A Mathematical Moment-Area Technique for Simply Supported Beams

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Abstract

Slope and Deflection in beams has a key role to play in determining the quality of a beam and they are thus paramount in conducting beam design. Moment-Area method of beam design is a versatile method of determining slope and deflection in beams as it can determine them in beams of varying cross-section unlike most mathematical methods of determining slope and deflection. This paper presents a novel Moment-Area method of determining slope and deflection in beams by analyzing various case scenarios of loading on a simply supported beam. The results of this novel method are validated using Macaulay's method of slope and beam deflection and are shown to be in unison.

Keywords: Bending Moment Diagram (BMD), Breakpoint, Deflection, Moment-Area, Turning Point

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1. Introduction

Beams are used to carry various types of loads that are loaded perpendicularly to the axis of the specific beam in question. It is inevitable that the loaded beam will sag under the weight of the load being carried, regardless of whether the span beam of the beam is wide. Many structural codes consider the deflection of structures in terms of safety and serviceability. Safety ensures that no casualties result from structural collapse. For this, it is necessary to provide sufficient time for people to identify deformations, such as deflection of the structure and to evacuate before destruction (Kim & Kim 2021). Several parameters are used to define the curvature of this sagged beam to determine its usefulness in engineering applications. The slope of the curvature is measured from the imaginary horizontal axis and the deflection of the curvature is measured from the nominal position of the beam when un-loaded.

Several methods have been devised to aid in calculating the slope and deflection in beams (Khurmi & Khurmi 2019), among them Macaulay's method, double integration method, Superposition method, Moment-Area method and Castigliano's method. However, Moment-Area method is superior to these other methods mentioned in that it in addition to analyzing deflection in conventional beams, it is also capable of analyzing the deflection in beams of varying cross-section like stepped shafts and as such it is more versatile.

Despite the fact that Mohr's theorems were seemingly invented a long time ago, they are still in use to this day as a graphical technique known as the Moment-Area method. Moment-Area method is a graphical technique that requires visual representation by graphical methods for analysis of slope and deflection of beams. From the conventional interpretation of Mohr I and Mohr II (collectively known as Mohr's theorems), one is required to have a graphical drawing of the Bending Moment Diagram before analyzing the slope and deflection of the beam.

The consequence of employing this graphical method as is currently interpreted from the Mohr's theorems is that it makes the method cumbersome and lengthy for engineers looking for a speedy and efficient method of analyzing beams. Further, the efficacy of the method in determining slope and deflection is also dependent on the accuracy of the graphs drawn for the analysis.

In an ideal situation, the determination of slope and deflection in beams should be simple, accurate and not prone to error. To remedy this, a mathematical technique is required to replace the graphical technique in use. It is much simpler, accurate and error-proof to use mathematical equations in determining the slope and deflection in beams than to employ graphs to do the same.

Considering Mohr I and Mohr II that form the backbone of Moment-Area method of beam deflection; to calculate slope between two points in a beam, one needs to determine the area under the Bending Moment Diagram (BMD) between these two points. On the other hand, to calculate deflection, you need to determine the first moment of area of the BMD between these two points i.e. the area between two points multiplied by the centroid of the region in question with respect to the right hand support.

When the BMD is triangular in nature due to the fact that the load applied is a point load, this makes it easy to determine the area under the BMD and the centroid of any region under the BMD. However, in the case the beam carries a Uniformly Distributed Load (UDL) or a Variable Distributed Load (VDL) or a combination of these loads with point loads; determining the area underneath the BMD is not straight forward nor is the determination of centroids of any specific region.

This paper proposes a novel mathematical Moment-Area technique to replace the graphical method that requires one to draw the BMD in the first place. The proposed technique will be versatile enough to calculate slope and deflection under all types of loadings on the beam without requiring one to actually draw the BMD.

2. WHEN THERE IS NO BREAKPOINT IN THE BMD, BUT THERE EXISTS A TURNING POINT ON THE DEFLECTION CURVE

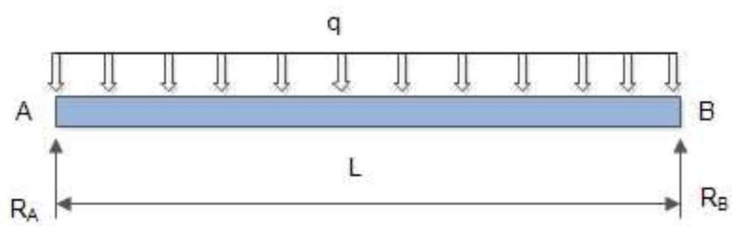

Figure 6:Beam without a Breakpoint in the BMD but with Turning Point on Deflection Curve

2.1 DETERMINING SLOPE AND DEFLECTION USING MACAULAY'S METHOD

$$R_A = R_B = \frac{qL}{2}$$

The bending moment along the length of the beam is given by:

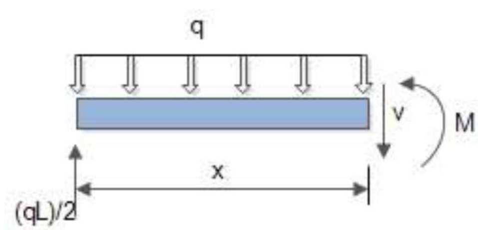

Figure 7: FBD for drawing the BMD

$$M(x) = \frac{-qx^2}{2} + \frac{qL}{2}x \qquad x \epsilon [0, L]$$

$$\frac{\partial^2 y}{\partial x^2} = \frac{M(x)}{EI} = \frac{-qx^2}{2EI} + \frac{qL}{2EI}x$$

$$\frac{\partial y}{\partial x} = \frac{1}{EI}\left(\frac{-qx^3}{6} + \frac{qL}{4}x^2 + C_1\right)$$

$$y = \frac{1}{EI}\left(\frac{-qx^4}{24} + \frac{qL}{12}x^3 + C_1 x + C_2\right)$$

To eliminate the constants of integration C_1 and C_2 we apply boundary conditions. From the beam, it is evident that the maximum slope will occur at the ends A and B and slope will be zero at mid-point. Similarly, deflection will be zero at the ends A and B and maximum deflection will occur at mid-point.

$$y(0) = \frac{1}{EI}(0 + 0 + (C_1 \times 0) + C_2) = 0 \xrightarrow{yields} C_2 = 0$$

$$y(L) = \frac{1}{EI}\left(\frac{-qL^4}{24} + \frac{qL}{12}L^3 + C_1 L\right) = 0 \xrightarrow{yields} C_1 = \frac{-qL^3}{24}$$

$$Alternatively, \quad \frac{\partial y}{\partial x}\left(\frac{L}{2}\right) = \frac{1}{EI}\left(\frac{-qL^3}{48} + \frac{qL}{16}L^2 + C_1\right) = 0 \xrightarrow{yields} C_1 = \frac{-qL^3}{24}$$

Therefore, the equations of deflection and slope become:

$$y'(x) = \frac{1}{EI}\left(\frac{-qx^3}{6} + \frac{qL}{4}x^2 + \frac{-qL^3}{24}\right)$$

$$y(x) = \frac{1}{EI}\left(\frac{-qx^4}{24} + \frac{qL}{12}x^3 + \frac{-qL^3}{24}x\right)$$

<u>Evaluating Slope:</u>

$$y'(0) = \frac{1}{EI}\left(\frac{-q0^3}{6} + \frac{qL}{4}0^2 + \frac{-qL^3}{24}\right) = \frac{-qL^3}{24EI}$$

The negative means the slope at A is measured clockwise from the horizontal

$$y'\left(\frac{L}{4}\right) = \frac{1}{EI}\left(\frac{-qL^3}{384} + \frac{qL}{64}L^2 + \frac{-qL^3}{24}\right) = \frac{-11qL^3}{24EI}$$

$$y'\left(\frac{L}{2}\right) = \frac{1}{EI}\left(\frac{-qL^3}{48} + \frac{qL}{16}L^2 + \frac{-qL^3}{24}\right) = 0$$

$$y'\left(\frac{3L}{4}\right) = \frac{1}{EI}\left(\frac{-27qL^3}{384} + \frac{9qL}{64}L^2 + \frac{-qL^3}{24}\right) = \frac{11qL^3}{24EI}$$

$$y'(L) = \frac{1}{EI}\left(\frac{-qL^3}{6} + \frac{qL}{4}L^2 + \frac{-qL^3}{24}\right) = \frac{qL^3}{24EI}$$

<u>Evaluating Deflection:</u>

$$y(0) = \frac{1}{EI}\left(\frac{-q0^4}{24} + \frac{qL}{12}0^3 + \frac{-qL^3}{24}0\right) = 0$$

$$y\left(\frac{L}{4}\right) = \frac{1}{EI}\left(\frac{-qL^4}{6144} + \frac{qL}{768}L^3 + \frac{-qL^3}{96}L\right) = \frac{57qL^4}{6144EI}$$

$$y\left(\frac{L}{2}\right) = \frac{1}{EI}\left(\frac{-qL^4}{384} + \frac{qL}{96}L^3 + \frac{-qL^3}{48}L\right) = \frac{5qL^4}{384EI}$$

$$y\left(\frac{3L}{4}\right) = \frac{1}{EI}\left(\frac{-81qL^4}{6144} + \frac{27qL}{768}L^3 + \frac{-3qL^3}{96}L\right) = \frac{57qL^4}{6144EI}$$

$$y(L) = \frac{1}{EI}\left(\frac{-qL^4}{24} + \frac{qL}{12}L^3 + \frac{-qL^3}{24}L\right) = 0$$

2.2 DETERMINING SLOPE AND DEFLECTION USING MOMENT-AREA METHOD

For this beam, the equation of BM will be the same when x is measured from either $Point\ A$ or $Point\ B$ and is given below. In the instance when they are not the same, it is important to generate equations of BM with x measured from the right hand support.

$$M(x) = \frac{-q}{2}x^2 + \frac{qL}{2}x$$

To determine the slope and deflection of the beam, consider the deflection of the loaded beam as shown in the diagram below.

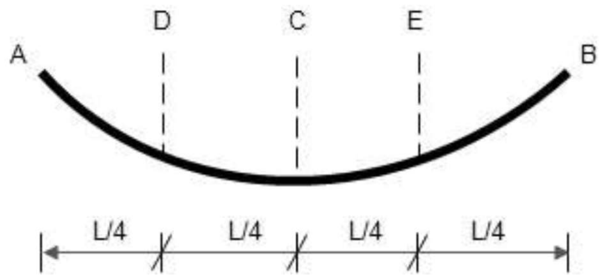

Figure 8: Deflection curve of the loaded beam

Analysis of Slope:
If C is the mid-point of the beam, we know from theory, that maximum slope along the length of the beam will be given by:

$Max\ Slope = \theta_{AC} = \theta_{BC}$ (*albeit with different signs!*)

However, let us derive a general equation to determine the slope at any point along the length of the beam with respect to the position of "$null - slope$".

$$From\ theory, \qquad \theta_{QP} = \frac{1}{EI}\int_P^Q M\,dx$$

$$\theta_{QP}(x) = \frac{1}{EI}\int_P^Q \left(\frac{-q}{2}x^2 + \frac{qL}{2}x\right)dx = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_P^Q$$

Thus, the slope of any point on the beam with respect to the slope at mid-span is given by:

$$\theta_{xC}(x) = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_C^Q = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_{\frac{L}{2}}^{Q = x\ measured\ from\ point\ B}$$

Hence:

$$\theta_{AC} = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_{\frac{L}{2}}^{L} = \frac{1}{EI}\left[\frac{qL^3}{12} - \left(\frac{qL^3}{24}\right)\right] = \frac{qL^3}{24EI}\ or\ \theta_{CA} = \frac{-qL^3}{24EI}$$

$(The\ negative\ sign\ indicates\ slope\ from\ Point\ C\ to\ A\ is\ measured\ closkwise)$

$$\theta_{DC} = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_{\frac{L}{2}}^{\frac{3L}{4}} = \frac{1}{EI}\left[\frac{27qL^3}{384} - \left(\frac{qL^3}{24}\right)\right] = \frac{11qL^3}{384EI}\ or\ \theta_{CD} = \frac{-11qL^3}{384EI}$$

$$\theta_{EC} = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_{\frac{L}{2}}^{\frac{L}{4}} = \frac{1}{EI}\left[\frac{5qL^3}{384} - \left(\frac{qL^3}{24}\right)\right] = \frac{-11qL^3}{384EI}\ or\ \theta_{CE} = \frac{11qL^3}{384EI}$$

$$\theta_{BC} = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_{\frac{L}{2}}^{0} = \frac{1}{EI}\left[0 - \left(\frac{qL^3}{24}\right)\right] = \frac{-qL^3}{24EI}\ or\ \theta_{CB} = \frac{qL^3}{24EI}$$

These results compare well with those obtained by Macaulay's Method of beam deflection and as such, it is evident that in addition to the already mentioned advantage, this method is quicker when determining the slope (*and as we shall see later, the deflection*) between two arbitrary points on a beam.

