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Frameworks for Teachers' Knowledge of Mathematics

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Abstract
This paper, describes seven teacher knowledge frameworks and relates these frameworks to the teaching and assessment of elementary teachers' mathematics knowledge. The frameworks classify teachers' knowledge and provide a vocabulary and common language through which knowledge can be discussed and assessed. These frameworks are categorized into two classes: content knowledge and content knowledge for teaching. Content knowledge frameworks include Instrumental and Relational Understandings, Procedural and Conceptual Understandings, Depth of Knowledge; and Cognitive Complexities. Content knowledge for teaching frameworks includes. Type of Teachers Knowledge and Mathematical Knowledge for Teaching. The Diagnostic Teacher Assessment of Mathematics and Science (DTAMS), a tool that assesses both mathematics teachers' depth of conceptual knowledge and pedagogical content knowledge, is used to concretely connect the frameworks. The paper concludes with examples of quantitative assessments of teachers' mathematics knowledge based on these teachers' knowledge frameworks.

Introduction
The need to both strengthen and increase teachers' mathematics knowledge can not be over emphasis and this led many researchers to design mathematics knowledge frameworks as ways to understand what teachers' mathematics knowledge looks like.(Hiebert & Carpenter, 1992; Hill & Ball, 2006; Abdullahi, 2009; Porter, 2002; Shulman, 1986; Petit & Hess, 2005, 2007; Skemp, 2002). Some frameworks describe teacher knowledge in terms of degree of cognitive knowledge exhibited (; Hiebert & Carpenter, 1992; Musa,2011, 2002; Porter, 2002) while others describe and classify teachers' pedagogical or specialized content knowledge (Ball, 2003; Shulman, 2010: Shiely 1995: Bloom, 2009). Such frameworks provide the means to plan for and promote higher order thinking behaviors—first in the teacher, then in the student. This article describes seven teacher knowledge frameworks and relates these frameworks to (a) the teaching of elementary mathematics and (b) assessment of that teaching through the Diagnostic Teacher Assessment of Mathematics and Science (DTAMS), a tool that assesses both mathematics teachers' depth of conceptual knowledge and pedagogical content knowledge and is used to concretely connect the frameworks. DTAMS items reflect different types of teacher content and pedagogical knowledge, including depths of content knowledge. Depth of knowledge measures three levels of cognitive difficulty: memorization, understanding, and problem solving/reasoning. The importance of describing these teacher knowledge frameworks is that they not only classify teachers' knowledge but also provide a vocabulary and common language through which knowledge can be discussed. Broadly speaking, these frameworks are categorized into two classes: content knowledge and content knowledge for teaching. Content knowledge frameworks include Bloom's Taxonomy (1999); Skemp's (2000) Instrumental and Relational Understandings; Hiebert and Carpenter's (1992) Procedural and Conceptual Understand. Content Knowledge Frameworks(Bloom's Taxonomy).

Historically, knowledge was first classified or framed by behavioral psychologists who sought to capture and classify intellectual learning. The result, Bloom's Taxonomy of Educational Objectives, was originally constructed as a way to identify desired student behavior useful for curriculum building and test construction (Bloom, et al., 1999). Bloom developed a classification method to describe the levels of difficulty that students would need to answer questions in the classroom, whether orally or on tests. Six levels of knowledge were identified, ranging from the simplest level, requiring very little difficulty, to the highest level, requiring abstract thought and a network of interconnected schemas. These levels, in order from least complex to greatest, are: (a) knowledge, referring to mental recall of previously learned material; (b) comprehension, referring to the ability to grasp or understand the meaning of material;(c) application, referring to the ability to use the information learned previously in new and concrete situations; (d) analysis, referring to the ability to break down material into its component parts in order to find new or hidden meaning; (e) synthesis, referring to the ability to put parts together to create something new; and (f) evaluation, referring to the ability make accurate judgments and assess value for a given purpose, Bloom's Taxonomy (2007)).The depth of content difficulty relates to the cognitive difficulty of the item, not student or teacher responses.

