A Common Fixed Point Theorems in Menger Space using Occasionally Weakly Compatible Mappings

Kamal Wadhwa, Jyoti Panthi and Ved Prakash Bhardwaj
Govt. Narmada Mahavidyalaya, Hoshangabad, (M.P) India

Abstract
In this paper we have improved the result of Saurabh Manro [7] by using the concept of occasionally weakly compatible Maps and proved some results on fixed points in menger space.

Key words: Menger space, Common fixed point, occasionally weakly compatible mappings.

1. Introduction:
In 1942 Menger [4] introduced the notion of a probabilistic metric space (PM-space) which is in fact, a generalization of metric space. The idea in probabilistic metric space is to associate a distribution function with a point pair, say (x, y), denoted by F(x, y; t) where t > 0 and interpret this function as the probability that distance between x and y is less than t, whereas in the metric space the distance function is a single positive number.

Sehgal [8] initiated the study of fixed points in probabilistic metric spaces. The study of these spaces was expanded rapidly with the pioneering works of Schweizer-Sklar [1]. A weakly compatible map in fuzzy metric space is generalized by A. Al. Thagafi and Nasser Shahzad [1] by introducing the concept of occasionally weakly compatible mappings. Our paper improves the result of Saurabh Manro [7] by using of occasionally weakly compatible Maps and proved some results on fixed points in menger space.

2. Preliminaries:
First, recall that a real valued function f defined on the set of real numbers is known as a distribution function if it is nondecreasing, continuous and \( \inf f(x) = 0, \sup f(x) = 1. \) We will denote by L, the set of all distribution functions.

Definition 2.1: A probabilistic metric space (PM-space) is a pair (X, F) where X is a set and F is a function defined on X x X to L such that if x, y and z are points of X, then

(F-1) \( F_{x,y}(t) = 1 \) for every \( t > 0 \) iff \( x = y, \)
(F-2) \( F_{x,y}(0) = 0, \)
(F-3) \( F_{x,y}(t) = F_{y,x}(t), \)
(F-4) if \( F_{x,y}(t) = 1 \) and \( F_{y,z}(s) = 1, \) then \( F_{x,z}(s + t) = 1 \) for all \( x, y, z \in X \) and \( s, t \geq 0. \)

For each \( x, y \in X \) and for each real number \( t > 0, F_{x,y}(t) \) is to be thought of as the probability that the distance between \( x \) and \( y \) is less than \( t. \)

It is interesting to note that, if \( (X, d) \) is a metric space, then the distribution function \( F(x, y; t) \) defined by the relation \( F(x, y; t) = H(t – d(x, y)) \) induces a PM-space where \( H(x) \) denotes the distribution function defined as follows:

\[
H(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
1 & \text{if } x > 0 
\end{cases}
\]

Definition 2.2: A t-norm is a 2-place function, \( t: [0,1] \times [0,1] \to [0,1] \) satisfying the following:

(i) \( t(0,0) = 0, \) (ii) \( t(0,1) = 1, \) (iii) \( t(a, b) = t(b, a), \) (iv) if \( a \leq c, b \leq d, \) then \( t(a, b) \leq t(c, d), \)
(v) \( t(t(a, b), c) = t(a, t(b, c)) \) for all \( a, b, c \in [0,1]. \)

Definition 2.3: A Menger PM-space is a triplet \( (X, F, t) \) where \( (X, F) \) is a PM-space and \( t \) is a t-norm with the following condition:

(F-5) \( F_{x,y}(s + p) \geq t(F_{x,y}(s), F_{y,z}(p)), \) for all \( x, y, z \in X \) and \( s, \) \( p \geq 0. \)

This inequality is known as Menger’s triangle inequality.

In our theory, we consider \( (X, F, t) \) to be a Menger PM-space with the additional following postulate: (F-6) \( \lim_{t \to 0} F_{x,y}(t) = 1 \) \( \forall x, y \in X. \)

Definition 2.4: A menger space \( (X, F, t) \) is said to be complete if and only if every Cauchy sequence in X is convergent.

In 1996, Jungck [2] introduced the notion of weakly compatible maps as follows:

Definition 2.5: A pair of self mappings \( (A, S) \) on set X is said to be weakly compatible if they commute at the coincidence points i.e. \( A(u) = Su \) for some \( u \in X, \) then \( SAu = ASu. \)

We need the following Lemmas due to Schweizer and Sklar [1] and Singh and Pant [6], in the proof of the theorems:
Lemma 2.1: Let $(X, F, t)$ be a menger space and if for a number $k \in (0,1)$ such that $F_{x,y}(t) \succeq F_{y,z}(t)$. Then $x = y$.