For instance, one can quickly determine the slope between two arbitrary points, say the slope of $Point\ E$ with respect to $Point\ A$ shown in the diagram below as follows:

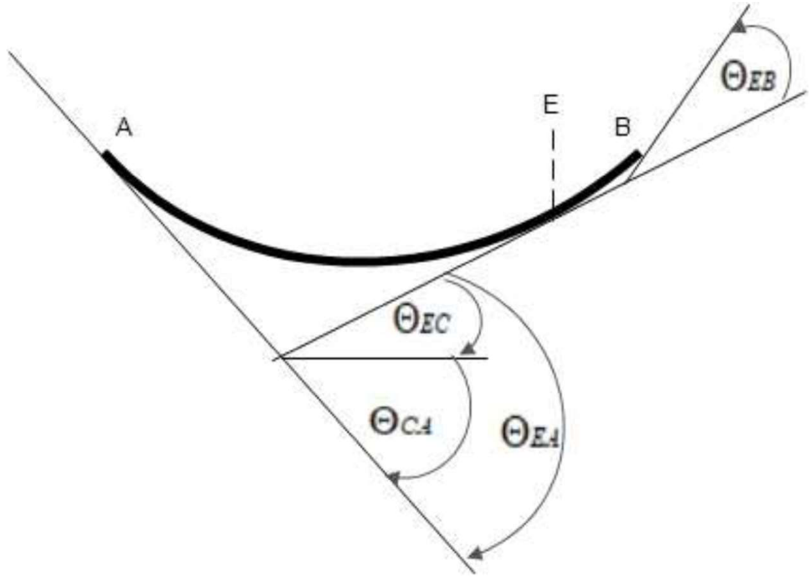

Figure 9:Determining slope between point E and point A

$$\theta_{EA} = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_A^E = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_L^{\frac{L}{4}} = \frac{1}{EI}\left[\frac{5qL^3}{384} - \frac{qL^3}{12}\right] = \frac{-27qL^3}{384EI}$$

$(-ve\ means\ slope\ is\ measured\ clockwise\ from\ tangent\ Line\ at\ E\ to\ tangent\ Line\ at\ A)$

$Alternatively, \quad \theta_{EA} = \theta_{EC} + \theta_{CA} = \frac{-11qL^3}{384EI} + \frac{-qL^3}{24EI} = \frac{-27qL^3}{384EI}$

Slope of Point E with respect to Point B can be given by:

$$\theta_{EB} = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_B^E = \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_0^{\frac{L}{4}} = \frac{1}{EI}\left[\frac{5qL^3}{384} - 0\right] = \frac{5qL^3}{384EI}$$

$It\ is\ evident\ in\ the\ results\ that, |\theta_{EA}| \neq |\theta_{EB}|\ hence, we\ can\ conclude\ that:$

$|\theta_{xA}| \neq |\theta_{xB}|\ \forall x \in \left\{\left[0, \frac{L}{2}\right) \cup \left(\frac{L}{2}, L\right]\right\}$

We can determine the nature of variation of Slope along the length of the beam as:

$\frac{d}{dx}\{\theta_{xC}\} = \frac{d}{dx}\left\{\frac{1}{EI}\int_C^x M\,dx\right\} = \frac{1}{EI} \times \frac{d}{dx}\left\{\int_C^x M\,dx\right\} = \frac{q}{2EI}[-x^2 + Lx] = \frac{q}{2EI}[x(-x + L)]$

$Thus, if\ \frac{d}{dx}\{\theta_{xC}\} = 0 \xrightarrow{yields} x = 0\ and\ x = L$

This means that the slope varies $quadratically$ along the length of the beam and $Maximum\ Slope$ with respect to the $Slope\ at\ Point\ C$ occurs when $x = 0\ and\ x = L$.

$\theta_{max} = \theta_{0C} = \theta_{BC}\ or\ \theta_{max} = \theta_{LC} = \theta_{AC}\ (which\ agrees\ with\ our\ earlier\ analysis)$

Analysis of Deflection:

Again, if C is the mid-point of the beam, it is important to note that maximum deflection of the beam will be given by:

$Max\ Deflection\ (y_{max}) = y_{CB} = \Delta_{CB}$

However, let us derive a general equation to determine the *Vertical Intercept* at any point along the length of the beam.

$$\textit{From previous theory,} \quad \Delta_{QP} = \frac{1}{EI}\int_P^Q x\, dA = \frac{1}{EI}\int_P^Q Mx\, dx$$

$$\Delta_{QP}(x) = \frac{1}{EI}\int_P^Q \left(\frac{-q}{2}x^3 + \frac{qL}{2}x^2\right) dx = \frac{1}{EI}\left[\frac{-qx^4}{8} + \frac{qLx^3}{6}\right]_P^Q$$

Thus, the Vertical Intercept of a *Point x* on the beam with respect to *Point C (mid-span)* is given by:

$$\Delta_{xC}(x) = \frac{1}{EI}\left[\frac{-qx^4}{8} + \frac{qLx^3}{6}\right]_C^Q = \frac{1}{EI}\left[\frac{-qx^4}{8} + \frac{qLx^3}{6}\right]_{\frac{L}{2}}^{Q = x \text{ measured from point } B}$$

Similarly, the Vertical Intercept of a *Point x* on the beam with respect to *Point B (Right-Hand Support)* is given by:

$$\Delta_{xB}(x) = \frac{1}{EI}\left[\frac{-qx^4}{8} + \frac{qLx^3}{6}\right]_B^{Q = x \text{ measured from point } B} = \frac{1}{EI}\left[\frac{-qx^4}{8} + \frac{qLx^3}{6}\right]_0^x$$

Thus, considering the points A, D, E and B along the beam shown on the deflection curve in the previous analysis, we determine their *Vertical Intercept* and *Deflection* with respect to $Point\ B$ as follows:

<u>*Deflection of Point D with respect to Point B*</u>

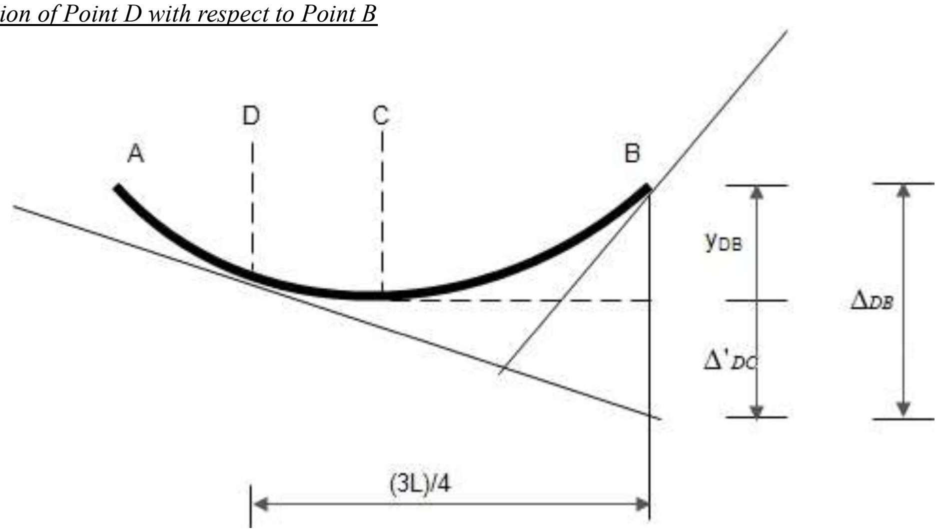

Figure 10:Schematic for determining deflection at point D

$$\underbrace{y_{DB}}_{\substack{\text{Deflection of} \\ \text{Point D wrt Point B}}} = \Delta_{DB} - \Delta'_{DC}$$

$$\Delta_{DB} = \frac{1}{EI}\left[\frac{-qx^4}{8} + \frac{qLx^3}{6}\right]_0^{\frac{3L}{4}} = \frac{1}{EI}\left[\frac{189qL^4}{6144} - 0\right] = \frac{63qL^4}{2048EI}$$

$$\Delta'_{DC} = \frac{3L}{4}|\theta_{DC}| = \frac{3L}{4}|\theta_{CD}| = \frac{3L}{4} \times \frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_{\frac{L}{2}}^{\frac{3L}{4}} = \frac{3L}{4EI}\left[\frac{27qL^3}{384} - \left(\frac{qL^3}{24}\right)\right]$$

$$\Delta'_{DC} = \frac{3L}{4EI}\left(\frac{11qL^3}{384}\right) = \frac{33qL^4}{1536EI}$$

$$y_{DB} = \frac{63qL^4}{2048EI} - \frac{33qL^4}{1536EI} = \frac{57qL^4}{6144EI}$$

<u>*Deflection of Point C with respect to Point B*</u>

$$y_{CB} = \Delta_{CB} - \Delta'_{CC}$$

$$\Delta_{CB} = \frac{1}{EI}\left[\frac{-qx^4}{8} + \frac{qLx^3}{6}\right]_0^{\frac{L}{2}} = \frac{1}{EI}\left[\frac{5qL^4}{384} - 0\right] = \frac{5qL^4}{384EI}$$

$$\Delta'_{DC} = \frac{L}{2}|\theta_{CC}| = \frac{L}{2}(0) = 0$$

$$y_{CB} = \frac{5qL^4}{384EI} - 0 = \frac{5qL^4}{384EI}$$

Deflection of Point E with respect to Point B

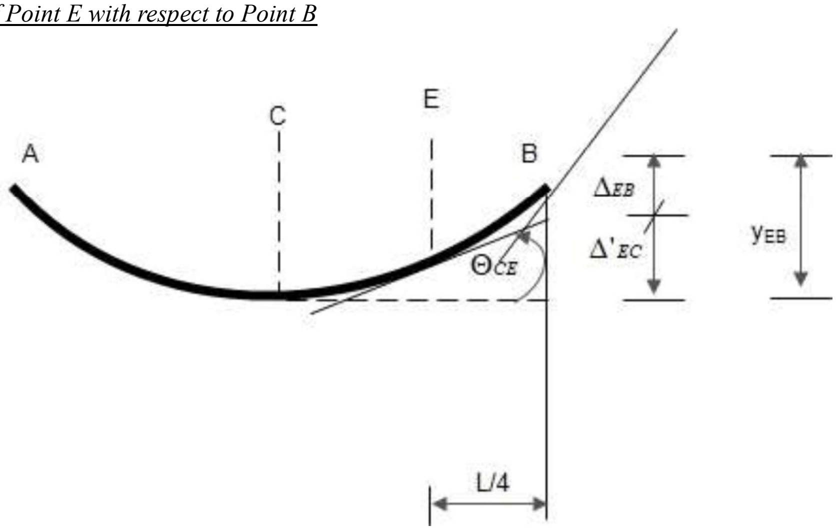

Figure 11: Schematic for determining deflection at point E

$$y_{EB} = \Delta_{EB} + \Delta'_{EC}$$

$$\Delta_{EB} = \frac{1}{EI}\left[\frac{-qx^4}{8} + \frac{qLx^3}{6}\right]_0^{\frac{L}{4}} = \frac{1}{EI}\left[\frac{13qL^4}{6144} - 0\right] = \frac{13qL^4}{6144EI}$$

$$\Delta'_{EC} = \frac{L}{4}|\theta_{EC}| = \frac{L}{4} \times \left|\frac{1}{EI}\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_{\frac{L}{2}}^{\frac{L}{4}}\right| = \frac{L}{4EI}\left|\frac{5qL^3}{384} - \left(\frac{qL^3}{24}\right)\right| = \left|\frac{-11qL^4}{1536EI}\right| = \frac{11qL^4}{1536EI}$$

$$y_{EB} = \frac{13qL^4}{6144EI} + \frac{11qL^4}{1536EI} = \frac{57qL^4}{6144EI}$$

Thus, keeping in mind that we are analyzing the beam from the Point B, the equations of deflection can be summarized as follows:

$$y_{xB} = \begin{cases} y_{xB} = \Delta_{xB} + \Delta'_{xc} \quad \forall x \in \left[0, \frac{L}{2}\right] \\ y_{xB} = \Delta_{xB} - \Delta'_{xc} \quad \forall x \in \left[\frac{L}{2}, L\right] \end{cases} \quad \Bigg| \quad \begin{array}{c} \Delta_{xB} = \frac{1}{EI}\left[\frac{-qx^4}{8} + \frac{qLx^3}{6}\right]_0^x \\ \hline \Delta'_{xc} = x|\theta_{xc}| = \frac{x}{EI}\left|\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_{\frac{L}{2}}^x\right| \end{array}$$

$$Where; \begin{cases} y_{xB} \text{ is the Deflection at any Point } x \text{ wrt to Point } B \\ \Delta_{xB} \text{ is the Vertical Intercept of any Point } x \text{ wrt Point } B \\ \Delta'_{xc} \text{ is the Modified Vertical Intercept of any Point } x \text{ wrt Point } C \end{cases}$$

