1) 0 \preceq d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$.
- $(d_2) d(x,y) = d(y,x)$ for all $x, y \in X$.
- $(d_3) d(x,y) \preceq d(x,z) + d(z,y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric [8] or $K$-metric [22] on $X$ and $(X, d)$ is called a cone metric [8] or $K$-metric space [22].

Example 1.1 (see [13]): Let $E = \mathbb{R}^2$ and $P = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}$, $X = \mathbb{R}^2$ and $d : X \times X \to E$ defined by $d(x, y) = \langle x - y, a(x - y) \rangle$, where $a \geq 0$ is a constant. Then $(X, d)$ is a cone metric space with normal cone $P$ where $K = 1$.

Example 1.2 (see [18]): Let $E = \mathbb{F}$, $P = \{x_n \}_{n=0}^{\infty} \subseteq E : x_n \geq 0$, for all $n$, $(X, P)$ a metric space, and $d : X \times X \to E$ defined by $d(x, y) = \langle p(x, y), 2z \rangle$ for some fixed $x \in X$.

Remarks 1.10: The concept of a cone metric space is more general than that of a metric space, because each metric space is a cone metric space where $E = \mathbb{R}$ and $P = \{0, +\infty\}$.

Definition 1.11 (see [13]): Let $(X, d)$ be a cone metric space. We say that $(X, d)$ is a Cauchy sequence if for every $0 < \varepsilon \in E$, there is an $N$ such that for all $n, m > N$, $d(x_n, x_m) \ll \varepsilon$.

Definition 1.9 (see [8.22]): Let $(X, d)$ be a cone metric space. We say that $(X, d)$ is a complete cone metric space if every Cauchy sequence in $(X, d)$ is convergent in $X$.

Remarks 1.5: A complete cone metric space $(X, d)$ is said to be complete if every Cauchy sequence in $X$ is convergent in $X$.

In the following $(X, d)$ will stand for a cone metric space with respect to a cone $P$ with $P \subseteq E$.

Theorem 3.1: Let $(X, d)$ be a complete cone metric space with respect to a cone $P$ contained in real Banach space $E$. Let $\{T_n\}$ be a sequence of self maps on $X$ satisfying generalized contractive condition (2.1) with for some $\alpha, \beta, \gamma, \delta, \eta, \mu \in [0, 1]$ for $x, y \in X$, let $x_n = T_n x_{n-1}$ for all $n$, then the sequence $\{x_n\}$ converges in $X$ and its limit $v$ is a common fixed point of all the maps of the sequence $\{T_n\}$. This common fixed point is unique if $\alpha + 2\eta + \mu < 1$.

Proof: Let $x = x_{n-1}$, $y = x_n$, $T = T_n$ and $T' = T_n + 1$ in (2.1) we have...
\begin{align*}
\text{d}(T_{x_n-1}x_n, T_{n+1}x_0) &\leq \alpha \text{d}(x_n-1, x_n) + \beta \text{d}(x_{n-1}, T_{x_n-1}x_n) \\
&\quad + \gamma \text{d}(x_n, T_{n+1}x_0) + \delta \text{d}(x_n, T_{x_n-1}x_n) \\
&\quad + \eta \text{d}(x_n-1, T_{n+1}x_0) + \mu \text{d}(x_n-1, T_{x_n-1}x_n)
\end{align*}

As \( x_n = T_{x_n-1}x_n \), we have
\begin{align*}
\text{d}(x_n, x_{n+1}) &\leq \alpha \text{d}(x_n-1, x_n) + \beta \text{d}(x_{n-1}, x_n) \\
&\quad + \gamma \text{d}(x_n, x_{n+1}) + \delta \text{d}(x_n, x_{n+1})
\end{align*}

Writing \( \text{d}(x_n, x_{n+1}) = \rho_n \), we have
\( \rho_n \leq (\alpha + \beta + \delta + \eta + 2\mu) \rho_{n-1} + (\gamma + \delta + \eta + \mu) \rho_n \)
\[ (1 - \gamma - \delta - \eta - \mu) \rho_n \leq (\alpha + \beta + \delta + \eta + 2\mu) \rho_{n-1} \]
This implies that
\[ \rho_n \leq t \rho_{n-1} \]
where
\[ t = \frac{\alpha + \beta + \delta + \eta + 2\mu}{1 - \gamma - \delta - \eta - \mu} \]

As \( (\alpha + \beta + 2\delta + \gamma + 2\eta + 3\mu < 1) \), we obtain that \( t < 1 \)

