Unique Fixed Point Theorem For Asymptotically Regular Maps In Hilbert Space

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Abstract: The object of this paper is to obtain unique fixed point theorems for asymptotically regular maps and sequence in Hilbert Space.

Keywords: Hilbert Space, Asymptotically Regular Map, Asymptotically Regular Sequence.

INTRODUCTION

Banach fixed point theorem and its applications are well known. Many authors have extended this theorem, introducing more general contractive conditions which imply the existence of a fixed point. Almost all of conditions imply the asymptotic regularity of the mappings under consideration. So the investigation of the asymptotically regular maps plays an important role in fixed point theory. Sharma and Yuel [8] and Guay and Singh [6] were among the first who used the concept of asymptotic regularity to prove fixed point theorems for wider class of mappings than a class of mappings introduced and studied by Ciric [4].

The purpose of this short paper is to study a wide class of asymptotically regular mappings which possess fixed points in Hilbert spaces. In present paper we extend the results of Gopal and Ranadive [5] in Hilbert space.

2. PRELIMINARIES

Definition 2.1: A self-mapping T on a metric space (X, d) is said to be asymptotically regular at a point x in X, if $d(T^n x, T^n Tx) \to 0$ as $n \to \infty$

Where $T^n x$ denotes the nth iterate of T at x.

Definition 2.2: Let C be a closed subset of a Hilbert space H. A sequence $\{x_n\}$ in C is said to be asymptotically T – regular if $\|x_n - Tx_n\| \to 0$ as $n \to \infty$

Definition 2.3: A self- mapping T on a closed subset C of a Hilbert space H is said to be asymptotically regular at a point x in C if

$$\|T^n x - T^{n+1} x\| \to 0 \text{ As } n \to \infty$$

Where $T^n x$ denotes the nth iterate of T at x.

The result of Dhananjay Gopal and A.S. Ranadive [5] is given below

Theorem 2.4. Let (X ,d) be a complete metric space and T a self- map satisfying the inequality :

$$d(TX, Ty) \leq \alpha \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$$

for all x, y in X , where $\alpha \in (0,1)$. If there exist an asymptotically T – regular sequence in X, then T has a unique fixed point.

Theorem (2.5) (Banach's contraction principle) Let (X, d) be a complete metric space, c∈(0,1) and f: X→X be a mapping such that for each x, y ∈X,
$d(fx, fy) \leq cd(x, y)$ Then f has a unique fixed point a ∈X, such that for each
x∈ X, $\lim_{n\to\infty} f^n(x) = a$.

After the classical result, Kannan [7] gave a subsequently new contractive mapping to prove the fixed point theorem. Since then a number of mathematicians have been worked on fixed point theory dealing with mappings satisfying various type of contractive conditions.

In 2002, A. Branciari [2] analysed the existence of fixed point for mapping f defined on a complete metric space (X,d) satisfying a general contractive condition of integral type.

Theorem (2.6) (Branciari) Let (X, d) be a complete metric space ,c $\in (0,1)$ and let f : X→ X be a mapping such that for each x, y ∈X,
$\int_0^{d(fx,fy)} \phi(t)dt \leq c \int_0^{d(x,y)} \phi(t)dt$ where ϕ:[0,+ ∞) →[0,+ ∞) is a Lebesgue integrable mapping which is summable on each compact subset of [0,+ ∞) , non-negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \phi(t)dt$, then f has a unique fixed point a ∈X, such that for each x ∈ X,
$\lim_{n\to\infty} f^n(x) = a$.

After the paper of Branciari, a lot of research works have been carried out on generalizing contractive condition of integral type for different contractive mappings satisfying various known properties. A fine work has been done by Rhoades [2] extending the result of Branciari by replacing the condition [1.2] by the following

$$\int_0^{d(fx,fy)} \phi(t)dt \leq \int_0^{\max\left\{d(x,y),d(x,fx),d(y,fy),\frac{d(x,fy)+d(y,fx)}{2}\right\}} \phi(t)dt.$$