Deflection of Point A with respect to Point B

$$y_{AB} = \frac{1}{EI}\left[\frac{-qx^4}{8} + \frac{qLx^3}{6}\right]_0^L - \frac{L}{EI}\left|\left[\frac{-qx^3}{6} + \frac{qLx^2}{4}\right]_{\frac{L}{2}}^L\right|$$

$$y_{AB} = \frac{1}{EI}\left[\frac{qL^4}{24}\right] - \frac{L}{EI}\left|\frac{-qL^3}{24}\right| = \frac{qL^4}{24EI} - \frac{qL^4}{24EI} = 0$$

We can further generalize the equation of Deflection further as follows:

$$y_{xB} = \begin{cases} y_{xB} = \Delta_{xB} + \Delta'_{xx_0} & \forall\, x \in [0, x_0] \\ y_{xB} = \Delta_{xB} - \Delta'_{xx_0} & \forall\, x \in [x_0, L] \end{cases} \quad \Delta_{xB} = \frac{1}{EI}\left(\frac{-qx^4}{8} + \frac{qLx^3}{6}\right) \quad x_0 \text{ is Point of "null} - \text{slope"}$$

$$\Delta'_{xx_0} = x\left|\theta_{xx_0}\right| = \frac{x}{EI}\left|\frac{-q(x^3 - x_0^3)}{6} + \frac{qL(x^2 - x_0^2)}{4}\right|$$

All distances measured from RHS

3. WHEN THE TURNING POINT ON THE DEFLECTION CURVE COINCIDES WITH THE BREAKPOINT IN THE BMD

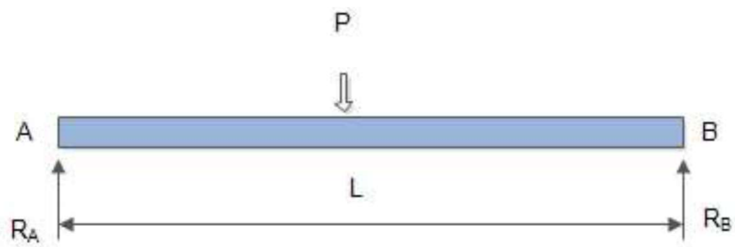

Figure 12: Beam with Turning Point in deflection curve coinciding with Breakpoint in BMD

3.1 DETERMINING SLOPE AND DEFLECTION USING MACAULAY'S METHOD

From previous knowledge, we know that $R_A = R_B = \frac{P}{2}$

$0 \le x \le \frac{L}{2}$

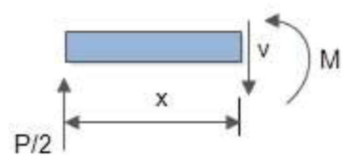

Figure 13: FBD for drawing the BMD

$$M(x) = \frac{P}{2}x$$

$\frac{L}{2} \le x \le L$

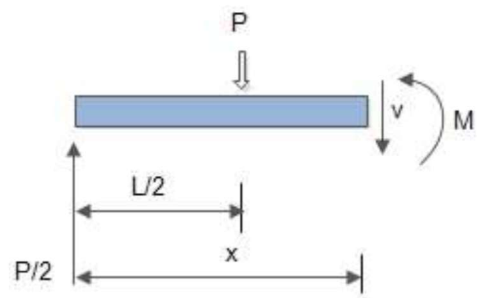

Figure 14: FBD for drawing the BMD

$$M(x) = \frac{P}{2}x - P\left(x - \frac{L}{2}\right)$$

Therefore, the total bending moment equation can be written as:

$$M(x) = \left(\frac{P}{2}x \;\middle|\; - P\left(x - \frac{L}{2}\right)\right)$$

The separation indicates the division between the two parts of the beam that have different equations for bending moment.

$$y''(x) = \frac{M}{EI} = \frac{1}{EI}\left(\frac{P}{2}x \;\middle|\; -P\left(x - \frac{L}{2}\right)\right)$$

$$y'(x) = \frac{1}{EI}\left(\frac{P}{4}x^2 + C_1 \;\middle|\; -\frac{P\left(x - \frac{L}{2}\right)^2}{2}\right)$$

$$y(x) = \frac{1}{EI}\left(\frac{P}{12}x^3 + C_1x + C_2 \;\middle|\; -\frac{P\left(x - \frac{L}{2}\right)^3}{6}\right)$$

The constants of integration are obtained by applying the boundary conditions similar to those in section 2.1.

$$y(0) = \frac{1}{EI}\left(\frac{P}{12}0^3 + C_1 0 + C_2 \;\middle|\; \right) \xrightarrow{yields} C_2 = 0$$

$$y(L) = \frac{1}{EI}\left(\frac{P}{12}L^3 + C_1L + \;\middle|\; -\frac{P\left(L - \frac{L}{2}\right)^3}{6}\right) = 0 \xrightarrow{yields} C_1 = \frac{-PL^2}{16}$$

Therefore, the equations for slope and deflection reduce to:

$$y'(x) = \frac{1}{EI}\left(\frac{P}{4}x^2 - \frac{PL^2}{16} \;\middle|\; -\frac{P\left(x - \frac{L}{2}\right)^2}{2}\right)$$

$$y(x) = \frac{1}{EI}\left(\frac{P}{12}x^3 - \frac{PL^2}{16}x \;\middle|\; -\frac{P\left(x - \frac{L}{2}\right)^3}{6}\right)$$

<u>Evaluating Slope:</u>

$$y'(0) = \frac{1}{EI}\left(\frac{P}{4}0^2 - \frac{PL^2}{16} \;\middle|\; \right) = -\frac{PL^2}{16EI}$$

$$y'\left(\frac{L}{4}\right) = \frac{1}{EI}\left(\frac{P}{64}L^2 - \frac{PL^2}{16} \;\middle|\; \right) = -\frac{3PL^2}{64EI}$$

$$y'\left(\frac{L}{2}\right) = \frac{1}{EI}\left(\frac{P}{16}L^2 - \frac{PL^2}{16} \;\middle|\; \right) = 0$$

$$y'\left(\frac{3L}{4}\right)=\frac{1}{EI}\left(\frac{9P}{64}L^2-\frac{PL^2}{16}\ \middle|\ -\frac{P\left(\frac{3L}{4}-\frac{L}{2}\right)^2}{2}\right)=\frac{3PL^2}{64EI}$$

$$y'(L)=\frac{1}{EI}\left(\frac{P}{4}L^2-\frac{PL^2}{16}\ \middle|\ -\frac{P\left(L-\frac{L}{2}\right)^2}{2}\right)=\frac{PL^2}{16EI}$$

Evaluating Deflection:

$$y(0)=\frac{1}{EI}\left(\frac{P}{4}0^3-\frac{PL^2}{16}0\ \middle|\ \right)=0$$

$$y\left(\frac{L}{4}\right)=\frac{1}{EI}\left(\frac{P}{768}L^3-\frac{PL^3}{64}\ \middle|\ \right)=-\frac{11PL^3}{768EI}$$

$$y\left(\frac{L}{2}\right)=\frac{1}{EI}\left(\frac{P}{96}L^3-\frac{PL^3}{32}\ \middle|\ \right)=-\frac{PL^3}{48EI}$$

$$y\left(\frac{3L}{4}\right)=\frac{1}{EI}\left(\frac{27P}{768}L^3-\frac{3PL^3}{64}\ \middle|\ -\frac{P\left(\frac{3L}{4}-\frac{L}{2}\right)^3}{6}\right)=-\frac{11PL^3}{768EI}$$

$$y(L)=\frac{1}{EI}\left(\frac{P}{12}L^3-\frac{PL^3}{16}\ \middle|\ -\frac{P\left(L-\frac{L}{2}\right)^3}{6}\right)=0$$

3.2 DETERMINING SLOPE AND DEFLECTION USING MOMENT-AREA METHOD

For this beam, the equation of BM with x is measured from $Point\ B$ is given by:

$$M(x)=\begin{cases}\dfrac{Px}{2}, & x\in\left[0,\dfrac{L}{2}\right]\\[2mm] \dfrac{-Px}{2}+\dfrac{PL}{2}, & x\in\left[\dfrac{L}{2},L\right]\end{cases}$$

To determine the slope and deflection of the beam, consider the deflection of the loaded beam as shown in the diagram below.

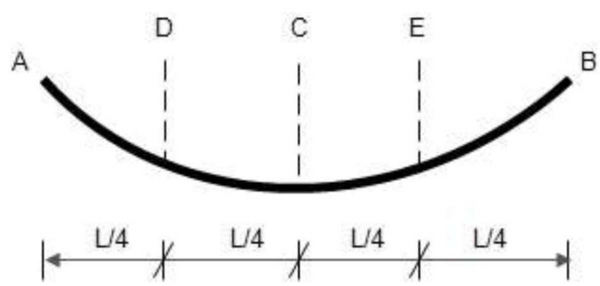

Figure 15: Deflection curve of the loaded beam

Analysis of Slope:

From previous analysis, know the slope will be zero at Point C (*mid-span*) when the load, P, is applied at C. Therefore, the equation for maximum slope along the length of the beam with respect to Point C is given as:

$Max\ Slope\ =\ \theta_{AC}\ =\ \theta_{BC}\,(albeit\ with\ different\ signs!)$

As in the previous example, we attempt to derive a general equation to determine the slope at any point along the length of the beam with respect to position of "$null-slope$" on the beam.

$$From\ previous\ theory,\qquad \theta_{QP}=\frac{1}{EI}\int_P^Q M\,dx$$

Slope at Point x wrt to the Slope at Point C when $x \in [0.5L\ , L]$

$$\theta_{QP}(x) = \frac{1}{EI}\int_P^Q \left(\frac{-Px}{2} + \frac{PL}{2}\right) dx = \frac{1}{EI}\left[\frac{-Px^2}{4} + \frac{PLx}{2}\right]_P^Q$$

$$\theta_{xC}(x) = \frac{1}{EI}\left[\frac{-Px^2}{4} + \frac{PLx}{2}\right]_C^Q = \frac{1}{EI}\left[\frac{-Px^2}{4} + \frac{PLx}{2}\right]_{\frac{L}{2}}^{Q = x \text{ measured from point } B} \Bigg| \forall\, x \in [0.5L\ , L]$$

Hence:

$$\theta_{AC} = \frac{1}{EI}\left[\frac{-Px^2}{4} + \frac{PLx}{2}\right]_{\frac{L}{2}}^{L} = \frac{1}{EI}\left[\left(\frac{PL^2}{4}\right) - \left(\frac{3PL^2}{16}\right)\right] = \frac{PL^2}{16EI} \text{ or } \theta_{CA} = \frac{-PL^2}{16EI}$$

(*the negative sign indicates that slope of A measured from C is clockwise*)

$$\theta_{DC} = \frac{1}{EI}\left[\frac{-Px^2}{4} + \frac{PLx}{2}\right]_{\frac{L}{2}}^{\frac{3L}{4}} = \frac{1}{EI}\left[\left(\frac{15PL^2}{64}\right) - \left(\frac{3PL^2}{16}\right)\right] = \frac{3PL^2}{64EI} \text{ or } \theta_{CD} = \frac{-3PL^2}{64EI}$$

Slope at Point x wrt the Slope at Point C when $x \in [0\ , 0.5L]$

$$\theta_{QP}(x) = \frac{1}{EI}\int_P^Q \left(\frac{Px}{2}\right) dx = \frac{1}{EI}\left[\frac{Px^2}{4}\right]_P^Q$$

$$\theta_{xC}(x) = \frac{1}{EI}\left[\frac{Px^2}{4}\right]_C^Q = \frac{1}{EI}\left[\frac{Px^2}{4}\right]_{\frac{L}{2}}^{Q = x \text{ measured from point } B} \Bigg| \forall\, x \in [0\ , 0.5L]$$

Hence:

$$\theta_{EC} = \frac{1}{EI}\left[\frac{Px^2}{4}\right]_{\frac{L}{2}}^{\frac{L}{4}} = \frac{1}{EI}\left[\left(\frac{PL^2}{64}\right) - \left(\frac{PL^2}{16}\right)\right] = \frac{-3PL^2}{64EI} \text{ or } \theta_{CE} = \frac{3PL^2}{64EI}$$

$$\theta_{BC} = \frac{1}{EI}\left[\frac{Px^2}{4}\right]_{\frac{L}{2}}^{0} = \frac{1}{EI}\left[0 - \left(\frac{PL^2}{16}\right)\right] = \frac{-PL^2}{16EI} \text{ or } \theta_{CB} = \frac{PL^2}{16EI}$$

Considering the variation of slope when $x \in [0\ , 0.5L]$, we can determine the nature of variation of Slope along the length of the beam as:

$$\frac{d}{dx}\{\theta_{xC}\} = \frac{d}{dx}\left\{\frac{1}{EI}\int_C^x M\, dx\right\} = \frac{1}{EI} \times \frac{d}{dx}\left\{\int_C^x M\, dx\right\} = \frac{1}{EI}\{M\} = \frac{Px}{2EI}$$

$$\text{Thus, if } \frac{d}{dx}\{\theta_{xC}\} = 0 \xrightarrow{\text{yields}} x = 0$$

This means that the slope varies *linearly* along the length of the beam and *Maximum Slope* with respect to the *Slope at Point C* occurs when $x = 0$.