Primary Student Question	Bloom's Taxonomy
34 10 + 12 56 Item 1: Look at James' responses. What makes his reasoning correct or incorrect? **Musa**: First I added 4 + 0 + 2. That was 6 Then I added 3 + 1 + 1 and got 5. 5 and 6 is 56	Item = Comprehension of Arithmetic
34 10 + 12 56 Item 2:Look at Jamie's response. What makes her reasoning correct or incorrect? **Mary**: First I added 30 + 10 + 10 and got 50. Then added 6 more. The answer is 56.	Item = Analysis of arithmetic (incorporated place value)

In order to understand Bloom's taxonomy, some clarification of terminology is necessary. When Bloom referred to understanding at the comprehension level, he was not referring to a deep, conceptual attainment of underlying principles. He was speaking only of a basic "how to" type of understanding, such as understanding the procedural algorithm for changing a fraction to a decimal. Table 1 better illustrates Bloom's degrees of understanding. (Bloom's Taxonomy: Test Construction, 2007). Students were asked to add three 2-digit numbers: 34 + 10 + 12, and explain their reasoning. The level of item difficulty is based on what the respondent is required to know in order to answer the question correctly. Item 1 requires identification of a standard algorithm. Item 2 requires the teacher to understand the role of place value in multi-digit addition and non-standard algorithms based on place value. Item difficulty is not based on the knowledge expressed by the teacher. A teacher may understand place value in Item 1, but since that knowledge was not necessary in order to answer the question, the question is deemed "less difficult" than Item 2. Because Item 2 could not be answered correctly without some reference to place value, the item is more complex. The item's requisite knowledge dictates the level of complexity.

Another clarification is required for Bloom's application level. When he discussed "applying" the learned information, Bloom did not mean simply making a calculation as James did in Table 1. In this classification, the key word is "new situations": the item must be written in a way that requires understanding and applying it in a new dimension.

DTAMS and Bloom's taxonomy

While Bloom's taxonomy was not specifically written for mathematics, it is appropriate for describing mathematics content depth of teachers' knowledge (Bloom's Taxonomy in the Classroom, 2007) as viewed through the Diagnostic Teacher Assessment of Mathematics and Science (DTAMS). DTAMS memorization questions require teachers to define and identify mathematics content (such as place value, prime numbers, and fraction equivalency). This is the lowest level of cognitive difficulty and is similar to Bloom's knowledge level. Understanding-type questions require teachers to demonstrate an understanding of whole-number concepts and procedures. Teachers compute using simple algorithms and describe mathematical relationships, such as those between numerator and denominator in fractional amounts. This is the second level of cognitive difficulty and is similar to both Bloom's comprehension and application levels. Problem solving/reasoning questions require teachers to reason and solve non-standard algorithm problems and to make connections between mathematical concepts in new and different ways.

Skemp's Instrumental and Relational Understanding Framework defined.

Skemp (1976, 1987, 1993) partitioned teachers' mathematics knowledge into two classes: relational understanding and instrumental understanding. Relational understanding encompasses a deep, conceptual understanding of material. For Skemp, there isreally only one type of "understanding," and anything less is not really understanding at all. This lesser degree of knowledge, termed instrumental, parrots or mimics true knowledge. Skemp uses the analogy of a parrot or dog to capture instrumental understanding. A parrot learns to mimic sound and "talk," but it really has no understanding of speech. A dog is trained to "walk" on two legs, but has no understanding of what it means to walk. The dog only mimics the behavior of his master. Skemp believed that instrumental and relational understandings are hierarchical—one is a lower, baser version of another. Rote memorization of facts and processes brings about instrumental understanding. Deliberate conceptual comprehension brings about relational understanding. Skemp believed that relational and instrumental understandings spawned from different cognitive activities and produced different cognitive outcomes. According to Skemp, an elementary mathematics teacher who teaches from an instrumental paradigm cannot produce students who learn mathematics conceptually or relationally. Relational understanding promotes conceptual understanding.

What truly defines relational understanding is the web of conceptual connections that undergird mathematics knowledge. The term "relational" accurately portrays understanding because the knowledge expressed is indeed schematic—embedded in the meaning are cognitive relationships or schemas, where background knowledge and related concepts are connected in the person's thinking. The web is the structured knowledge of concepts from which problem solving can flourish. Skemp maintained that, when students understand the meaning behind a mathematical concept, problem solving techniques can be employed to garner correct answers and foster further understanding. (Skemp, 1976, 1993).Skemp divided relational understanding into three depth categories. The cognitive difficulty or level of abstraction required in each category loosely falls into Bloom's application to evaluation levels. Category 1 includes traditional word problems or, as Skemp (1993) termed them, "problems in applied mathematics." These problems are akin to application tasks in Bloom's taxonomy.Category 2 incorporates problems, projects, or tasks, which Skemp termed "problems in pure math or mathematical puzzles." These tasks align roughly with Bloom's analysis level. Finally, Category 3 includes "problems outside our present domain" and "problems outside of our frontier zone." The level of abstraction is highest for these tasks, as new concepts and theories are derived from the old (1993). Figure 3 illustrates Skemp's types of understanding as related to Bloom's taxonomy.