3. Main Results

Theorem 3.1. Let C be a closed subset of a Hilbert space H and T be a self -mapping of C, satisfying

$$\int_0^{\|Tx-Ty\|^2} \phi(t)dt \leq \alpha \int_0^{\max\left\{\|x-y\|^2, \|x-Tx\|^2, \|y-Ty\|^2, \frac{\|x-Ty\|^2+\|y-Tx\|^2}{2}\right\}} \phi(t)dt$$

$$+\beta \int_0^{\|Tx-y\|\|x-Ty\|} \phi(t)dt + \gamma \int_0^{\|x-Ty\|\|x-y\|} \phi(t)dt$$

$$+\delta \int_0^{\|x-y\|^2} \phi(t)dt$$

For all x, y $\in$ C, where α , $\beta, \gamma, \delta, \eta$ are non-negative reals with $0 < 8\alpha + 4\beta + 4\gamma + 4\delta < 1$. Also $\emptyset$:[0,+ ∞) $\rightarrow$[0,+ ∞) is a Lebesgue integrable mapping which is summable on each compact subset of [0,+ ∞) , non- negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$, If there exist an asymptotically T – regular sequence in X , then T has a unique fixed point.

Proof: Let $\{x_n\}$ be an asymptotically T – regular sequence in closed subset C of Hilbert Space H then

$$\|x_n - x_m\|^2 \leq 2\,\|x_n - Tx_n\|^2 + 2\|Tx_n - x_m\|^2$$

[By parallelogram law $\|x + y\|^2 \leq 2\,\|x\|^2 + 2\|y\|^2$]

Now,

$$\int_0^{\|x_n-x_m\|^2} \emptyset(t)dt \leq 2\int_0^{\|x_n-Tx_n\|^2} \emptyset(t)dt + 2\int_0^{\|Tx_n-x_m\|^2} \emptyset(t)dt$$

$$\leq 2\int_0^{\|x_n-Tx_n\|^2} \emptyset(t)dt + 4\int_0^{\|Tx_n-Tx_m\|^2} \emptyset(t)dt$$

$$+4\int_0^{\|Tx_m-x_m\|^2} \emptyset(t)dt$$

$$\leq 2\int_0^{\|x_n-Tx_n\|^2} \emptyset(t)dt$$

$$+4\alpha\int_0^{\max\left\{\|x_n-x_m\|^2,\|x_n-Tx_n\|^2,\|x_m-Tx_m\|^2,\frac{\|x_n-Tx_m\|^2+\|x_m-Tx_n\|^2}{2}\right\}}\emptyset(t)dt$$

$$+4\beta\int_0^{\|Tx_n-x_m\|\|x_n-Tx_m\|}\emptyset(t)dt + 4\gamma\int_0^{\|x_n-Tx_m\|\|x_n-x_m\|}\emptyset(t)dt$$

$$+4\int_0^{\|Tx_m-x_m\|^2} \emptyset(t)dt + 4\delta\int_0^{\|x_n-x_m\|^2} \emptyset(t)dt$$

$$\leq 2\int_0^{\|x_n-Tx_n\|^2} \emptyset(t)dt + 4\alpha$$

$$\int_0^{\max\left\{\|x_n-x_m\|^2,\|x_n-Tx_n\|^2,\|x_m-Tx_m\|^2,\frac{2\|x_n-x_m\|^2+2\|x_m-Tx_m\|^2+2\|x_n-x_m\|^2+2\|x_n-Tx_n\|^2}{2}\right\}}\emptyset(t)dt$$

$$+4\beta\int_0^{(\|x_n-Tx_n\|+\|x_n-x_m\|)(\|x_m-Tx_m\|+\|x_n-x_m\|)}\emptyset(t)dt$$

$$+4\gamma\int_0^{(\|x_m-Tx_m\|+\|x_n-x_m\|)\|x_n-x_m\|}\emptyset(t)dt + 4\int_0^{\|Tx_m-x_m\|^2} \emptyset(t)dt$$

$$+4\delta\int_0^{\|x_n-x_m\|^2} \emptyset(t)dt$$

Taking limit as n $\to \infty$, then asymptotically T – regularity of $\{x_n\}$ gives

$$\int_0^{\|x_n-x_m\|^2} \emptyset(t)dt \leq (8\alpha + 4\beta + 4\gamma + 4\delta) \int_0^{\|x_n-x_m\|^2} \emptyset(t)dt$$

$$\int_0^{\|x_n-x_m\|^2} \emptyset(t)dt = 0 \text{ as } 8\alpha + 4\beta + 4\gamma + 4\delta < 1 \quad \text{---------------- (3.1.1)}$$

Hence $\{x_n\}$ is a Cauchy sequence, since C is a closed subset of H so every sequence converges in it , put

$\lim_{n\to\infty} x_n = z$

Existence of Fixed Point: We claim that z is a fixed point of T in C.