$\theta_{max} = \theta_{0C} = \theta_{BC}$ (*which agrees with our earlier analysis*)

Alternatively, we can arrive at the same conclusion by considering the variation of slope when $x \in [0.5L\ , L]$ as follows:

$$\frac{d}{dx}\{\theta_{xC}\} = \frac{d}{dx}\left\{\frac{1}{EI}\int_C^x M\, dx\right\} = \frac{1}{EI} \times \frac{d}{dx}\left\{\int_C^x M\, dx\right\} = \frac{M}{EI} = \frac{-Px}{2EI} + \frac{PL}{2EI}$$

$$\text{Thus, if } \frac{d}{dx}\{\theta_{xC}\} = 0 \xrightarrow{\text{yields}} x = L$$

Hence, $\theta_{max} = \theta_{LC} = \theta_{AC}$ (*which agrees with our earlier analysis*)

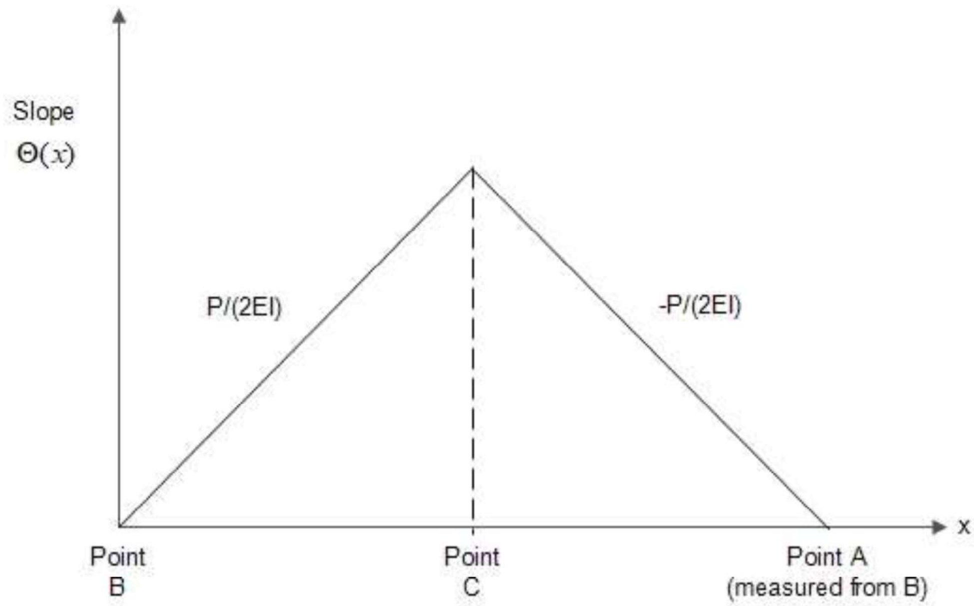

Figure 16: Variation of Slope

Analysis of Deflection:
We have already established that the slope varies linearly with the distance *x* measured from *Point B*. With this information, we attempt to derive a general equation to determine the *Vertical Intercept* at a *Point x* along the length of the beam with respect to the right-hand support *(Point B)*.

$$\text{From previous theory,}\qquad \Delta_{QP} = \frac{1}{EI}\int_{P}^{Q} x\, dA = \frac{1}{EI}\int_{P}^{Q} Mx\, dx$$

Vertical Intercept and Deflection of Point x with respect to Point B when $x \in [0\,,0.5L]$

$$\Delta_{QP}(x) = \frac{1}{EI}\int_{P}^{Q}\left(\frac{Px^2}{2}\right) dx = \frac{1}{EI}\left[\frac{Px^3}{6}\right]_{P}^{Q}$$

Thus, the Vertical Intercept of a *Point x* on the beam with respect to the *Right-Hand Support (i.e. Point B)* is given by:

$$\Delta_{xB}(x) = \frac{1}{EI}\left[\frac{Px^3}{6}\right]_{B}^{Q=x\ \text{measured from point}\ B} = \frac{1}{EI}\left[\frac{Px^3}{6}\right]_{0}^{x}$$

Thus, considering the points $A, D, E\ and\ B$ along the beam shown on the deflection curve in the previous analysis, we determine their *Vertical Intercept* and *Deflection* with respect to $Point\ B$ as follows:
Deflection of Point E with respect to Point B

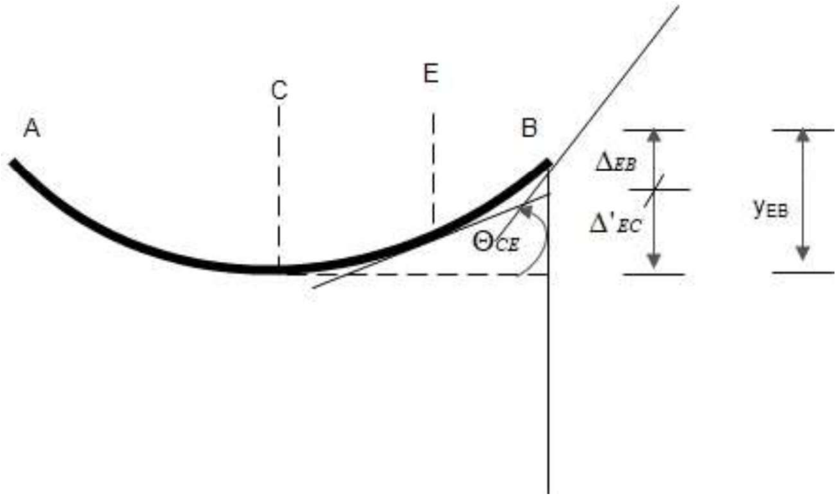

Figure 17: Schematic for determining deflection at point E

$$y_{EB} = \Delta_{EB} + \Delta'_{EC}$$

$$\Delta_{EB} = \frac{1}{EI}\left[\frac{Px^3}{6}\right]_0^{\frac{L}{4}} = \frac{1}{EI}\left(\frac{PL^3}{384}\right) = \frac{PL^3}{384EI}$$

$$\Delta'_{EC} = \frac{L}{4}|\theta_{EC}| = \frac{L}{4}|\theta_{CE}| = \frac{L}{4} \times \frac{1}{EI}\left|\left[\frac{Px^2}{4}\right]_{\frac{L}{2}}^{\frac{L}{4}}\right| = \frac{L}{4EI}\left|\left(\frac{PL^2}{64}\right) - \left(\frac{PL^2}{16}\right)\right|$$

$$\Delta'_{EC} = \frac{L}{4EI}\left|\frac{-3PL^2}{64}\right| = \frac{3PL^3}{256EI}$$

$$Thus, y_{EB} = \frac{PL^3}{384EI} + \frac{3PL^3}{256EI} = \frac{11PL^3}{768EI}$$

Deflection of Point C with respect to Point B

$$y_{CB} = \Delta_{CB} + \Delta'_{CC}$$

$$\Delta_{CB} = \frac{1}{EI}\left[\frac{Px^3}{6}\right]_0^{\frac{L}{2}} = \frac{1}{EI}\left(\frac{PL^3}{48}\right) = \frac{PL^3}{48EI}$$

$$\Delta'_{CC} = 0$$

$$Thus, y_{CB} = \frac{PL^3}{48EI} + 0 = \frac{PL^3}{48EI}$$

Vertical Intercept and Deflection of Point x with respect to Point B when $x \in [0.5L, L]$

$$\Delta_{QP}(x) = \frac{1}{EI}\int_P^Q \left(\frac{-Px^2}{2} + \frac{PLx}{2}\right) dx = \frac{1}{EI}\left[\frac{-Px^3}{6} + \frac{PLx^2}{4}\right]_P^Q$$

Thus, the Vertical Intercept of a *Point x* on the beam with respect to the *Right-Hand Support (i.e. Point B)* is given by:

$$\Delta_{xB}(x) = \frac{1}{EI}\left[\frac{-Px^3}{6} + \frac{PLx^2}{4}\right]_B^{Q = x \text{ measured from point } B} = \frac{1}{EI}\left[\frac{-Px^3}{6} + \frac{PLx^2}{4}\right]_0^x$$

Thus, considering the points A, D, E and B along the beam shown on the deflection curve in the previous analysis, we determine their *Vertical Intercept* and *Deflection* with respect to $Point\ B$ as follows:

Deflection of Point D with respect to Point B

$D_{DB} = \Delta_{DB} - \Delta'_{DC}$

Note that in our previous example, even though the tangent to the deflection curve changes when $x \geq 0.5L$, the slope is still determined by the same equation of bending moment. In this example however, the slope is defined by a different equation of bending moment $when\ x \leq 0.5L\ and\ when\ x \geq 0.5L$. Since the *Point C* is common to both equations of bending moment, the actual Vertical Intercept of a Point at a distance $x \geq 0.5L$ can be calculated as shown below.

$$\Delta_{DB} = \frac{1}{EI}\left[\frac{-Px^3}{6} + \frac{PLx^2}{4}\right]_0^{\frac{3L}{4}} - \left(\frac{PL^3}{48EI}\right) = \left(\frac{27PL^3}{384EI} - 0\right) - \left(\frac{PL^3}{48EI}\right) = \frac{19PL^3}{384EI}$$

We shall denote the term $\left(\frac{PL^3}{48EI}\right)$ as *Correcting Factor*, since it is used to *adjust* the value of *Vertical Intercept* obtained.

$$\Delta'_{DC} = \frac{3L}{4}|\theta_{DC}| = \frac{3L}{4}|\theta_{CD}| = \frac{3L}{4} \times \frac{1}{EI}\left[\frac{-Px^2}{4} + \frac{PLx}{2}\right]_{\frac{L}{2}}^{\frac{3L}{4}} = \frac{3L}{4EI}\left[\left(\frac{15PL^2}{64}\right) - \left(\frac{3PL^2}{16}\right)\right]$$

$$\Delta'_{DC} = \frac{3L}{4EI}\left(\frac{3PL^2}{64}\right) = \frac{9PL^3}{256EI}$$

$$Hence, y_{DB} = \frac{19PL^3}{384EI} - \frac{9PL^3}{256EI} = \frac{38PL^3 - 27PL^3}{768EI} = \frac{11PL^3}{768EI}$$

<u>*Deflection of Point A with respect to Point B*</u>

$$y_{AB} = \Delta_{AB} - \Delta'_{AC}$$

$$\Delta_{AB} = \frac{1}{EI}\left[\frac{-Px^3}{6} + \frac{PLx^2}{4}\right]_0^{L} - \left(\frac{PL^3}{48EI}\right) = \left(\frac{PL^3}{12EI} - 0\right) - \left(\frac{PL^3}{48EI}\right) = \frac{3PL^3}{48EI} = \frac{PL^3}{16EI}$$

$$\Delta'_{AC} = L|\theta_{AC}| = L|\theta_{CA}| = L \times \frac{1}{EI}\left[\frac{-Px^2}{4} + \frac{PLx}{2}\right]_{\frac{L}{2}}^{L} = \frac{L}{EI}\left[\left(\frac{PL^2}{4}\right) - \left(\frac{3PL^2}{16}\right)\right] = \frac{L}{EI}\left(\frac{PL^2}{16}\right)$$

$$Thus, \quad \Delta'_{AC} = \frac{PL^3}{16EI}$$

$$Hence, \quad y_{AB} = \frac{PL^3}{16EI} - \frac{PL^3}{16EI} = 0$$

<u>*Deflection of Point C with respect to Point B*</u>

We have already determined the deflection of *Point C* when $x \leq 0.5L$. However, we stated that this point is common to both equations of bending moment. Hence, the deflection of the same point when $x \geq 0.5L$ can be given as:

$$y_{CB} = \Delta_{CB} - \Delta'_{CC}$$

$$\Delta_{CB} = \frac{1}{EI}\left[\frac{-Px^3}{6} + \frac{PLx^2}{4}\right]_0^{\frac{L}{2}} - \underbrace{\left(\frac{PL^3}{48EI}\right)}_{\substack{Correcting\\Factor\\(CF)}} = \left(\frac{PL^3}{24EI} - 0\right) - \left(\frac{PL^3}{48EI}\right) = \frac{PL^3}{48EI}$$

$CF = max\ Vertical\ Intercept\ of\ preceding\ equation\ of\ BM\ wrt\ Point\ B$

$$Thus\ if\ \Delta'_{CC} = 0\ then\ y_{CB} = \frac{PL^3}{48EI} - 0 = \frac{PL^3}{48EI}\ (which\ agrees\ with\ our\ previous\ analysis)$$

In summary, we can modify our general equation of deflection by incorporating the *Correcting Factor (CF)* as shown below.