Now \( \rho_n \leq t \rho_{n-1} \leq t^2 \rho_{n-2} \leq \ldots \leq t^n \rho_0 \)

Where \( \rho_0 = \text{d}(x_0, x_1) \), also for \( n > m \) we have
\begin{align*}
\text{d}(x_n, x_m) &\leq \text{d}(x_n, x_{n-1}) + \text{d}(x_{n-1}, x_{n-2}) + \ldots + \text{d}(x_{m+1}, x_m) \\
&\leq (t^{n-1} + t^{n-2} + \ldots + t^m) \text{d}(x_1, x_0) \\
&\leq \frac{t^m}{1-t} \rho_0
\end{align*}

As \( t < 1 \) and \( p \) is closed, thus we obtain that
\[ \text{d}(x_n, x_m) \leq \frac{t^m}{1-t} \rho_0 \quad (3.2) \]

Now for \( \epsilon \in p \), there exists \( r > 0 \) such that \( \epsilon \cdot y \in p \), if \( \| y \| < r \). Choose a positive integer \( N \in \mathbb{N} \) such that for all \( n > N \)
\[ \| y \| < r \text{ which implies } \epsilon - \frac{t^m}{1-t} \rho_0 \in p \]

So we have \( \epsilon - \text{d}(x_n, x_m) \in p \) for all \( n > N \), and for all \( m \) by definition(1.11), this implies \( \text{d}(x_n, x_m) \in p \) for all \( n > N \)
and for all \( m \) hence \( \{ x_n \} \) is a Cauchy sequence in \( X \). By the completeness of \( X \), there exists \( z \in X \) such that \( x_n \rightarrow z \) as \( n \rightarrow \infty \). For an arbitrary fixed \( m \) we show that \( T_mz=z \). Now
\[ \text{d}(T_mz, z) \leq \text{d}(T_mz, T_nx_0) + \text{d}(T_nx_0, z) \]
\[ \quad = \text{d}(x_0, z) + \text{d}(T_mz, T_nx_0) \]
Using (2.1) we have
\[ \text{d}(T_mz, z) \leq \text{d}(T_mz, T_nx_0) + \text{d}(T_nx_0, z) \]
have a unique common fixed point, say \( z \).

This fixed point is unique if \( \alpha + 2\eta + \mu < 1 \).

Theorem 3.2:- Let \( \{T_n\} \) be a sequence of self maps in \( X \) satisfying for some \( \alpha, \beta, \gamma, \delta, \eta, \mu \in [0,1] \) with \( \alpha + \beta \gamma + 2\delta + 2\eta + 3\mu < 1 \) and \( \alpha + 2\eta + \mu < 1 \) there exists positive integer \( m \) such that for all \( x, y \in X \),

\[
\begin{align*}
\alpha d(x, y) &\leq d(x, z) + \beta d(x, s^m_i x) + \gamma d(y, s^m_i y) \\
&\quad + \delta d(x, s^m_i x) + d(y, s^m_i y) + \eta d(x, s^m_i x) + d(x, s^m_i x) \\
&\quad + \mu d(x, s^m_i y) + d(x, s^m_i y) \\
&\quad + \mu d(x, s^m_i x) + d(x, s^m_i x)
\end{align*}
\]

Then all the maps of the sequence \( \{s_n\} \) have a unique common fixed point in \( X \).

Proof: - from theorem 3.1 all the maps of the sequence \( \{S^m_i\} \) have a unique common fixed point, say \( z \).

hence \( S^m_i z = z \) for all \( i \).

For all \( i \), \( S^m_i z = z \) implies \( s_i z = z \) taking \( x = s_i z, y = z \), \( i = 1 \) and \( j = 2 \) in (3.3), we have \( s_i z = z \) continuing in similar way it follows that \( s_i z = z \) for all \( i \). Thus \( z \) is a common fixed point of all the maps of the sequence \( \{s_i\} \). Its uniqueness follows from the fact that \( s_i z = z \) implies \( S^m_i z = z \) for all \( i \).

In theorem 3.1 taking \( T_1 = T_2 = \ldots = T_N = \ldots = T \), we get the following general form of Banach contraction principle in a cone metric space which is not necessarily normal.