Purpose of taxonomy

Skemp's (1976) purpose for classifying mathematics knowledge into instrumental and relational understanding was to influence mathematics instruction in the classroom. The value judgment is inherent in his claim that instrumental understanding is inferior to relational understanding. Skemp valued and advocated relational understanding for both teachers and students. To Skemp, the instrumental approach is detrimental to teaching mathematics effectively. If mathematics is not learned relationally, Skemp (1987) claimed that students do not analyze and make new connections; they are bound by the rules. "If the teacher asks a question that does not quite fit the rule, of course they will get it wrong".

Skemp found that many common student misconceptions are a result of misapplying rules and not understanding concepts a hazard of the instrumental approach to teaching and learning. Extrapolating and building connections when conceptual understanding is absent causes problems. Skemp gave an example of a young man who erroneously applied the multiplication decimal rule learned instrumentally to division, giving him an incorrect answer: "When multiplying two decimal fractions, drop the decimal point, multiplying as for whole numbers, and re-inserting the decimal point to give the same total as there were before" "By this method 4.8 ÷ 0.6 came to 0.08" In this case, like many others, the child made connections, but because he did not have relational understanding, his relational schemas were faulty. Relational understanding requires fewer rules and more principles or concepts behind the rule. Problem-solving activities are coupled with structured knowledge theory. Advanced-level mathematical learning becomes easier to remember than a plethora of isolated facts or rules alone. Additionally, Skemp asserted that relational understanding promotes enjoyment of mathematics, builds confidence, and produces self-reflective. Students

DTAMS and Skemp's understandings Instrumental understanding is closely related to the memorization level of cognitive difficulty. These items do not require understanding of the procedure or process but simply a rote adherence to an algorithm. DTAMS requires teachers to determine the number in a set that is not equivalent to others: a. 2/7; b. 0.0285; c. 28.5%; or d. 57/200.The item requires teachers to change fractions into decimals and percents into decimals, but it does not require an understanding of why division in one process requires division but in another requires moving the decimal two places to the right (Skemp, 1993). DTAMS understanding and problem solving/reason are both types of relational understanding.

Hiebert and Carpenter's Procedural and Conceptual Understandings Framework defined