$$y_{xB} = \Delta_{xB} + \Delta'_{xx_0} \quad \forall\, x \in [0, x_0] \quad \Bigg| \quad \Delta_{xB} = \frac{1}{EI}\left(\frac{Px^3}{6}\right) \quad CF = \Delta_{x_0 B} = \frac{Px_0^3}{6EI} \quad ; \quad \Delta'_{xx_0} = x\left|\theta_{xx_0}\right| = \frac{x}{EI}\left|\frac{P(x^2 - x_0^2)}{4}\right|$$

All distances measured from RHS

$$y_{xB} = \Delta_{xB} - \Delta'_{xx_0} - CF \quad \forall\, x \in [x_0, L] \quad \Bigg| \quad \Delta_{xB} = \frac{1}{EI}\left(\frac{-Px^3}{6} + \frac{PLx^2}{4}\right) \quad ; \quad \Delta'_{xx_0} = x\left|\theta_{xx_0}\right| = \frac{x}{EI}\left|\frac{-P(x^2 - x_0^2)}{4} + \frac{PL(x - x_0)}{2}\right|$$

All distances measured from RHS

4. WHEN THE TURNING POINT ON THE DEFLECTION CURVE DOES NOT COINCIDE WITH THE BREAKPOINT IN THE BMD

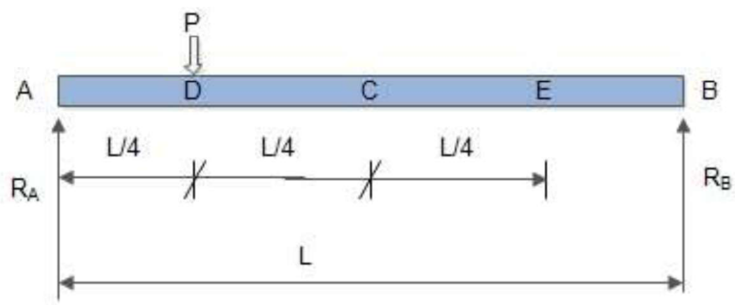

Figure 18: Beam where Turning Point in deflection curve does not coincide with Breakpoint in BMD

4.1 DETERMINING SLOPE AND DEFLECTION USING MACAULAY'S METHOD

$$\text{Slope and Deflection} \begin{cases} y'(x) = \dfrac{P}{2LEI}\left[bx^2 - \dfrac{b}{3}(L^2 - b^2)\right| - L(x-a)^2\Big] \\ y(x) = \dfrac{P}{6LEI}\left[bx^3 - b(L^2 - b^2)x\right| - L(x-a)^3\Big] \end{cases}$$

$$\text{Max Deflection and Position} \begin{cases} y_{max} = \dfrac{-Pb}{9\sqrt{3}\,LEI}(L^2 - b^2)^{\frac{3}{2}} \\ \quad at\ x_{max} = \sqrt{\dfrac{L^2 - b^2}{3}} \end{cases}$$

Value and Location of Maximum Deflection:

$$y_{max} = \frac{-Pb}{9\sqrt{3}\,LEI}(L^2 - b^2)^{\frac{3}{2}} = \frac{-P}{12\sqrt{3}\,EI}\left(\frac{7L^2}{16}\right)^{\frac{3}{2}} = \frac{-P}{12\sqrt{3}\,EI}\left(\frac{7L^2}{16}\right)\left(\frac{7L^2}{16}\right)^{\frac{1}{2}}$$

$$y_{max} = \frac{-P}{12\sqrt{3}\,EI}\left(\frac{7L^2}{16}\right)\left(\frac{L\sqrt{7}}{4}\right) = \frac{-7\sqrt{7}\,PL^3}{768\sqrt{3}\,EI}$$

$$y_{max}\ occurs\ at\ x_{max} = \sqrt{\frac{L^2 - b^2}{3}} = \sqrt{\frac{7L^2}{48}} = L\sqrt{\frac{7}{48}} = 0.3819L\ (between\ Points\ D\ and\ C)$$

Slope of Selected Points with Respect to position of "null-slope"
Slope of Point A

$$y'_A = \frac{P}{2LEI}\left[0 - \frac{3L}{12}\left(\frac{7L^2}{16}\right)\right] = \frac{P}{2LEI}\left[0 - \left(\frac{21L^3}{192}\right)\right] = \frac{-21PL^2}{384EI}$$

$(-ve\ sign\ indicates\ its\ meaured\ clockwise\ from\ "null - slope"\ i.e.\ horizontal)$

Slope of Point D

$$y'_D = \frac{P}{2LEI}\left[\frac{3L}{4}\left(\frac{L^2}{16}\right) - \left(\frac{21L^3}{192}\right)\right] = \frac{P}{2LEI}\left[\left(\frac{3L^3}{64}\right) - \left(\frac{21L^3}{192}\right)\right] = \frac{P}{2LEI}\left[\frac{9L^3 - 21L^3}{192}\right] = \frac{-12PL^2}{384EI}$$

$$y'_D = \frac{-PL^2}{32EI}$$

Slope of Point C

$$y'_C = \frac{P}{2LEI}\left[\frac{3L}{4}\left(\frac{L^2}{4}\right) - \left(\frac{21L^3}{192}\right) - L\left(\frac{L}{4}\right)^2\right] = \frac{P}{2LEI}\left[\left(\frac{3L^3}{16}\right) - \left(\frac{21L^3}{192}\right) - \left(\frac{L^3}{16}\right)\right]$$

$$y'_C = \frac{3PL^2}{384EI} = \frac{PL^2}{128EI}$$

$(+ve\ sign\ indicates\ its\ meaured\ counterclockwise\ from\ "null - slope"\ i.e.\ horizontal)$

Slope of Point E

$$y'_E = \frac{P}{2LEI}\left[\frac{3L}{4}\left(\frac{9L^2}{16}\right) - \left(\frac{21L^3}{192}\right) - L\left(\frac{L}{2}\right)^2\right] = \frac{P}{2LEI}\left[\left(\frac{27L^3}{64}\right) - \left(\frac{21L^3}{192}\right) - \left(\frac{L^3}{4}\right)\right]$$

$$y'_E = \frac{12PL^2}{384EI} = \frac{PL^2}{32EI}$$

Slope of Point B

$$y'_B = \frac{P}{2LEI}\left[\frac{3L}{4}(L^2) - \left(\frac{21L^3}{192}\right) - L\left(\frac{3L}{4}\right)^2\right] = \frac{P}{2LEI}\left[\left(\frac{3L^3}{4}\right) - \left(\frac{21L^3}{192}\right) - \left(\frac{9L^3}{16}\right)\right] = \frac{15PL^2}{384EI}$$

Deflection of Selected Points with Respect to the Supports

Deflection of Point A

$$y_A = \frac{P}{6LEI}[bx^3 - b(L^2 - b^2)x] = \frac{Pb}{6LEI}[x^3 - (L^2 - b^2)x] = \frac{3P}{24EI}[0 - 0] = 0$$

$(-ve\ sign\ indicates\ deflection\ is\ meaured\ downwards\ from\ the\ supports)$

Deflection of Point D

$$y_D = \frac{Pb}{6LEI}[x^3 - (L^2 - b^2)x] = \frac{3P}{24EI}\left[\left(\frac{L^3}{64}\right) - \left(\frac{7L^2}{16}\right)\frac{L}{4}\right] = \frac{P}{8EI}\left(\frac{-3L^3}{32}\right) = \frac{-3PL^3}{256EI}$$

Deflection of Point C

$$y_C = \frac{P}{6LEI}\left[\frac{3L}{4}\left(\frac{L^3}{8}\right) - \frac{3L}{4}\left(\frac{7L^2}{16}\right)\frac{L}{2} - L\left(\frac{L^3}{64}\right)\right] = \frac{P}{6EI}\left[\left(\frac{3L^3}{32}\right) - \left(\frac{21L^3}{128}\right) - \left(\frac{L^3}{64}\right)\right] = \frac{-11PL^3}{768EI}$$

Deflection of Point E

$$y_E = \frac{P}{6LEI}\left[\frac{3L}{4}\left(\frac{27L^3}{64}\right) - \frac{3L}{4}\left(\frac{7L^2}{16}\right)\frac{3L}{4} - L\left(\frac{L^3}{8}\right)\right] = \frac{P}{6EI}\left[\left(\frac{81L^3}{256}\right) - \left(\frac{63L^3}{256}\right) - \left(\frac{L^3}{8}\right)\right]$$

$$y_E = \frac{-14PL^3}{1536EI} = \frac{-7PL^3}{768EI}$$

Deflection of Point B

$$y_B = \frac{P}{6LEI}\left[\frac{3L}{4}(L^3) - \frac{3L}{4}\left(\frac{7L^2}{16}\right)L - L\left(\frac{27L^3}{64}\right)\right] = \frac{P}{6EI}\left[\left(\frac{3L^3}{4}\right) - \left(\frac{21L^3}{64}\right) - \left(\frac{27L^3}{64}\right)\right] = 0$$

4.2 DETERMINING SLOPE AND DEFLECTION USING MOMENT-AREA METHOD

$$R_A = \frac{Pb}{L}\ and\ R_B = \frac{Pa}{L}$$

Analyzing the beam from the RHS, we generate the equations of BM as:

$$M(x) = \begin{cases} R_B x = \dfrac{Pa}{L}x = \dfrac{P}{4}x, & x \in [0\,, b] \\ R_B x - P(x-b) = \dfrac{Pb}{L}(L-x) = \dfrac{3P}{4}(L-x), & x \in [b\,, L] \end{cases}$$

From these equations we draw super-imposed diagrams of *Bending Moment* and *Slope* as shown below.

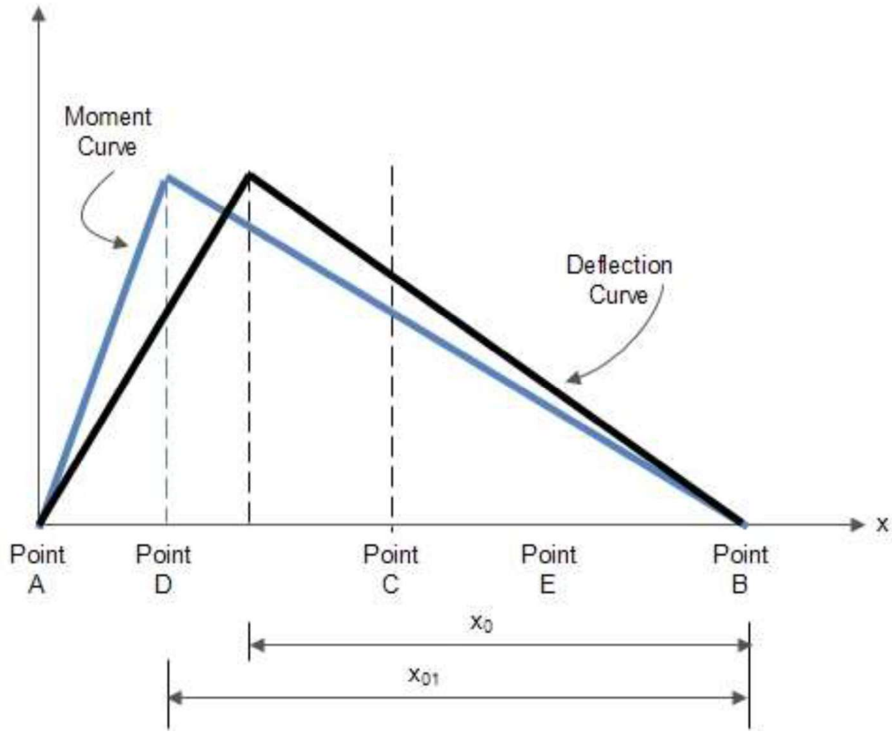

Figure 19: Deflection curve and Moment curve

In a slight deviation from previous analysis, the turning point of the Slope Curve and Moment Curve do not coincide $i.e.\ x_0 < x_{01}$. Before we can determine how this affects equations of *Slope and Vertical Intercept* we do some general derivations as shown below.