Consider

$$\int_0^{\|Tz-z\|^2} \emptyset(t)dt \leq 2\int_0^{\|Tz-Tx_n\|^2} \emptyset(t)dt + 4\int_0^{\|Tx_n-x_n\|^2} \emptyset(t)dt$$

$$+4\int_0^{\|x_n-z\|^2} \emptyset(t)dt$$

[By parallelogram law $\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$]

$$\int_0^{\|Tz-z\|^2} \emptyset(t)dt \leq 2\alpha \int_0^{\max\left\{\|z-x_n\|^2, \|z-Tz\|^2, \|x_n-Tx_n\|^2, \frac{\|z-Tx_n\|^2+\|x_n-Tz\|^2}{2}\right\}} \emptyset(t)dt$$

$$+2\beta \int_0^{\|Tz-x_n\|\|z-Tx_n\|} \emptyset(t)dt + 2\gamma \int_0^{\|z-Tx_n\|\|z-x_n\|} \emptyset(t)dt$$

$$+2\delta \int_0^{\|x_n-z\|^2} \emptyset(t)dt + 4\int_0^{\|Tx_n-x_n\|^2} \emptyset(t)dt + 4\int_0^{\|x_n-z\|^2} \emptyset(t)dt$$

$$\int_0^{\|Tz-z\|^2} \emptyset(t)dt \leq$$

$$2\alpha \int_0^{\max\left\{\|z-x_n\|^2, \|z-Tz\|^2, \|x_n-Tx_n\|^2, \frac{2\|z-x_n\|^2+2\|x_n-Tx_n\|^2+2\|x_n-z\|^2+2\|z-Tz\|^2}{2}\right\}} \emptyset(t)dt$$

$$+2\beta \int_0^{(\|z-x_n\|+\|z-Tz\|)(\|z-x_n\|+\|x_n-Tx_n\|)} \emptyset(t)dt$$

$$+2\gamma \int_0^{(\|z-x_n\|+\|x_n-Tx_n\|)\|z-x_n\|} \emptyset(t)dt + 2\delta \int_0^{\|x_n-z\|^2} \emptyset(t)dt$$

$$+4\int_0^{\|Tx_n-x_n\|^2} \emptyset(t)dt + 4\int_0^{\|x_n-z\|^2} \emptyset(t)dt$$

Taking limit as n $\to \infty$, then asymptotically T – regularity of $\{x_n\}$ gives

$$(1-2\alpha)\int_0^{\|Tz-z\|^2} \emptyset(t)dt \leq 0$$

$$\int_0^{\|Tz-z\|^2} \emptyset(t)dt = 0 \qquad (\text{ as } 8\alpha + 4\beta + 4\gamma + 4\delta < 1)$$

Tz = z

Hence z is a fixed point of T.

Uniqueness: To show the uniqueness let z and z_1 be two distinct fixed points of T then

$$\int_0^{\|z-z_1\|^2} \emptyset(t)dt = \int_0^{\|Tz-Tz_1\|^2} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\max\left\{\|z-z_1\|^2, \|z-Tz\|^2, \|z-Tz_1\|^2, \frac{\|z-Tz_1\|^2+\|z_1-Tz\|^2}{2}\right\}} \emptyset(t)dt$$

$$+\beta \int_0^{\|Tz-z_1\|\|z-Tz_1\|} \emptyset(t)dt + \gamma \int_0^{\|z-Tz_1\|\|z-z_1\|} \emptyset(t)dt + \delta \int_0^{\|z-z_1\|^2} \emptyset(t)dt$$

$$\int_0^{\|z-z_1\|^2} \emptyset(t)dt \leq (\alpha+\beta+\gamma+\delta)\int_0^{\|z-z_1\|^2} \emptyset(t)dt$$

$$\int_0^{\|z-z_1\|^2} \emptyset(t)dt = 0 \qquad (\text{ as } 8\alpha + 4\beta + 4\gamma + 4\delta < 1)$$

Z = z_1

This completes the proof .