Hiebert and Carpenter also classified learning using understanding and described two types of mathematics knowledge (Hiebert, 1989; Hiebert & Carpenter, 1992). They classified knowledge into procedural or conceptual understanding. Procedural knowledge is the knowledge gained from formal language or symbolic representations and is the knowledge of rules, algorithms, and procedures (Hiebert et al., 2000; Carpenter, Madison, Franke, & Zringue, 2005). Conceptual knowledge involves understanding relationships among concepts and principles, including concepts and schemas behind a concept. Unlike in Skemp's (1976, 1987, 1993) model, procedural and conceptual understanding are not divergent; in fact, they support each other. "Competence in mathematics requires children to develop and link their knowledge of concepts and procedures they reinforce each other" (Rittle-Johnson, Siegler, & Alibali, 2001, p. 246). Hiebert and Carpenter (1992) noted that "it is important to emphasize that both kinds of knowledge are required for mathematical expertise" (Hiebert & Lefevre, 1986, p. 78). In a study of elementary students' acquisition of procedural and conceptual understanding of decimal fractions, Rittle-Johnson (2001) confirmed that understanding correct mathematical representations (the number line) improved procedural knowledge and that both procedural and conceptual knowledge grew together, with neither one preceding or following the other. She asserts that the reciprocal relationship of conceptual / procedural knowledge is that initial conceptual knowledge predicts procedural knowledge and those gains predict improvements in conceptual knowledge. While Hiebert and Carpenter (1992) promoted classroom instruction that uses every level of Bloom's taxonomy, they do not contend that the higher, abstract thinking is dependent on mastery of the lower levels of the taxonomy as did Bloom. Their framework is not hierarchical. Teachers, under Hiebert and Carpenter's framework, incorporate the two types of understandings spirally. The line is mathematical knowledge, which is ever growing and increasing in depth of understanding and conceptual knowledge (circle size) while seemingly looping back as new procedural knowledge is gained (Carpenter, Franke, & Levi, 2003, 2005). Therefore, Hiebert and Carpenter (1992) suggested that teachers introduce conceptual understanding prior to procedural in some in stances. In fact, they proposed open-ended, problem-based learning instruction and even suggested initially introducing conceptual knowledge through problem-solving activities prior to introducing procedural steps. In this way, students learn to understand concepts and the "how to" of solving problems simultaneously. For instance, in an introductory elementary mathematics lesson on slope as a rate of change, students could work in groups on a discovery activity measuring the length of a slinky that is attached to a small cup when the weight of the cup is manipulated (through the addition and subtraction of M&M's). Students then explore linear functions as a rate of change and the concept of the y-intercept. This investigation occurs before formal definitions and terminology are introduced. As Hiebert and Carpenter maintained, "Growth in understanding is accomplished as the students reorganize and adjoin new representations to existing networks;" these are the loops and spirals of integrating rocedural and conceptual knowledge (p. 70).When students acquire only procedural knowledge in the classroom, Hiebert and Carpenter (1992) contended that misconceptions can more easily develop. This situation is similar to the problems inherent in Skemp's instrumental understanding, in which the process is not understood. The same problems arise when no connection is made with the reason behind a process or procedure and "off-the-wall" answers appear acceptable. For example, a student might write 10 + 12 = 112, yet if the student had ten blocks and added twelve more blocks to it, he or she would know the answer was not 112. Many systematic errors are diffused by conceptual understanding. Hiebert and Carpenter encouraged teachers to connect concepts to procedures when correcting student misconceptions. Poor and imprecise mathematical vocabulary is an often unforeseen consequence of students learning only procedurally. Teachers must teach the meaning behind the symbols. In order to do so, teachers must understand mathematics both procedurally and conceptually. Particularly, Hiebert and Carpenter (1992) cautioned against confusing students by teaching single content-based meanings for symbols for which multiple meanings exist. The fraction numerator/denominator is a ratio representation as well as a division operation. Students derive meaning from symbols in multiple ways including by making connections with other representations such as physical objects, pictures, and spoken language and by creating connections within the symbol system (Hiebert & Carpenter). Hiebert and Carpenter's types of understanding can best be summed up as mathematical orientations. A teacher with a procedural orientation has students perform a task; a teacher with a conceptual orientation has a student understand a task. In Hiebert and Carpenter's framework, the best scenario is to have a teacher with a blend of orientations in order to develop students who have sufficient depth of understanding to allow both the efficient and elegant execution of mathematical tasks and assignments.

Hiebert and Carpenter's types of understanding and DTAMS

Procedural understanding is closely related to the memorization level of cognitive difficulty. These items require performing steps or algorithms in order to get an answer. Item 3 on the DTAMS requires teachers to identify which operations hold true for a given number property (see Figure 5). While the problem does not necessitate understanding basic number properties, it does require understanding key components of a definition. DTAMS understanding and problem solving/reasoning are both cases of relational understanding

Content Knowledge and Standards Frameworks.

Throughout the discussion on teacher knowledge frameworks, the term depth of knowledge has been loosely used to refer to the degree of understanding or level of cognitive complexity of a task. In Bloom's taxonomy, the deeper the knowledge, the higher the taxonomy level. Skemp's (1976) deeper knowledge reflected relational understanding, as did Hiebert and Carpenter's (1992) procedural and conceptual knowledge. However, Webb (1997) used the term depth of knowledge (DOK) to describe four levels of standards and assessments in mathematics. Similarly, Porter (2000) offered a slightly different classification of the cognitive complexity of standard statements and assessments Webb's Depth of Knowledge Framework.

Framework defined "Standards and assessments can be aligned not only on the basis of the category of content covered by each, but also on the basis of the complexity of knowledge required by each" (Webb, 2002, p.

5). This complexity is what Webb termed depth of knowledge (p. 5). “Depth-of-knowledge consistency between standards and assessment indicates alignment if what is elicited from students on the assessment is as demanding cognitively as what students are expected to know and do as stated in the standards” (p. 5). In order to interpret and assign the DOK levels, Webb developed a rubric. The four levels of the rubric are similar to Bloom’s taxonomy levels. Webb’s DOK levels are as follows: Level 1: Recall; Level 2: Skill/concept; Level 3: Strategic thinking; and Level 4: Extended thinking (p. 6). Table 2 delineates each level. Level 1 included recall and reproduction, such as stating the associative property; Level 2 included basic understanding, such as graphing data or classifying quadrilaterals; Level 3 included complex reasoning, such as explaining how changes in dimensions affect area and perimeter of geometric figures or justifying a geometric proof; and Level 4 included extended reasoning, such as designing and conducting an experiment or completing a unit of formal geometric constructions, such as nine-point circles or the Euler line. (Webb, 2002). An example of how standards and DOK levels are checked for alignment is shown in Table 3. MA-05-5.1.1 is a DOK level 3 Kentucky Department of Education standard assessing pattern; the test item should also be at DOK level 3 (Support Materials for CCA version 4.1, 2007 KDOE KYofEd). Only the highlighted item would be an appropriate question for this standard DOK and DTAMS.