$When\ x \in [0\,, x_{01}]$

$$\theta_{QP} = \frac{1}{EI}\int_P^Q M(x)\ dx = \frac{1}{EI}\int_P^Q \frac{Px}{4}dx = \frac{1}{EI}\left[\frac{Px^2}{8}\right]_P^Q$$

$$\Delta_{QP} = \frac{1}{EI}\int_P^Q xM(x)\ dx = \frac{1}{EI}\int_P^Q \frac{Px^2}{4}dx = \frac{1}{EI}\left[\frac{Px^3}{12}\right]_P^Q$$

$When\ x \in [x_{01}\,, L]$

$$\theta_{QP} = \frac{1}{EI}\int_P^Q M(x)\ dx = \frac{1}{EI}\int_P^Q \frac{3P}{4}(L-x)\ dx = \frac{3P}{4EI}\int_P^Q (-x+L)\ dx = \frac{3P}{4EI}\left[\frac{2Lx - x^2}{2}\right]_P^Q$$

$$\Delta_{QP} = \frac{1}{EI}\int_P^Q xM(x)\ dx = \frac{1}{EI}\int_P^Q \frac{3P}{4}(Lx - x^2)\ dx = \frac{3P}{4EI}\left[\frac{Lx^2}{2} - \frac{x^3}{3}\right]_P^Q = \frac{3P}{4EI}\left[\frac{3Lx^2 - 2x^3}{6}\right]_P^Q$$

Applying these general equations to our specific beam:

$$\left\{ \theta_{x x_0} = \begin{cases} x \in [0\,, x_{01}],\ \dfrac{1}{EI}\left[\dfrac{Px^2}{8}\right]_{x_0}^{x} \\ x \in [x_{01}\,, L],\ \dfrac{1}{EI}\left[\dfrac{3P(2Lx - x^2)}{8}\right]_{x_0}^{x} \end{cases} \middle|\ \Delta_{xB} = \begin{cases} x \in [0\,, x_{01}],\ \dfrac{1}{EI}\left[\dfrac{Px^3}{12}\right]_0^{x} \\ x \in [x_{01}\,, L],\ \dfrac{1}{EI}\left[\dfrac{P(3Lx^2 - 2x^3)}{8}\right]_0^{x} \end{cases} \right\}$$

From these derived equations, we determine the expression for *Vertical Intercept of Point A wrt Point B* as follows:

$$\Delta_{AB} = \Delta_{AD} + \Delta_{DB} = \frac{P}{8EI}[3Lx^2 - 2x^3]_{\frac{3L}{4}}^{L} + \frac{P}{12EI}[x^3]_0^{\frac{3L}{4}} = \frac{P}{8EI}\left[L^3 - \frac{27L^3}{32}\right] + \frac{P}{12EI}\left(\frac{27L^3}{64}\right)$$

$$\Delta_{AB} = \left(\frac{5PL^3}{256EI}\right) + \left(\frac{27PL^3}{768EI}\right) = \frac{21PL^3}{384EI}$$

Similarly, the expression of *Slope of Point A wrt Point B* can be determined as:

$$\theta_{AB} = \theta_{AD} + \theta_{DB} = \frac{3P}{8EI}[2Lx - x^2]_{\frac{3L}{4}}^{L} + \frac{P}{8EI}[x^2]_0^{\frac{3L}{4}} = \frac{3PL^2}{128EI} + \frac{9PL^2}{128EI} = \frac{3PL^2}{32EI}$$

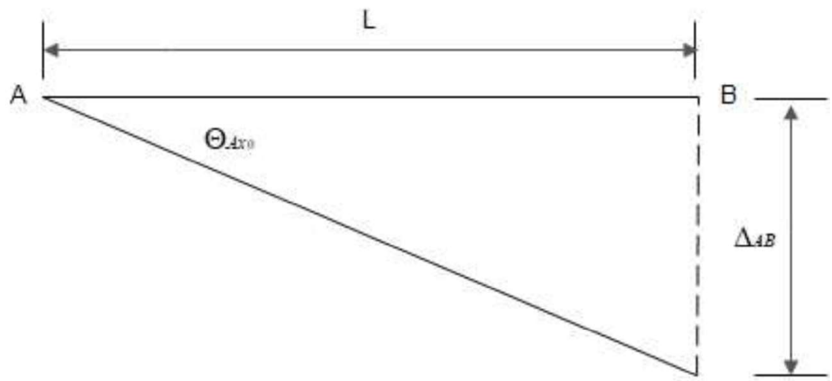

Figure 20: Schematic for calculating slope

From the diagram above, $\Delta_{AB} = L\left|\theta_{A\,x_0}\right|$

$$\therefore \left|\theta_{Ax_0}\right| = \frac{\Delta_{AB}}{L} = \frac{21PL^2}{384EI}$$

$$But\ we\ know, \theta_{AB} = \theta_{A\,x_0} - \theta_{B x_0} = \theta_{A\,x_0} + \theta_{x_0 B} = \left|\theta_{Ax_0}\right| + \left|\theta_{B\,x_0}\right|$$

$$\therefore \left|\theta_{Bx_0}\right| = \frac{3PL^2}{32EI} - \frac{21PL^2}{384EI} = \frac{15PL^2}{384EI}$$

To determine the location of Maximum Deflection, we utilize two principles. Firstly, we know *maximum deflection* will occur at the point of *"null-slope"* i.e. *Point* x_0 and secondly, this point of *"null-slope"* will occur before *Point D* $i.e.\ x_0 < x_{01}$. We can now determine the value of x_0 as:

$$\theta_{B\,x_0} = \frac{1}{EI}\left[\frac{Px^2}{8}\right]_{x_0}^{0} = \frac{-Px_0^2}{8EI} \xrightarrow{yields} \left|\theta_{B\,x_0}\right| = \frac{Px_0^2}{8EI} = \frac{15PL^2}{384EI}$$

$$x_0^2 = \frac{120}{384}L^2 \xrightarrow{yields} x_0 = L\sqrt{\frac{120}{384}} \quad from\ the\ Point\ B$$

$(We\ discard\ the\ negative\ root\ as\ it\ is\ out\ of\ our\ region\ of\ interest)$

Having determined the value of x_0 where the maximum deflection occurs, we now determine the value of maximum deflection itself as:

$$y_{max} = y_{x_0 B} = \Delta_{x_0 B} + \Delta'_{x_0 x_0} = \Delta_{x_0 B} = \frac{P}{12EI}[x^3]_0^{x_0} = \frac{Px_0^3}{12EI} = \frac{PL^3}{12EI} \times \frac{120}{384}\sqrt{\frac{120}{384}}$$

$$y_{max} = \frac{PL^3}{EI} \times \frac{10}{384}\sqrt{\frac{5}{16}} = \frac{10\sqrt{5}\,PL^3}{1536\,EI} = \frac{5\sqrt{5}\,PL^3}{768\,EI}$$

Slope of Selected Points with Respect to position of "null-slope"

Slope of Point D

$$\theta_{D x_0} = \frac{1}{EI}\left[\frac{Px^2}{8}\right]_{x_0}^{\frac{3L}{4}} = \frac{P}{8EI}[x^2]_{x_0}^{\frac{3L}{4}} = \frac{P}{8EI}\left[\frac{9L^2}{16} - x_0^2\right] = \frac{9PL^2}{128EI} - \frac{P}{8EI}(x_0^2)$$

$But\ we\ already\ determined\ the\ value\ of\ x_0\ as\ L\sqrt{\frac{120}{384}}\ therefore:$

$$\theta_{D x_0} = \frac{9PL^2}{128EI} - \frac{P}{8EI}\left(\frac{120L^2}{384}\right) = \frac{9PL^2}{128EI} - \frac{15PL^2}{384EI} = \frac{12PL^2}{384EI} = \frac{PL^2}{32EI}$$

$(+ve\ sign\ indicates\ the\ slope\ at\ Point\ D\ is\ measured\ clockwise\ wrt\ the\ slope\ at\ Point\ x_0)$

$However, we\ already\ determined\ that\ \frac{15PL^2}{384EI} = -\theta_{B x_0} = \theta_{x_0 B}\ therefore:$

$$\theta_{D x_0} = \frac{1}{EI}\left[\frac{Px^2}{8}\right] - |\theta_{B x_0}|\ where\ x = \frac{3L}{4}$$

Thus, the general equation for Slope when $x \in [0, x_{01}]$ $is\ given\ by:$

$$\theta_{x x_0} = \frac{1}{EI}\left[\frac{Px^2}{8}\right] - |\theta_{B x_0}|\ \forall x \in [0, x_{01}]$$

Slope of Point C

$$Similarly, if\ \theta_{x x_0} = \frac{1}{EI}\left[\frac{Px^2}{8}\right] - |\theta_{B x_0}|\ then\ \theta_{C x_0} = \frac{P}{8EI}\left[\frac{L^2}{4}\right] - \frac{15PL^2}{384EI} = \frac{-3PL^2}{384EI}$$

$$\theta_{C x_0} = \frac{-PL^2}{128EI}$$

$(-ve\ sign\ indicates\ the\ slope\ at\ Point\ C\ is\ measured\ counterclockwise\ wrt\ the\ slope\ at\ Point\ x_0)$

Slope of Point E

$$Similarly, if\ \theta_{x x_0} = \frac{1}{EI}\left[\frac{Px^2}{8}\right] - |\theta_{B x_0}|\ then\ \theta_{E x_0} = \frac{P}{8EI}\left[\frac{L^2}{16}\right] - \frac{15PL^2}{384EI} = \frac{PL^2}{128EI} - \frac{15PL^2}{384EI}$$

$$\theta_{E x_0} = \frac{-PL^2}{32EI}$$

Slope of Point B

$$Similarly, if\ \theta_{x x_0} = \frac{1}{EI}\left[\frac{Px^2}{8}\right] - |\theta_{B x_0}|\ then\ \theta_{B x_0} = \frac{P}{8EI}[0] - \frac{15PL^2}{384EI} = \frac{-15PL^2}{384EI}$$

Slope of Point A

$$\theta_{A x_0} = \frac{1}{EI}\left[\frac{3P(2Lx - x^2)}{8}\right]_{x_0}^{L} = \frac{3P}{8EI}[2Lx - x^2]_{x_0}^{L} = \frac{3P}{8EI}[(L^2) - (2Lx_0 - x_0^2)]$$

$$But\ we\ know\ that\ \theta_{Ax_0} = \underbrace{\theta_{AD}}_{\substack{Slope\ of\ point\ A\\ wrt\ to\ point\ D}} + \underbrace{\theta_{Dx_0}}_{\substack{Slope\ of\ Point\ D\\ wrt\ point\ x_0}}$$

$$\theta_{xD} = \frac{3P}{8EI}[2Lx - x^2]_{x_{01}=\frac{3L}{4}}^{L} = \frac{3P}{8EI}[2Lx - x^2] - \frac{3P}{8EI}\left[\left(\frac{6L^2}{4}\right) - \left(\frac{9L^2}{16}\right)\right]$$

$$\theta_{xD} = \frac{3P}{8EI}[2Lx - x^2] - \frac{45PL^2}{128EI}$$

$$\theta_{D x_0} = \frac{1}{EI}\left[\frac{Px^2}{8}\right] - |\theta_{B x_0}| = \frac{9PL^2}{128EI} - |\theta_{B x_0}|$$

$$\therefore\ \theta_{x x_0} = \frac{3P}{8EI}[2Lx - x^2] - \frac{45PL^2}{128EI} + \frac{9PL^2}{128EI} - |\theta_{B x_0}|$$

$$\theta_{xx_0} = \frac{3P}{8EI}[2Lx - x^2] - |\theta_{Bx_0}| - \underbrace{\frac{36PL^2}{128EI}}_{\substack{Correcting\ Factor \\ for\ Slope\ CF_{slope}}}\ where\ x \in [x_{01}, L]$$

The constant term, $\frac{36PL^2}{128EI}$ is known as the correcting factor for slope and is calculated using the two equations of bending moment at Point D as follows:

$$CF_{slope} = \frac{1}{EI}\left[\frac{3P(2Lx - x^2)}{8}\right] - \frac{1}{EI}\left[\frac{Px^2}{8}\right] = \frac{3P}{8EI}[2Lx - x^2] - \frac{P}{8EI}[x^2]$$

$$CF_{slope} = \frac{3P}{8EI}\left(2Lx - x^2 - \frac{x^2}{3}\right) = \frac{3P}{8EI}\left(2Lx - \frac{4x^2}{3}\right)\ where\ in\ this\ case, x = x_{01} = \frac{3L}{4}$$

$$\therefore, in\ our\ case, CF_{slope} = \frac{3P}{8EI}\left(\frac{6L^2}{4} - \frac{36L^2}{48}\right) = \frac{9PL^2}{32EI}$$

Therefore, we can write a general equation of calculating slope when $x > x_{01}\,as$:

$$\theta_{xx_0} = \frac{3P}{8EI}[2Lx - x^2] - |\theta_{Bx_0}| - CF_{slope}\ Where\ CF_{slope} = \frac{3P}{8EI}\left(2Lx - \frac{4x^2}{3}\right)\ \forall x \in [x_{01}, L]$$

Thus, using our equation in the example, we obtain the slope of Point A with respect to the Point of *"null-slope"* as:

$$\theta_{A x_0} = \frac{3PL^2}{8EI} - \frac{15PL^2}{384EI} - \frac{9PL^2}{32EI} = \frac{144PL^2 - 15PL^2 - 108PL^2}{384EI} = \frac{21PL^2}{384EI}$$

<u>*Summary of Slope Analysis:*</u>

The equations of slope for the entire length of the beam can be summarized as shown below:

$$\theta_{x x_0}(x) = \begin{cases} \frac{P}{8EI}[x^2] - |\theta_{B x_0}| & \forall x \in [0, x_{01}] \\ \frac{3P}{8EI}[2Lx - x^2] - |\theta_{Bx_0}| - CF_{slope} & \forall x \in [x_{01}, L] \\ Where\ CF_{slope} = \frac{3P}{8EI}\left(2Lx - \frac{4x^2}{3}\right) = \frac{3P}{8EI}\left(2L x_{01} - \frac{4x_{01}^2}{3}\right) & \end{cases}$$