Theorem 3.3:- Let \( (X,d) \) be a complete cone metric space with respect to a cone \( \pi \) contained in real Banach space \( E \). Let \( T \) be a self map in \( X \) satisfying generalized contractive condition (2.1) with \( \alpha + \beta \gamma + 2\delta + 2\eta + 3\mu < 1 \) and for some \( \alpha, \beta, \gamma, \delta, \eta, \mu \in [0,1] \) then for each \( x \in X \) sequence \( \{T^n_x\} \) converges in \( X \) and its limit is a fixed point \( T \). This fixed point is unique if \( \alpha + 2\eta + \mu < 1 \).
Theorem 3.4: Let $(X, d)$ be a complete cone metric space with respect to a cone $P$ contained in a real Banach space $E$. Suppose the mapping $T: X \rightarrow X$ satisfying for some positive integer $n$
\[d(T^n x, T^n y) \leq \alpha d(x, y) + \beta d(x, T^n x) + \gamma d(y, T^n y) + \delta d(x, T^n x) + \eta d(y, T^n x) + \mu d(x, T^n y) + \nu d(y, T^n y)\]

For all $x, y \in X$ and $\alpha, \beta, \gamma, \delta, \eta, \mu, \nu \in [0, 1]$ are constants such that $\alpha + \beta + \gamma + 2\delta + 2\eta + 3\mu < 1$ then $T$ has a unique fixed point in $X$.

Proof: from Theorem 3.3 $T^n$ has a unique fixed point $u$. But $T^n(Tu) = T^n u = Tu$, so $Tu$ is also a fixed point of $T^n$ hence $Tu = u$ is a fixed of $T$. Since the fixed point of $T$ is unique.

Corollary 3.5: Let $(X, d)$ be a complete cone metric space with respect to a cone $P$ contained in a real Banach space $E$. Suppose the mapping $T: X \rightarrow X$ satisfies for some positive integer $m, n$.

\[d(T^n x, T^n y) \leq \alpha d(x, y) + \beta d(x, T^n x) + \gamma d(y, T^n y) + \delta d(x, T^n x) + \eta d(y, T^n x) + \mu d(x, T^n y) + \nu d(y, T^n y)\]

For all $x, y \in X$ and $\alpha, \beta, \gamma, \delta, \eta, \mu, \nu \in [0, 1]$ are constants such that $\alpha + \beta + \gamma + 2\delta + 2\eta + 3\mu < 1$ and $\delta = \eta$ then $T$ has a unique fixed point in $X$.

Proof: by Theorem 3.4 we get $x \in X$ such that $T^n x = T^n y = x$. The result then follows from the fact that $d(T^n x, x) = d(T^n y, y) = d(T^{n+1} x, T^{n+1} y)$

\[d(T^n x, x) = d(T^n y, y) = d(T^{n+1} x, T^{n+1} y) = 0\]

Which implies $T^n x = x$.

Theorem 1[8] and Theorem 2.3[20]: Let $(X, d)$ be a complete cone metric space. Suppose the mapping $T: X \rightarrow X$ satisfies the contractive condition
\[d(T^n x, T^n y) = k d(x, y)\]

For all $x, y \in X$ where $k \in [0, 1]$, is a constant. Then $T$ has a unique fixed point in $X$. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 3[8] and Theorem 2.6[20]: Let $(X, d)$ be a complete cone metric space. Suppose the mapping $T: X \rightarrow X$ satisfies the contractive condition
\[d(T^n x, T^n y) = k \left( d(x, T^n x) + d(y, T^n y) \right)\]

For all $x, y \in X$ where $k \in [0, 1/2]$, is a constant. Then $T$ has a unique fixed point in $X$. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 4[8] and Theorem 2.7[20]: Theorem 1[8] and Theorem 2.3[20]. Let $(X, d)$ be a complete cone metric space. Suppose the mapping $T: X \rightarrow X$ satisfies the contractive condition
\[d(T^n x, T^n y) = k \left( d(x, T^n x) + d(y, T^n y) \right)\]

For all $x, y \in X$ where $k \in [0, 1]$, is a constant. Then $T$ has a unique fixed point in $X$. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.

Remark 3.8: Above theorems os [8] and [20] follows Theorem 3.3 of this paper by taking

(i) $\alpha, \gamma, \delta, \eta, \mu$ and $\alpha = k$
(ii) $\alpha, \gamma, \delta, \eta, \mu$ and $\beta = k$
(iii) $\alpha, \beta, \gamma, \delta, \eta, \mu$ and $\gamma = k$
(iv) $\alpha, \beta, \gamma, \delta, \eta, \mu$ and $\delta = k$
(v) $\alpha, \beta, \gamma, \delta, \mu$ and $\eta = k$
(vi) $\alpha, \beta, \gamma, \delta, \eta$ and $\mu = k$

Precisely, Theorem 3.3 synthesize and generalizes all the results of [9] and [20] for a non normal cone metric space. Theorem 3.2 is a generalized form of Banach contraction principle in a complete cone metric space which is not necessarily normal

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