The DTAMS depth-of-knowledge levels are closely aligned with Webb’s (1997) DOK levels. Memorization is a Level 1 process; understanding is a Level 2; and problem solving/reasoning is predominately a Level 3 process but can extend to Level 4. See Figure 6 below for a comparison of DOK and DTAMS. In both Webb’s DOK and DTAMS, the lowest level of cognitive difficulty involves correctly completing a simple one-step practice problem or following an algorithmic procedure to perform a task. The second level of difficulty (understanding and skills/concept) requires responses above the rote level, touse recalled information in a new manner or demonstrate decisions based on the learned material. The third level activities for both DTAMS and Webb (problem solving/reasoning and strategic thinking) are more open-ended, requiring greater analysis and complex reasoning skills. Item 15 (Figure 6) require teachers to prove a standard fraction computation algorithm Porter’s Cognitive Complexity Framework

Framework definedPorter (2002) and Porter & Chester (2002), like Webb (1997), were interested in standard-content alignment and in identifying the degree of cognitive demand in both assessments and standards. To better assess the degree of alignment between content and standards, Porter developed “uniform descriptors of topics and categories of cognitive demand that together describe the content of instruction” (Porter, 2002,). In his earlier work, Porter (2000) identified three elementary cognitive demands: conceptual understanding, skills, and applications. After his 1993 year-long study of over 63 mathematics and science teachers, he increased these to nine; however, some of the categories could be subsumed into others. Eventually, Porter settled on six (Porter, Kirst, Ostholff, Smithson, & Schneider, 1993; Porter 2002). The descriptors—action verbs that define or identify five cognitive behaviors—range from least cognitively complex to the most complex. They are Level A, memorize; Level B, perform procedures; Level C, communicate understanding; Level D, solve non-routine problems; and Level E, conjecture/generalize/prove (Porter, 2002, 2007b).Level A includes memorizing facts definitions, and formulas, such as reciting the rules for fraction division; Level B involves performing procedures and solving routine problems ,such as dividing fractions in practice exercises; Level C involves communicating understanding of concepts conceptually, such as solving a one-step word problem requiring students to divide fractions and explain their reasoning to the class; Level D involves solving non-routine problems and making connections, such as solving a problem requiring dividing fractions using two different strategies; and Level E involves making and investigating mathematical conjectures, such as making a model to explain (justify) why a division of fractions strategy works and determining in what circumstances it would not work (Porter, 2007a, 2007b). Porter’s and Webb’s (1997) knowledge frameworks are very similar in the range of cognitive behaviors that each covers. Figure 7 shows the comparison between Webb’s depth of knowledge and Porter’s cognitive demands. Porter’s cognitive demands and DTAMS. The most interesting difference between Webb’s DOK and Porter’s cognitive demands is the breakdown of the DOK 2 levels into two: B and C. Because the DTAMS are so closely aligned with DOK, the item distinction is of most interest here. The construction of the DTAMS understanding level items mirrors this breakdown. Multiple-choice understanding items require teachers to perform procedures; Level B and open-response understanding items require teachers to communicate their understanding of a concept. In Item 10, teachers identify the prime factors of a given number and add up the resulting factors. In Item 18a, teachers are given credit if they demonstrate conceptual understanding of associative, commutative, and/or distributive properties and non-standard algorithms. See Figure 8 for the comparison of Porter’s cognitive demands and DTAM.Summation

The above frameworks were similar in that they classified content knowledge in terms of cognitive difficulty for the purpose of improving pedagogy and driving instruction. Whether the frameworks were created for assessment purposes or to aid in classroom instruction, each framework provided the means to plan for and promote higher order thinking behaviors—first in the teacher, then in the student. Some frameworks classified content knowledge hierarchically—that is, by the item’s complexity of cognitive behavior, moving from simple to complex behaviors (left to right) or incorporating levels of knowledge within one category.