Definition: Let $X$ be a set, $f$ and $g$ selfmaps of $X$. A point $x \in X$ is called a coincidence point of $f$ and $g$ iff $fx = gx$. We shall call $w = fx = gx$ a point of coincidence of $f$ and $g$.

Definition 2.6[3]: Two self mappings $A$ and $S$ of a non-empty set $X$ are OWC iff there is a point $x \in X$ which is a coincidence point of $A$ and $S$ at which $A$ and $S$ commute. The notion of OWC is more general than weak compatibility (see [5]).

Lemma 2.2[3]: Let $X$ be a non-empty set, $A$ and $B$ are occasionally weakly compatible self maps of $X$. If $A$ and $B$ have a unique point of coincidence, $w = Ax = Bx$, then $w$ is the unique common fixed point of $A$ and $B$.

3. Main Results:
In our result, we used the following implicit relation:

Definition (Implicit Relation): Let $I= [0, 1]$ and $\Omega$ be the set of all real continuous functions $\phi: I^6 \rightarrow \mathbb{R}$ satisfying the condition:

(i) $\phi$ is non increasing or non decreasing in third and fourth argument and
(ii) If we have $\phi(u, v, 1, 1, v, v) \succeq 1$, for all $u, v \in (0, 1)$, then $u \succeq v$.

Example: We define $\phi: I^6 \rightarrow \mathbb{R}$ by $\phi(u_1, v_1, v_2, v_4, v_5) = u_1 - v_1 + v_2 - v_3 + v_4 - v_5$.

Then clearly continuous function such that if we have $\phi(u, v, 1, 1, v, v) \succeq 1$, for all $u, v \in (0, 1)$, then $\phi(u, v, 1, 1, v, v) = u - v + 1 - 1 + v - v = u - v \succeq 1 \Rightarrow u \succeq v$.

Theorem 3.1: Let $(X, F, t)$ be a Menger space. Let $A$, $B$, $S$ and $T$ be self maps of $X$ satisfying the following conditions:

1. $(A, S)$ and $(B, T)$ are OWC.
2. There exist $k \in (0, 1)$ and $\phi \in \Omega$ such that

\[ \phi \left( F_{Ax,By}(kt) \right), \left( F_{sx,ty}(t) \right), \left( F_{Ax,tx}(t) \right), \left( F_{By,ty}(t) \right), \left( F_{Ax,ty}(t) \right), \left( F_{By,sx}(t) \right) \succeq 1 \quad (I) \]

for all $x, y \in X$ and $t > 0$.

Then there exists a unique point $w \in X$ such that $Aw = Sw = w$ and a unique point $z \in X$ such that $Bz = Tz = z$.

Moreover, $z = w$, so that there is a unique common fixed point $A$, $B$, $S$ and $T$ in $X$.

Proof: Since the pairs $(A, S)$ and $(B, T)$ are OWC, there exist points $x, y \in X$ such that $Ax = Sx$, $ASx = SAx$ and $By = Ty$, $BTy = TBy$. Now we show that $Ax = By$.

Then we have by inequality (I),

\[ \phi \left( F_{Ax,By}(kt) \right), \left( F_{sx,ty}(t) \right), \left( F_{Ax,tx}(t) \right), \left( F_{By,ty}(t) \right), \left( F_{Ax,ty}(t) \right), \left( F_{By,sx}(t) \right) \succeq 1 \]

Thus by lemma 2.1 $Ax = By$. Therefore $Ax = Sx = By = Ty$. Moreover, if there is another point $z$ such that $Az = Sz$. Then using inequality (I) it follows that $Az = Sz = By = Ty$, or $Ax = Az$.

Hence $w = Ax = Sx$ is the unique point of coincidence of $A$ and $S$. By lemma 2.2, $w$ is the unique common fixed point of $A$ and $S$. Similarly, there is a unique point $z \in X$ such that $z = Bz = Tz$. Suppose that $w \neq z$ and using inequality (I), we get

\[ \phi \left( F_{w,x}(kt) \right), \left( F_{w,x}(t) \right), \left( F_{w,w}(t) \right), \left( F_{x,w}(t) \right), \left( F_{z,w}(t) \right) \succeq 1 \]

Thus by lemma 2.1 $w = z$. Therefore $z = Sz = Tz = Az = Bz$.