Theorem 3.2. Let C be a closed subset of a Hilbert space H and T be a self -mapping of C, satisfying

$$\int_0^{\|Tx-Ty\|^2} \emptyset(t)dt \leq \alpha \int_0^{\max\left\{\|x-y\|^2, \|x-Tx\|^2, \|y-Ty\|^2, \frac{\|x-Ty\|^2+\|y-Tx\|^2}{2}\right\}} \emptyset(t)dt$$

$$+\beta \int_0^{\|Tx-y\|\|x-Ty\|} \emptyset(t)dt + \gamma \int_0^{\|x-Ty\|\|x-y\|} \emptyset(t)dt$$

$$+\delta \int_0^{\|Tx-Ty\|^2} \emptyset(t)dt$$

For all x, y $\in$ C , where $\alpha, \beta, \gamma, \delta, \eta$ are non-negative reals with $0 < 8\alpha + 4\beta + 4\gamma + 4\delta < 1$. Also $\emptyset$:[0,+ ∞) $\rightarrow$[0,+ ∞) is a Lebesgue integrable mapping which is summable on each compact subset of [0,+ ∞) , non- negative ,and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \emptyset(t)dt$, If T is an asymptotically regular at some point x of C , then there exists a unique fixed point of T .

Proof: Let T be an asymptotically regular at $x_0 \in C$. Consider the sequence

$\{T^n x_0\}$ then for all m, n ≥ 1

$$\int_0^{\|T^m x_0 - T^n x_0\|^2} \emptyset(t)dt = \int_0^{\|TT^{m-1} x_0 - TT^{n-1} x_0\|^2} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\max\left\{\|T^{m-1}x_0 - T^{n-1}x_0\|^2, \|T^{m-1}x_0 - T^m x_0\|^2, \|T^{n-1}x_0 - T^n x_0\|^2, \frac{\|T^{m-1}x_0 - T^n x_0\|^2 + \|T^{n-1}x_0 - T^m x_0\|^2}{2}\right\}} \emptyset(t)dt$$

$$+ \beta \int_0^{\|T^m x_0 - T^{n-1} x_0\| \|T^{m-1} x_0 - T^n x_0\|} \emptyset(t)dt$$

$$+ \gamma \int_0^{\|T^{m-1} x_0 - T^n x_0\| \|T^{m-1} x_0 - T^{n-1} x_0\|} \emptyset(t)dt + \delta \int_0^{\|T^{m-1} x_0 - T^{n-1} x_0\|^2} \emptyset(t)dt$$

$$\leq \alpha \int_0^{\max\left\{\begin{array}{c} 2\|T^{m-1}x_0 - T^m x_0\|^2 + 4\|T^m x_0 - T^n x_0\|^2 + 4\|T^n x_0 - T^{n-1}x_0\|^2, \|T^{m-1}x_0 - T^m x_0\|^2, \|T^{n-1}x_0 - T^n x_0\|^2 \\ , \frac{2\|T^{m-1}x_0 - T^m x_0\|^2 + 2\|T^m x_0 - T^n x_0\|^2 + 2\|T^{n-1}x_0 - T^n x_0\|^2 + \|T^n x_0 - T^m x_0\|^2}{2} \end{array}\right\}} \emptyset(t)dt$$

$$+ \beta \int_0^{(\|T^m x_0 - T^n x_0\| + \|T^n x_0 - T^{n-1} x_0\|)(\|T^{m-1} x_0 - T^m x_0\| + \|T^m x_0 - T^n x_0\|)} \emptyset(t)dt$$

$$+ \gamma \int_0^{(\|T^{m-1} x_0 - T^m x_0\| + \|T^m x_0 - T^n x_0\|)(\|T^{m-1} x_0 - T^m x_0\| + \|T^m x_0 - T^n x_0\| + \|T^n x_0 - T^{n-1} x_0\|)} \emptyset(t)dt$$

$$+ \delta \int_0^{2\|T^{m-1} x_0 - T^m x_0\|^2 + 4\|T^m x_0 - T^n x_0\|^2 + 4\|T^n x_0 - T^{n-1} x_0\|^2} \emptyset(t)dt$$

[By parallelogram law $\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$]