The following frameworks depart from depth of content knowledge and describe teachers' knowledge. The frameworks describe the type of teacher knowledge displayed—for example, content knowledge, pedagogical content knowledge (PCK), curricular knowledge, or specialized knowledge. Unlike content frameworks, the following frameworks focus on teachers' knowledge, not the degree of cognitive complexity of items. The following section describes these frameworks and defines the types of teacher knowledge.Shulman's Types of Teacher Knowledge Framework

Framework defined Shulman (1986) developed a framework for describing teacher knowledge— the information in the minds of teachers. He wanted to answer the question, "What kinds of knowledge do teachers use as they reason?" What knowledge is necessary to allow teachers to communicate effectively the "most useful forms of representation the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, (knowing) the ways of formulating the subject that make it comprehensible to others" (p. 9).Shulman's framework divided teacher content knowledge into three main dimensions: content knowledge, PCK, and curricular knowledge. Subject matter knowledge is the knowledge of the content of the specified discipline; for mathematics it would be the knowledge that teachers possess about mathematics. This knowledge is broken down into two categories: (a) substantive and (b) syntactical knowledge. In elementary mathematics, substantive knowledge includes the understanding and explanation of key facts, concepts, and principles, such as place value, base ten, and properties of addition and subtraction (Shulman & Grossman, 1988). Syntactical knowledge is the ability to "speak" of the structures underlying the mathematical concepts; it requires understanding the grammar, rules, and proofs of mathematics that underlie the particular topic under engagement. An example of substantive and syntactic knowledge is found in the problem "Simplify and justify your steps: 22(11x)." Corcoran (2005), when studying the substantive and syntactic knowledge of Irish pre-service teachers, explained this problem succinctly: The problem required multiplication calculations that almost all students performed correctly (substantive knowledge), but it also required knowledge of the application of the associative property of multiplication, and this piece of syntactic mathematical subject knowledge appears to have escaped 93.3% of students . Shulman (1986) defined PCK as a particular form of content knowledge that embodies the aspect of content most germane to its teach ability. As the name suggests, pedagogical content knowledge is the integration of what was previously regarded as two distinct types of knowledge—content knowledge and pedagogical knowledge. When the two types of knowledge are kept distinct, teachers learn about what to teach and best-practice principles that govern how to teach, but not how to specifically teach the "what." In an interview conducted by Dennis Sparks on the merging of content knowledge and pedagogy, Shulman and Sparks (1992) gave two examples of the hazard of disconnecting the two branches of learning:

Example 1: What do you have to know about mathematics and about your students to fashion the appropriate anticipatory set before you begin a unit on signed numbers in mathematics?

We know a great deal about the problems of understanding positive and negative numbers, about the common misunderstandings students bring to the study of such a topic before the instruction begins, and about the subsequent consequences for future learning if students fail to grasp the essential nature of signed numbers. We also know a great deal about a whole host of strategies for teaching signed numbers. Unfortunately, when the generic staff development is done, people are often left with a general grasp of what it means to establish an anticipatory set, but none of the particulars. Teachers need a substantial amount of subject-specific examples, analyses, and practice within their staff development programs. Example 2: Explaining why we invert and multiply to divide fractions by fractions demands a store of topic-specific examples and clarifications. Just knowing that you should "check for understanding" doesn't get you too far (p. 14–15). The benefit of PCK is that it conveys how best to specifically teach mathematics. "It represents the blending of content and pedagogy into an understanding of how particular topics, problems, or issues are organized, represented, and adapted to the diverse interests and abilities of learners, and presented for instruction" (Shulman, 1999,). Intertwined closely with PCK is Shulman's third category of knowledge, curricular knowledge, which is the ability to understand and appropriately choose and use the instructional materials (curriculum) necessary to teach. These include "alternate texts, software, programs, visual materials, single-concept films, laboratory demonstrations, or 'invitations to enquiry'" (Shulman, 1986, p.10). As Shulman questioned, "Would we trust a physician who did not really understand the alternate ways of dealing with categories of infectious disease, but who knew only one way?" (p. 10). Without a doubt, teachers must be able to intelligently select and manipulate curricula appropriately to meet the needs of the individual student. "Making judgments about the mathematical quality of instructional materials and modifying as necessary; judging and correcting textbook treatments of particular topics; and connecting mathematical ideas within and across other mathematical topics" are required skills for elementary teachers and are check-listed in Bush's (2005) "Assessing the Mathematics Knowledge of Teachers," an amalgamated checklist. Effective elementary mathematics teaching in Shulman's (1986) framework is a balance of deep, conceptual understanding in each of the three content knowledge areas: subject matter (or content), pedagogical content, and curricular knowledge. Elementary teachers should have a deep understanding of the concepts and principles that govern mathematics, know what relevant topics (standards) should be taught, and know how best to teach them. The cognitive frameworks can be used within each of Shulman's categories to identify and guide higher order and complex classroom instruction. The particular framework used to identify the degree of cognitive complexity attained by the teachers—whether it is Bloom's Taxonomy or Webb's depths of knowledge—is immaterial. What is important is the marriage of the cognitive frameworks with the content frameworks (Hammerness, Darling-Hammond, Grossman, Rust, & Shulman, 2005).