To prove uniqueness, let $u$ and $v$ are two common fixed points of $A$, $B$, $S$ and $T$ in $X$. Therefore, by definition, $Au = Bu = Tu = Su = u$ and $Av = Bv = Tv = Sv = v$.

Then by (I), take $x = u$ and $y = v$, we get

\[ \phi \left( F_{u,v}(kt) \right), \left( F_{u,v}(t) \right), \left( F_{u,u}(t) \right), \left( F_{v,v}(t) \right), \left( F_{u,v}(t) \right), \left( F_{v,u}(t) \right) \succeq 1 \]

Thus by lemma 2.1 $w = z$. Therefore $z = Sz = Tz = Az = Bz$. 
Now by inequality (3.1.2), we have (at \( x = r \) and \( y = s \))

\[
\phi \left( F_{u,v}(kt) \right) \geq F_{u,v}(t)
\]

Therefore, by lemma 2.1, \( u = v \).

Hence the self maps \( A, B, S \) and \( T \) have a unique common fixed point in \( X \).

**Theorem 3.2:** Let \( (X, M, t) \) be a menger space and let \( A, B, S, T, P \) and \( Q \) be self maps of \( X \) satisfying the following conditions:

- (3.1.1) the pairs \((A, SP)\) and \((B, TQ)\) are owc;
- (3.2.2) there exists \( k \in (0, 1) \) and \( \phi \in \Omega \) such that \( \phi \left( F_{A,B}(kt), F_{S,P}(t), F_{A,S}(t), F_{B,T}(t), F_{A,B}(t), F_{B,T}(t) \right) \geq 1, \forall x, y \in X \) and \( t > 0 \),

(3.1.3) the pairs \((A, P), (S, P), (B, Q)\) and \((T, Q)\) are commuting;

then \( A, B, S, T, P \) and \( Q \) have a unique common fixed point in \( X \).

**Proof:** Since the pairs \((A, SP)\) and \((B, TQ)\) are owc, so there are points \( x, y \in X \) such that \( Ax = SPx \) implies \( A(SP)x \) and \( By = TQy \) implies \( B(TQ)y \).

We claim that \( Az = Bz \). Now by inequality (3.2.2)

\[
\phi \left( F_{A,B}(kt), F_{S,P}(t), F_{A,S}(t), F_{B,T}(t), F_{A,B}(t), F_{B,T}(t) \right) \geq 1,
\]

\( \Rightarrow \) \( F_{A,B}(kt) \geq F_{A,B}(t) \), thus by lemma 2.1 \( Ax = Bx \).

Therefore \( Ax = SPx = Bx = TQy = z \) (say), then \( Az = SPz \) and \( Bz = TQz \).

We claim that \( Az = Bz \). Now by inequality (3.2.2)

\[
\phi \left( F_{A,B}(kt), F_{S,P}(t), F_{A,S}(t), F_{B,T}(t), F_{A,B}(t), F_{B,T}(t) \right) \geq 1,
\]

\( \Rightarrow F_{A,B}(kt) \geq F_{A,B}(t) \), thus by lemma 2.1 \( Az = Bz \).

Hence the self maps \( A, B, S, T, P \) and \( Q \) have a unique common fixed point in \( X \).

**To prove uniqueness:** let \( r \) and \( s \) be two distinct common fixed points of \( A, B, S, T, P \) and \( Q \).

Then \( Ar = Br = Sr = Tr = Pr = Qr = r \) and \( As = Bs = SS = TS = PS = QS = s \),

Now by inequality (3.1.2), we have \( \phi \) \( \left( F_{r,s}(kt), F_{r,s}(t), F_{r,s}(t), F_{r,s}(t), F_{r,s}(t) \right) \geq 1 \),

\( \phi \left( F_{r,s}(kt), F_{r,s}(t), F_{r,s}(t) \right) \geq 1, \)

\( \Rightarrow F_{r,s}(kt) \geq F_{r,s}(t) \), thus by lemma 2.1 \( r = s \).

This completes the proof of the theorem.

**Conclusion:** Our theorem is an improvement of theorem 3.1 of saurabh manro [7]. In our theorem we do not require the completeness & continuity of the space and also condition (1) of [7, theorem 3.1]. Our theorem is true for any continuous t-norm. In our result we do not require to define many implicit relations.

**4. References:**


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