For verification, we apply the results of this derivation to the previous example of a simply supported beam with Concentrated Load at mid-span (*Example 2*):

$$CF_{slope} = \frac{1}{EI}\left(\frac{-Px^2}{4} + \frac{PLx}{2}\right) - \frac{Px^2}{4EI} = \frac{-2Px^2}{4EI} + \frac{PLx}{2EI} = \frac{P}{2EI}(Lx - x^2)$$

$$If\ x = x_0 = x_{01} = 0.5L, \quad then\quad CF_{slope} = \frac{P}{2EI}\left(\frac{L^2}{2} - \frac{L^2}{4}\right) = \frac{PL^2}{8EI}$$

$$While\ if, \quad |\theta_{B x_0}| = \frac{PL^2}{16EI}\ \ therefore\ Equations\ of\ Slope\ become$$

$$\theta_{xx_0}(x) = \begin{cases} \frac{Px^2}{4EI} - |\theta_{B x_0}| & \forall x \in [0, 0.5L] \\ \frac{1}{EI}\left(\frac{-Px^2}{4} + \frac{PLx}{2}\right) - |\theta_{B x_0}| - CF_{slope} & \forall x \in [0.5L, L] \\ Where\ CF_{slope} = \frac{P}{2EI}(Lx - x^2) = \frac{P}{2EI}(Lx_0 - x_0^2) & \end{cases}$$

We check the results of Slope at two sample points, which we already know to have the same value of Slope.

$$\theta_{xx_0}\left(\frac{L}{4}\right)=\frac{PL^2}{64EI}-\frac{PL^2}{16EI}=\frac{-3PL^2}{64EI}$$

$$\theta_{xx_0}\left(\frac{3L}{4}\right)=\frac{1}{EI}\left(\frac{-9PL^2}{64}+\frac{3PL^2}{8}\right)-\frac{PL^2}{16EI}-\frac{PL^2}{8EI}=\frac{15PL^2}{64EI}-\frac{PL^2}{16EI}-\frac{PL^2}{8EI}=\frac{3PL^2}{64EI}$$

$$\left|\theta_{xx_0}\left(\frac{L}{4}\right)\right|=\left|\theta_{xx_0}\left(\frac{3L}{4}\right)\right| \quad hence, the\ derivation\ is\ valid$$

Deflection of Selected Points with Respect to the Supports

Deflection of Point B

Using the results of our previous derivations:

$$If\ y_{xB}=\Delta_{xB}+\Delta'_{xx_0}\quad \forall\, x\in[0\,,x_{01}],\qquad then\quad y_{BB}=\Delta_{BB}+\Delta'_{Bx_0}=\Delta_{BB}+x\left|\theta_{Bx_0}\right|$$

$$\therefore,\qquad y_{BB}=0$$

Deflection of Point E

$$From\ previous\ derivations, we\ have\ shown\ that: y_{xB}=\Delta_{xB}+\Delta'_{xx_0}\qquad \forall\, x\in[0\,,x_{01}],$$

$$\therefore\ y_{xB}=\Delta_{xB}+x\left|\theta_{xx_0}\right|\quad \forall\, x\in[0\,,x_{01}]$$

$$In\ the\ analysis\ of\ slope, we\ showed:\ \theta_{xx_0}=\frac{P}{8EI}(x^2)-\left|\theta_{Bx_0}\right|\qquad \forall x\in[0\,,x_{01}]$$

$$\therefore, y_{xB}=\frac{P}{12EI}(x^3)+x\left|\frac{P}{8EI}(x^2)-\left|\theta_{Bx_0}\right|\right|$$

$$If\ \left|\theta_{Bx_0}\right|=\frac{15PL^2}{384EI}\ then\ deflection\ of\ Point\ E\ is: y_{EB}=\frac{P}{12EI}\left(\frac{L}{4}\right)^3+\frac{L}{4}\left|\frac{P}{8EI}\left(\frac{L}{4}\right)^2-\frac{15PL^2}{384EI}\right|$$

$$y_{EB}=\frac{PL^3}{768EI}+\frac{L}{4}\left|\frac{PL^2}{128EI}-\frac{15PL^2}{384EI}\right|=\frac{PL^3}{768EI}+\frac{L}{4}\left|\frac{-PL^2}{32EI}\right|=\frac{PL^3}{768EI}+\frac{PL^3}{128EI}=\frac{7PL^3}{768EI}$$

Deflection of Point C:

$$Similarly, if\ \ y_{xB}=\frac{P}{12EI}(x^3)+x\left|\frac{P}{8EI}(x^2)-\left|\theta_{Bx_0}\right|\right|\ then\ deflection\ of\ Point\ C\ is:$$

$$y_{CB}=\frac{P}{12EI}\left(\frac{L}{2}\right)^3+\frac{L}{2}\left|\frac{P}{8EI}\left(\frac{L}{2}\right)^2-\frac{15PL^2}{384EI}\right|=\frac{PL^3}{96EI}+\frac{L}{2}\left|\frac{PL^2}{32EI}-\frac{15PL^2}{384EI}\right|=\frac{PL^3}{96EI}+\frac{L}{2}\left|\frac{-PL^2}{128EI}\right|$$

$$y_{CB}=\frac{PL^3}{96EI}+\frac{PL^3}{256EI}=\frac{8PL^3+3PL^3}{768EI}=\frac{11PL^3}{768EI}$$

Deflection of Point D:

When analyzing the deflection of this point, we bear in mind that it occurs ahead of the point of maximum deflection *"null-slope"* $(x_{01}>x_0)$. This condition modifies the slope equation. Further, we are still using the same equation of bending moment as in points B, E and C and hence, we do not to apply the need a "*Correcting Factor for Vertical Intercept*". Our equation of deflection therefore becomes:

$$y_{xB}=\Delta_{xB}-\Delta'_{xx_0}\qquad \forall\, x\in[x_0\,,x_{01}],$$

$$\therefore\ y_{xB}=\Delta_{xB}-x\left|\theta_{xx_0}\right|\quad \forall\, x\in[x_0\,,x_{01}]$$

$$Equation\ of\ Slope\ remains\ constant\ as:\ \theta_{xx_0}=\frac{P}{8EI}(x^2)-\left|\theta_{Bx_0}\right|\qquad \forall x\in[0\,,x_{01}]$$

$$\therefore, y_{xB}=\frac{P}{12EI}(x^3)-x\left|\frac{P}{8EI}(x^2)-\left|\theta_{Bx_0}\right|\right|$$

$$Hence,\qquad the\ deflection\ of\ Point\ D\ is\ determined\ as:$$

$$y_{DB}=\frac{P}{12EI}\left(\frac{3L}{4}\right)^3-\frac{3L}{4}\left|\frac{P}{8EI}\left(\frac{3L}{4}\right)^2-\frac{15PL^2}{384EI}\right|=\frac{27PL^3}{768EI}-\frac{3L}{4}\left|\frac{9PL^2}{128EI}-\frac{15PL^2}{384EI}\right|$$

$$y_{DB} = \frac{27PL^3}{768EI} - \frac{3L}{4}\left|\frac{PL^2}{32EI}\right| = \frac{9PL^3}{256EI} - \frac{3PL^3}{128EI} = \frac{9PL^3 - 6PL^3}{256EI} = \frac{3PL^3}{256EI}$$

It is important to keep our equations of deflection simple. Therefore, for purposes of simplifying our equations, we can re-write them using the fact that θ_{xx_0} $is - ve\ when\ x < x_0\ and + ve\ when\ x > x_0$ as follows:

$$y_{xB} = \Delta_{xB} - \Delta'_{xx_0} = \Delta_{xB} - x(\theta_{x x_0}) \qquad \forall\, x \in [0\,, x_{01}].$$

We can verify this by re-calculating the deflection of Points on either side of where maximum deflection occurs with this equation; say Points C and D as follows:

$$y_{CB} = \Delta_{CB} - \frac{L}{2}(\theta_{C x_0}) = \frac{PL^3}{96EI} - \frac{L}{2}\left(\frac{-PL^2}{128EI}\right) = \frac{PL^3}{96EI} + \frac{PL^3}{256EI} = \frac{11PL^3}{768EI}$$

$$y_{DB} = \Delta_{DB} - \frac{3L}{4}(\theta_{D x_0}) = \frac{27PL^3}{768EI} - \frac{3L}{4}\left(\frac{PL^2}{32EI}\right) = \frac{9PL^3}{256EI} - \frac{3PL^3}{128EI} = \frac{3PL^3}{256EI}$$

Thus, our simplified equation of deflection holds.

Deflection of Point A:

From previous analysis, the deflection of *Point A* is given by the equation:

$$y_{xB} = \Delta_{xB} - \Delta'_{xx_0} - CF_{\substack{Vertical\\Intercept}} \quad \forall\, x \in [x_{01}\,, L]$$

We analyze the *"Correction Factor of Vertical Intercept"* in a similar way as the *"Correction Factor of Slope"*. Hence, we obtain the value of $CF_{\substack{Vertical\\Intercept}}$ $as:$

$$CF_{\substack{Vertical\\Intercept}} = \frac{P}{8EI}(3Lx^2 - 2x^3) - \frac{P}{12EI}(x^3) = \frac{P}{EI}\left(\frac{3Lx^2}{8} - \frac{2x^3}{8} - \frac{x^3}{12}\right) = \frac{P}{EI}\left(\frac{3Lx^2}{8} - \frac{x^3}{3}\right)$$

$$CF_{\substack{Vertical\\Intercept}} = \frac{P}{24EI}(9Lx^2 - 8x^3) \quad where\ in\ our\ case, x = \frac{3L}{4}$$

$$\therefore, for\ our\ analysis\ CF_{\substack{Vertical\\Intercept}} = \frac{P}{24EI}\left(\frac{81L^3}{16} - \frac{216L^3}{64}\right) = \frac{P}{24EI}\left(\frac{108L^3}{64}\right) = \frac{9PL^3}{128EI}$$

The equation of deflection therefore becomes:

$$y_{xB} = \Delta_{xB} - x(\theta_{x x_0}) - \frac{9PL^3}{128EI} \qquad \forall\, x \in [x_{01}\,, L]$$

Therefore, the deflection of *Point A wrt to Point B* is:

$$y_{AB} = \Delta_{AB} - L(\theta_{A x_0}) - \frac{9PL^3}{128EI} = \frac{P}{8EI}(L^3) - L\left(\frac{21PL^2}{384EI}\right) - \frac{9PL^3}{128EI} = \frac{PL^3}{8EI} - \frac{21PL^3}{384EI} - \frac{9PL^3}{128EI}$$

$$y_{AB} = \frac{48PL^3 - 21PL^3 - 27PL^3}{384EI} = 0$$

Note that the *Point D* exists on both equations of bending moment and as such, we can use this equation to verify the deflection of *Point D* calculated earlier as:

$$y_{DB} = \Delta_{DB} - \frac{3L}{4}(\theta_{D x_0}) - \frac{9PL^3}{128EI} = \frac{P}{8EI}\left(\frac{27L^3}{16} - \frac{54L^3}{64}\right) - \frac{3L}{4}\left(\frac{PL^2}{32EI}\right) - \frac{9PL^3}{128EI}$$

$$y_{DB} = \frac{27PL^3}{256EI} - \frac{3PL^3}{128EI} - \frac{9PL^3}{128EI} = \frac{3PL^3}{256EI} \ which\ agrees\ with\ our\ earlier\ analysis$$

Summary of Deflection Analysis:

The equations of deflection for the entire length of the beam can be summarized as shown below:

$$y_{xB}(x) = \begin{cases} \Delta_{xB} - \Delta'_{xx_0} = \dfrac{P}{12EI}(x^3) - x(\theta_{xx_0}) & \forall\, x \in [0\,, x_{01}] \\ \Delta_{xB} - \Delta'_{xx_0} - CF_{\substack{Vertical\\Intercept}} = \dfrac{P}{8EI}(3Lx^2 - 2x^3) - x(\theta_{xx_0}) - CF_{\substack{Vertical\\Intercept}} & \forall\, x \in [x_{01}\,, L] \\ Where\ CF_{\substack{Vertical\\Intercept}} = \dfrac{P}{24EI}(9Lx^2 - 8x^3) = \dfrac{P}{24EI}(9Lx_0^2 - 8x_0^3) & \end{cases}$$

For verification purposes, we apply these equations to our previous analysis (*Section 3.2*) as shown below:

$$CF_{\substack{Vertical\\Intercept}} = \frac{1}{EI}\left(\frac{-Px^3}{6} + \frac{PLx^2}{4}\right) - \frac{1}{EI}\left(\frac{Px^3}{6}\right) = \frac{1}{EI}\left(\frac{PLx^2}{4} - \frac{Px^3}{3}\right) = \frac{P}{12EI}(3Lx^2 - 4x^3)$$

$$\therefore,\ CF_{\substack{Vertical\\Intercept}} = \frac{P}{12EI}(3Lx_0^2 - 4x_0^3) = \frac{P}{12EI}\left(\frac{3L^3}{4} - \frac{4L^3}{8}\right) = \frac{P}{12EI}\left(\frac{L^3}{4}\right) = \frac{PL^3}{48EI}$$