Ball, Bass, and Hill Mathematical Knowledge for Teaching Framework Framework defined Like Shulman (1986), Ball, Hill, and Bass (2005) attempted to answer the same question about teacher knowledge: What do teachers do in teaching mathematics, and in what ways does what they do demand mathematical reasoning, insight, understanding, and skill? Ball, Bass, and Hill expanded and re-partitioned Shulman's teacher content-knowledge divisions (Ball & Bass, 2000; Hill, Ball & Schilling, 2008). They defined mathematical knowledge for teaching or specialized content knowledge (Hill & Ball, 2006; Ball & Bass, 2003a, 2003b) as "the mathematical knowledge used to carry out the work of teaching mathematics" (Hill, Rowan & Ball, 2005, p. 3). This knowledge is gained by observing and cataloging the specific knowledge required by classroom teachers to perform their jobs; it refers to a specific body of knowledge distinct from mathematical content knowledge that teachers must know and understand to enable them to teach effectively. This specialized teacher knowledge goes beyond knowing a body of content that is common to all mathematicians; it is being able to effectively teach that content, analyze and correct student understanding, and relate content in practice conceptually. Ball, Bass, and Hill categorized mathematical knowledge for teaching into four domains: (a) common content knowledge (CCK), (b) specialized content knowledge (SCK), (c) knowledge of students and content (KSC), and (d) knowledge of teaching and content (KTC).

To illustrate these four domains, consider the difference between calculating the answer to a multi-digit multiplication problem (CCK), analyzing calculation errors for the problem (SCK), identifying student thinking that is likely to have produced such errors (KSC), and recognizing which manipulative would best highlight place-value features of the algorithm (KTC) (Ball, Sleep & Thames, 2007, CCK is the mathematics knowledge that any educated professional would know and includes both procedural and conceptual knowledge as defined by Hiebert and Carpenter (1992). The procedural knowledge is illustrated when teachers display their mathematical knowledge of methods for adding or subtracting, computing area or perimeter, determining the mean, median, or mode, calculating independent and dependent probability events, and so on. Teachers reveal conceptual knowledge when they demonstrate understanding of a variety of methods to accomplish a task or link different concepts to uncover or form a new option. Bass and Ball (2003) and Hill et al. (2008) presented an example in which several students devised alternate methods for multiplying a simple 2-digit multiplication problem. See Figure 9 for the sample problem. In this example, the teacher must (a) understand which methods work and under what circumstances and (b) have a deep conceptual knowledge of mathematics—being able to discern that Student A used place value and the commutative property and Student C grouped using the distributive property. While some may confuse this knowledge with pedagogical knowledge, it is not. Ball and Bush made a clear distinction between content knowledge and PCK. This example demonstrates knowledge that any mathematics professional should know. In order to bridge into the PCK, the teacher would have had to identify (SCK) or correct (KSC) student misconceptions. The only requisite in this scenario is a sound knowledge of basic elementary mathematics (Hill, Schilling, & Ball, 2004; Hill, 2006).CCK, SCK, and KSC are all types of knowledge that Shulman (1986) would have categorized as subject matter knowledge. SCK and KSC both incorporate what Shulman would have categorized as PCK. See Figure 10 for a comparison of the frameworks from Ball, Bass, and Hill (2005) and Shulman.