Since T is an asymptotically regular

$$\int_0^{\|T^m x_0 - T^n x_0\|^2} \emptyset(t)dt \leq 4\alpha \int_0^{\|T^m x_0 - T^n x_0\|^2} \emptyset(t)dt$$

$$+ \beta \int_0^{\|T^m x_0 - T^n x_0\|^2} \emptyset(t)dt + \gamma \int_0^{\|T^m x_0 - T^n x_0\|^2} \emptyset(t)dt$$

$$+4\delta \int_0^{\|T^m x_0 - T^n x_0\|^2} \emptyset(t)dt$$

$$(1 - 4\alpha - \beta - \gamma - 4\delta) \int_0^{\|T^m x_0 - T^n x_0\|^2} \emptyset(t)dt \leq 0$$

$$\int_0^{\|T^m x_0 - T^n x_0\|^2} \emptyset(t)dt = 0 \quad (\text{ as } 8\alpha + 4\beta + 4\gamma + 4\delta < 1)$$

Hence $\{T^n x_0\}$ is a Cauchy sequence then

$$\lim_{n\to\infty} T^n x_0 = z$$

Existence of Fixed Point: We claim that z is a fixed point of T in C.

Consider

$$\int_0^{\|Tz-z\|^2} \emptyset(t)dt \leq 2\int_0^{\|Tz-T^n x_0\|^2} \emptyset(t)dt + 2\int_0^{\|T^n x_0 - z\|^2} \emptyset(t)dt$$

[By parallelogram law $\|x+y\|^2 \leq 2\|x\|^2 + 2\|y\|^2$]

$$\int_0^{\|Tz-z\|^2} \emptyset(t)dt \leq 2\alpha$$

$$\int_0^{\max\left\{\|z-T^{n-1}x_0\|^2, \|z-Tz\|^2, \|T^{n-1}x_0-T^n x_0\|^2, \frac{\|z-T^n x_0\|^2+\|T^{n-1}x_0-Tz\|^2}{2}\right\}} \emptyset(t)dt$$

$$+2\beta\int_0^{\|Tz-T^{n-1}x_0\|\|z-T^n x_0\|} \emptyset(t)dt + 2\gamma\int_0^{\|z-T^n x_0\|\|z-T^{n-1}x_0\|} \emptyset(t)dt$$

$$+2\delta\int_0^{\|z-T^{n-1}x_0\|^2} \emptyset(t)dt$$

Since $\{T^{n-1}x_0\}$ is a subsequence of $\{T^n x_0\}$ then $\{T^{n-1}x_0\} \to z$ as $n \to \infty$

Taking limit as $n \to \infty$, we get

$$(1-2\alpha)\int_0^{\|Tz-z\|^2} \emptyset(t)dt \leq 0$$

$$\int_0^{\|Tz-z\|^2} \emptyset(t)dt = 0 \quad (\text{ as } 8\alpha + 4\beta + 4\gamma + 4\delta < 1)$$

Tz = z

Hence z is a fixed point of T.

Uniqueness: To show the uniqueness let z and z_1 be two distinct fixed points of T then

$$\int_0^{\|z-z_1\|^2} \emptyset(t)dt = \int_0^{\|Tz-Tz_1\|^2} \emptyset(t)dt$$

$$\leq \alpha\int_0^{\max\left\{\|z-z_1\|^2, \|z-Tz\|^2, \|z-Tz_1\|^2, \frac{\|z-Tz_1\|^2+\|z_1-Tz\|^2}{2}\right\}} \emptyset(t)dt$$

$$+\beta \int_0^{\|Tz-z_1\|\|z-Tz_1\|} \emptyset(t)dt + \gamma \int_0^{\|z-Tz_1\|\|z-z_1\|} \emptyset(t)dt$$

$$+\delta \int_0^{\|z-z_1\|^2} \emptyset(t)dt$$

$$\int_0^{\|z-z_1\|^2} \emptyset(t)dt \leq (\alpha+\beta+\gamma+\delta) \int_0^{\|z-z_1\|^2} \emptyset(t)dt$$

$$\int_0^{\|z-z_1\|^2} \emptyset(t)dt = 0 \qquad (\text{ as } 8\alpha + 4\beta + 4\gamma + 4\delta < 1)$$

Z = z_1

This completes the proof.

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