We then use the formula to calculate the value of deflection at two Points on either side of the Point x_0 that we know to have the same value of deflection as follows:

$$y_{EB}\left(\frac{L}{4}\right) = \frac{1}{EI}\left(\frac{PL^3}{384}\right) - \frac{L}{4}\left(\frac{-3PL^2}{64EI}\right) = \frac{PL^3}{384EI} + \frac{3PL^3}{256EI} = \frac{11PL^3}{768EI}$$

$$y_{DB}\left(\frac{3L}{4}\right) = \frac{1}{EI}\left(\frac{9PL^3}{64} - \frac{27PL^3}{384}\right) - \frac{3L}{4}\left(\frac{3PL^2}{64EI}\right) - \frac{PL^3}{48EI} = \frac{27PL^3}{384EI} - \frac{9PL^3}{256EI} - \frac{PL^3}{48EI} = \frac{11PL^3}{768EI}$$

$$y_{DB} = y_{EB} = \frac{11PL^3}{768EI}\ which\ agrees\ with\ our\ analysis\ in\ section\ 3.2$$

From our analysis of *Section 3.2* and *Section 4.2*, we do not expect our equations to change when the $Point\ x_{01}\ precedes\ the\ Point\ x_0\ i.e\ when\ x_{01} < x_0$. We can therefore summarize the equations of *Slope (wrt to pint of "null-slope")* and *Deflection (wrt RHS support)* at different Points on Simply Supported Beams as shown below.

$$\theta_{xx_0}(x) = \begin{cases} \dfrac{P}{8EI}[x^2] - |\theta_{Bx_0}| & \forall x \in [0\,, x_{01}] \\ \dfrac{3P}{8EI}[2Lx - x^2] - |\theta_{Bx_0}| - CF_{slope}(x_{01}) & \forall x \in [x_{01}\,, L] \\ Where\ CF_{slope} = \dfrac{3P}{8EI}\left(2Lx - \dfrac{4x^2}{3}\right) = \dfrac{3P}{8EI}\left(2Lx_{01} - \dfrac{4x_{01}^2}{3}\right) & \\ All\ distances\ measured\ from\ RHS\ Support & \end{cases}$$

$$y_{xB}(x) = \begin{cases} \Delta_{xB} - x(\theta_{xx_0}) = \frac{P}{12EI}(x^3) - x(\theta_{xx_0}) & \forall x \in [0\,, x_{01}] \\ \Delta_{xB} - x(\theta_{xx_0}) - CF_{\substack{Vertical\\Intercept}} = \frac{P}{8EI}(3Lx^2 - 2x^3) - x(\theta_{xx_0}) - CF_{\substack{Vertical\\Intercept}}(x_{01}) & \forall x \in [x_{01}\,, L] \\ Where\ CF_{\substack{Vertical\\Intercept}} = \frac{P}{24EI}(9Lx^2 - 8x^3) = \frac{P}{24EI}(9Lx_{01}^2 - 8x_{01}^3) & \\ All\ distances\ measured\ from\ RHS\ Support & \end{cases}$$

5. WHEN THERE EXIST MULTIPLE BREAKPOINTS IN THE BMD

So far, we have looked at analysis of *Slope* and *Deflection* when the BMD has only one Break-Point. However, when *multiple loads* are applied on the beam, the BMD is bound to have multiple break-points.

For this analysis, we make the three assumptions: Firstly, the beam bears multiple break-points in the BMD *(say two break-points)*; Secondly, the *first* break-point *(that which is closest to the RHS support)* is denoted as "x_{01}" ; Finally, if the furthest Point on the beam *(towards the right)* from the point of "$null - slope$" $i.e.\ x_0$ is denoted as *"B"*, then the *Slope* at this Point *(wrt the horizontal)* can be denoted as θ_{Bx_0} . Thus, if the *length* of the beam is *"L"*, the equations of *Slope* and *Deflection* can be obtained as follows:

Analysis of Slope in Presence of Multiple Break-Points:

The equations of *Slope* when there are only *two* break-points in the BMD are:

$$\theta_{x\,x_0}(x) = \begin{cases} \frac{1}{EI}\int_0^x M(x)\,dx - |\theta_{B\,x_0}| & \forall x \in [0\,, x_{01}] \\ \frac{1}{EI}\int_0^x M(x)\,dx - |\theta_{B\,x_0}| - \sum_{i=1}^{1} CF_{slope}(x_{0i}) & \forall x \in [x_{01}\,, x_{02}] \\ \frac{1}{EI}\int_0^x M(x)\,dx - |\theta_{B\,x_0}| - \sum_{i=1}^{2} CF_{slope}(x_{0i}) & \forall x \in [x_{02}\,, L] \end{cases} \quad (1)$$

Alternatively, we may simplify the equations as shown below:

$$\theta_{x\,x_0}(x) = \begin{cases} \frac{1}{EI}\int_0^x M(x)\,dx + \theta_{B\,x_0} & \forall x \in [0\,, x_{01}] \\ \frac{1}{EI}\int_0^x M(x)\,dx + \theta_{B\,x_0} - \sum_{i=1}^{1} CF_{slope}(x_{0i}) & \forall x \in [x_{01}\,, x_{02}] \\ \frac{1}{EI}\int_0^x M(x)\,dx + \theta_{B\,x_0} - \sum_{i=1}^{2} CF_{slope}(x_{0i}) & \forall x \in [x_{02}\,, L] \end{cases} \quad (2)$$

Thus in the presence of a finite number of break-points $(say\ "n")$ in the BMD, we need $("n+1")$ equations of *Slope*. This is shown below:

$$\theta_{x\,x_0}(x) = \begin{cases} \frac{1}{EI}\int_0^x M(x)\,dx + \theta_{B\,x_0} & \forall x \in [0\,, x_{01}] \\ \frac{1}{EI}\int_0^x M(x)\,dx + \theta_{B\,x_0} - \sum_{i=1}^{1} CF_{slope}(x_{0i}) & \forall x \in [x_{01}\,, x_{02}] \\ \frac{1}{EI}\int_0^x M(x)\,dx + \theta_{B\,x_0} - \sum_{i=1}^{2} CF_{slope}(x_{0i}) & \forall x \in [x_{02}\,, x_{03}] \\ \vdots & \\ \frac{1}{EI}\int_0^x M(x)\,dx + \theta_{B\,x_0} - \sum_{i=1}^{n-2} CF_{slope}(x_{0i}) & \forall x \in [x_{0\,n-2}\,, x_{0\,n-1}] \\ \frac{1}{EI}\int_0^x M(x)\,dx + \theta_{B\,x_0} - \sum_{i=1}^{n-1} CF_{slope}(x_{0i}) & \forall x \in [x_{0\,n-1}\,, x_{0\,n}] \\ \frac{1}{EI}\int_0^x M(x)\,dx + \theta_{B\,x_0} - \sum_{i=1}^{n} CF_{slope}(x_{0i}) & \forall x \in [x_{0\,n}\,, L] \end{cases} \quad (3)$$

If $\exists$ $"n"$ number of break-points on the BMD such that $x_{0\,1}$ is the break point nearest to the RHS-Support and $x_{0\,n}$ is the break-point furthest from the RHS-support, then the slope at the left-hand support $(when\ x = x_{0\,n+1} = L)$ wrt to the right-hand support $(when\ x = x_{00} = 0)$ is given by:

$|\theta_{AB}| = |\theta_{A\,x_0}| + |\theta_{B\,x_0}|$

$$|\theta_{AB}| = \frac{1}{EI}\int_{0}^{x=x_{0\,n+1}=L} M(x)\,dx - \frac{1}{EI}\int_{0}^{x=x_{00}=0} M(x)\,dx - \sum_{i=1}^{n} CF_{slope}(x_{0i}) \qquad (4)$$

Analysis of Deflection in Presence of Multiple Break-Points:

When *Deflection is wrt to the furthest Point to right from the point of "null-slope" i.e. Point B or* $(x_{ref} = 0)$ then:

Considering the equations of *Deflection* when there are only *two* break-points in the BMD:

$$y_{xB}(x) = \begin{cases} \frac{1}{EI}\int_0^x xM(x)\,dx - x(\theta_{xx_0}) & \forall x \in [0, x_{01}] \\ \frac{1}{EI}\int_0^x xM(x)\,dx - x(\theta_{xx_0}) - \sum_{i=1}^{1} CF_{\substack{Vertical\\Intercept}}(x_{0i}) & \forall x \in [x_{01}, x_{02}] \\ \frac{1}{EI}\int_0^x xM(x)\,dx - x(\theta_{xx_0}) - \sum_{i=1}^{2} CF_{\substack{Vertical\\Intercept}}(x_{0i}) & \forall x \in [x_{02}, L] \end{cases} \qquad (5)$$

Thus in the presence of a finite number of break-points $(say\ "n")$ in the BMD, we need $("n+1")$ equations of *Deflection*. This is shown below:

$$y_{xB}(x) = \begin{cases} \frac{1}{EI}\int_0^x xM(x)\,dx - x(\theta_{xx_0}) & \forall x \in [0, x_{01}] \\ \frac{1}{EI}\int_0^x xM(x)\,dx - x(\theta_{xx_0}) - \sum_{i=1}^{1} CF_{\substack{Vertical\\Intercept}}(x_{0i}) & \forall x \in [x_{01}, x_{02}] \\ \frac{1}{EI}\int_0^x xM(x)\,dx - x(\theta_{xx_0}) - \sum_{i=1}^{2} CF_{\substack{Vertical\\Intercept}}(x_{0i}) & \forall x \in [x_{02}, x_{03}] \\ \vdots & \\ \frac{1}{EI}\int_0^x xM(x)\,dx - x(\theta_{xx_0}) - \sum_{i=1}^{n-2} CF_{\substack{Vertical\\Intercept}}(x_{0i}) & \forall x \in [x_{0\,n-2}, x_{0\,n-1}] \\ \frac{1}{EI}\int_0^x xM(x)\,dx - x(\theta_{xx_0}) - \sum_{i=1}^{n-1} CF_{\substack{Vertical\\Intercept}}(x_{0i}) & \forall x \in [x_{0\,n-1}, x_{0\,n}] \\ \frac{1}{EI}\int_0^x xM(x)\,dx - x(\theta_{xx_0}) - \sum_{i=1}^{n} CF_{\substack{Vertical\\Intercept}}(x_{0i}) & \forall x \in [x_{0\,n}, L] \end{cases} \qquad (6)$$

5. CONCLUSION

A mathematical form of Moment-Area method of beam analysis has been derived. This mathematical Moment-Area method has been applied on a beam with only a turning point in the deflection curve; in beams where the turning point on the deflection curve and the breakpoint in the BMD coincide and in beams where the turning point in the deflection curve and breakpoint in the BMD do not coincide. The results obtained using the mathematical Moment-Area method were shown to agree with those obtained on the same beams while using Macaulay's method, thus the novel Moment-Area method was validated.

With knowledge obtained from these three scenarios, a case for beams with only one turning point in the deflection curve but with two breakpoints in the BMD was advanced. Finally, a generic case with one turning point in the deflection curve and multiple (more than two) breakpoints in the BMD was advanced to replace the conventional Moment-Area technique.

References

Arai, T., Aiyama, Y., Sugi, M. & Ota, J. (2001), "Holonic Assembly System with Plug and Produce", *Computers in Industry* **46**, Elsevier, 289-299.

Bell, G.A., Cooper, M.A., Kennedy, M. & Warwick, J. (2000), "The Development of the Holon Planning and Costing Framework for Higher Education Management", Technical Report, SBU-CISM-11-00, South Bank University, 103 Borough Road, London, SE1 0AA.

Bongaerts, L. (1998), "Integration of Scheduling and Control in Holonic Manufacturing Systems", *PhD Thesis*, PMA Division, K.U.Leuven.

Deen, S.M. (1993), "Cooperation Issues in Holonic Manufacturing Systems", *Proceedings of DIISM'93 Conference*, 410-412.

Techawiboonwong, A., Yenradeea, P. & Das, S. (2006). A Master Scheduling Model with Skilled and Unskilled Temporary Workers", *Production Economics* **103**, Elsevier, 798-809.

Valckenaers, P., Van Brussel, H., Bongaerts, L. & Wyns, J. (1997), "Holonic Manufacturing Systems", *Integrated Computer Aided Engineering* **4**(3), 191-201.

Van Brussel, H., Wyns, J., Valckenaers, P., Bongaerts, L. & Peters, P. (1998), "Reference Architecture for Holonic Manufacturing Systems: PROSA", *Computers in Industry* **37**(3), 255-274.