The next steps—Assessment and Professional Development Centered Frameworks While these cognitive and teaching knowledge frameworks aide in understanding depth of teacher's mathematics knowledge, they also provide a basis from which to assess that knowledge. To date, only two instruments quantitatively measure mathematics knowledge for teaching as defined by these teacher knowledge frameworks. They are Learning for Mathematics Teaching (LMT) from the University of Michigan (which is based on Ball, Bass, and Hill's frameworks) and the Diagnostic Teachers' Assessment for Mathematics and Science (DTAMS) from the University of Louisville (based on Webb's Depth of Knowledge frameworks). These assessments measure not only mathematics content knowledge, but also PCK necessary for teaching. PCK items focus on explaining terms and concepts to students; interpreting students' statements and solutions; assessing students' mathematics learning and taking the next steps; interpreting and making mathematical and pedagogical judgments about students' questions, solutions, problems, and insights (both predictable and unusual); assisting students in building mathematical structures; and helping students abstract and generalize mathematical ideas (schemas) (Ball, 2003a; Hill & Ball, 2006). Both assessments measure teachers' knowledge with regard to whole and rational numbers. The LMT provides teachers with one total score that reflects all types of teachers' mathematics knowledge. DTAMS also provides sub-scores that reflect both DOK and PCK scores. The DOK scores are divided into three levels: memorization, understanding, and problem solving/reasoning. In addition to these quantitative assessments, Bush (2005) formulated a mathematics-specific checklist of teacher behaviors based on Ball, Bass, and Hill's (2005) implications for assessing mathematical knowledge for teaching and Shulman's (1986) PCK. The behaviors were broken down into three knowledge type categories: instructional strategies, the knowledge and skill set necessary to effectively instruct students; student learning, the knowledge and skill set necessary for teachers to interact and influence student learning; and curricula, being able to choose the materials necessary to facilitate conceptual learning. These examples only scratch the surface of practical uses for teaching frameworks in mathematics instruction. Shulman wrote, "the great promise of assessment is its deployment in the service of instruction, its capacity to inform the judgment of faculty and students regarding how they can best advance the quality of learning" (200 7, p. 23). In order to serve the classroom, these mathematics frameworks should be used to improve teacher classroom instruction. In the same way that teachers are adapting data driven instruction to further enhance student achievement in mathematics, educators could assess themselves and others using these knowledge frameworks. Then, not only will the question "At what depth is a mathematical concept understood?" be answered, but more importantly, "At what depth is it being displayed in the classroom?" Therein lies improved teacher instruction. Since classroom instruction directly impacts student learning, it is beneficial to be able to talk about, measure, and improve the depth at which mathematics teachers teach.

Research on teacher professional developments predicated around these frameworks for the purpose of improving mathematics teachers' classroom instruction is relatively new. Vicki-Lynn Holmes, Chelsea Miedema, and Lindsay Niewkoop (2010, in press) have been running a longitudinal study using Webb's Depth of Knowledge framework as the basis for developing Algebraic based professional developments for middle and high school teachers. Thirty-eight teachers were given a derivative of the DTAMS test to determine the depth of their function family pedagogical content knowledge prior to a three-day workshop. The teacher responses to "How would you correct the student misconception?" were broadly categorized into a conceptual or procedural / algorithmic response, and then further dissected into a recall, skill/concept, or strategic thinking response. Two major concern areas were uncovered: (a) teachers tended to correct individual student computational errors, rather than misconceptions; and (b) teachers were not linking algebraic concepts such as multiplying polynomials to their two-digit multiplication, place-value roots. The ensuing professional development addressed the first concern by providing teachers practice in correcting student scenarios while training them to categorize their responses via Webb's taxonomy. By sharing the Depth of Knowledge diagnostics results, the teachers became more conscious of how they addressed their students at what level of mathematics depth and literacy. This has direct implications for classroom instruction. Mark Thames and Deborah Ball (2010) promote teacher education and professional developments that "center more directly on the mathematical knowledge (frameworks) on which effective teaching draws. The next step is moving forward from using the frameworks discussed in this article as a means of talking about and assessing teachers' mathematics knowledge in the classroom, to providing professional developments centered around the frameworks addressing teachers weak areas and highlighting their. It is the hope of the author that by describing and comparing these seven frameworks for analyzing knowledge of mathematics, educators will not only employ the common language, but utilize the frameworks both in assessing teachers' classroom knowledge and in designing professional developments based on those outcomes

Glossary of Acronyms
CCA---- Core Center for Assessment
CCK---- Common Content Knowledge
DOK---- Depth of Knowledge
DTAMS- Diagnostic Teacher Assessment of Mathematics and Science
KDOE– Kentucky Department of Education
KSC– Knowledge of Students and content
KTC– Knowledge of Teaching and Content
LMT– Learning for Mathematics Teaching
MA – Mathematics
PCK – Pedagogical Content Knowledge
SCK– Specialized Content Knowledge

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