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Generation of New Julia Sets and Mandelbrot Sets for Tangent Function

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Abstract
The generation of fractals and study of the dynamics of transcendental function is one of emerging and interesting field of research nowadays. We introduce in this paper the complex dynamics of tangent function of the type $\{\tan(z^n)+c\}$, where $n \geq 2$ and applied Ishikawa iteration to generate new Relative Superior Mandelbrot sets and Relative Superior Julia sets. Our results are entirely different from those existing in the literature of transcendental function.
Keywords: Complex dynamics, Relative Superior Julia set, Relative Superior Mandelbrot set.

1. Introduction

Extracting qualitative information from data is a central goal of experimental science. In dynamical systems, for example, the data typically approximate an attractor or other invariant set and knowledge of the structure of these sets increases our understanding of the dynamics. The most qualitative description of an object is in terms of its topology — whether or not it is connected? Based on this objective, this paper studies the dynamical behavior of tangent function.

The study of transcendental function has emerged out as discrete dynamical systems in numerical and complex analysis. It forms a rich dynamics for well known Julia sets and Mandelbrot sets (Devaney 1989). On the other hand, the dynamics of iterated polynomials are one of the greatest pioneering work of Doaudy and Hubbard (1984, 1985). Given a polynomial of degree $n \geq 2$, the most important set is the Julia set J consisting of the points $z \in C$ which have no neighborhood in family of iterates, forms a normal family. Specially for the polynomials, one can start with the set of points I which converge to infinity under iteration (escaping points) and its complement $K = C / I$ is known as filled in Julia sets and it consists of points with bounded orbits. In other words, the Julia set J_c of the function Q_c where $Q_c = z^2 + c$ is either totally disconnected or connected. Its counterpart, Mandelbrot set for a family Q_c is defined as $M_c = \{c \in C : \text{orbit of } 0 \text{ under iteration by } Q_c \text{ is bounded}\}$ For $|c| > 2$, orbit of 0 escapes to ∞ so only $|c| \le 2$ is considered. For any n, $|Q_c^{(n)}(0)| > 2$, then the orbit of 0 tends to infinity (Devaney 1989) .

The key feature of this paper is to show that the tangent function, which falls under category of transcendental function, is an example, where Julia set is all of C. There is a great difference between the dynamics of polynomials and transcendental functions. Picard's Theorem (Schleicher 2007) tells us that for a transcendental function f , given any "neighborhood of infinity" $U = [\{z; |z| > r\},\ r \in R]$ $f(U)$ covers C with exception of at most one point. This is certainly not true for polynomials because we find a neighborhood of U so that $f(U) \subset U$.

The study of dynamical behavior of the transcendental functions were initiated by Fatou (1926). For transcendental function, points with unbounded orbits are not in Fatou sets but they must lie in Julia sets. Attractive points of a function have a basin of attraction, which may be disconnected. A point z in Julia for cosine function has an orbit that satisfies $|\operatorname{Im} z| \ge 50$

A Julia set thus, satisfies the following properties:

(i) Closed
(ii) Nonempty
(iii) Forward invariant t(If $z \in J(F)$, then $F(z) \in J(F)$, where F is the function).
(iv) Backward invariant
(v) Equal to the closure of the set of repelling cycles of F .

On the other hand, Fatou Set is the complement set of Julia set, also stated as stable set. Attracting cycles and their basins of attraction lie in the Fatou set, since iterates here tend to cycle and thus forms a normal family.

Thus, the iteration of complex analytic function F decompose the complex plane into two disjoint sets

1. Stable Fatou sets in which iterates are well behaved.
2. Julia sets on which the map is chaotic.

In trigonometric function, $S(z) = \sin z$, 0 is defined as fixed point for S . If $x_0 \in R$, then either $S(x_0) = 0$ or $S^n(x_0) \to 0$. Also, we have the points lying on the imaginary axis have their orbits that tend to infinity since $\sin(iy) = i \sin(hy)$. On the other hand for cosine function, if $[C_\lambda^n(z)] \to \infty$ as $n \to \infty$, then orbits which escapes do so, with the increase in the imaginary part. Here, Mandelbrot plane will contain infinitely many critical points given by $n^2\pi^2$, where $n \in N$. Sine and cosine functions are thus declared as "Topologically complete" (McMullen 1987).

The fixed point in topology, $z = z_0$ is declared as

(i) Attracting if $0 < |F'(z_0)| < 1$.
(ii) Superattracting if $F'(z_0) = 0$
(iii) Repelling if $|F'(z_0)| > 1$
(iv) Neutral if $F'(z_0) = e^{i2\pi\theta_0}$

If θ_0 is rational, then z_0 is rationally indifferent or parabolic, otherwise z_0 is irrationally indifferent.

The dynamics of cosine and sine function as revealed in the past literature states that the points that converge to ∞ under iteration are organized in the form of rays. It is well known that the set of escaping points is an open neighborhood of ∞, which can be parameterized by dynamic rays. As the tangent function is compromised of sine and cosine function, thus it will undertake most of the properties of both the functions. For the entire transcendental functions, the point ∞ is an essential singularity (rather than super attracting point). Ereneko (1989) studied that for every entire transcendental functions, the set of escaping points is always non-empty. His query was answered in an affirmative way by R. L. Devaney (1984, 1986), for the special case of Exponential function, where every escaping point can be connected to ∞, along with unique curve running entirely through the escaping points.

A dynamic ray is connected component of escaping set, removing the landing points. It turns out to be union of all uncountable many dynamic rays, having Hausdroff dimension equal to one. However by a result of McMullen (1987) the set of escaping points of a cosine family has an infinite planar Lesbegue measure. Therefore the entire measure of escaping points sits in the landing points of those rays which land at the escaping points.

In this past literature the sine and cosine functions were considered in the following manner:

- $\sin(z^n + c), where \ n \geq 2$
- $f(z) = (e^{iz} - e^{-iz})/2$
- $\cos(z^n + c), where \ n \geq 2$
- $f(z) = (e^{iz} + e^{-iz})/2$

We are introducing in this paper tangent function of the type $\{\tan(z^n) + c\}$, where $n \geq 2$ and applied Relative Superior Ishikawa iterates to develop an entirely new class of fractal images of this transcendental function. Escape criteria of polynomials are used to generate Relative Superior Mandelbrot Sets and Relative Superior Julia Sets. Our results are quite different from existing results in literature as we determined the connectivity of the Julia Sets using Ishikawa iterates.

2. Preliminaries:

The process of generating fractal images from $z \to \tan(z^n) + c$ is similar to the one employed for the self-squared function (Peitgen, Richter 1986).

Briefly, this process consists of iterating this function up to N times. Starting from a value z_0 we obtain $z_1, z_2, z_3, z_4, \ldots$ by applying the transformation $z \to \tan(z^n) + c$.

Definition2.1: Ishikawa Iteration (Ishikawa1974)**:** Let X be a subset of real or complex numbers and $f : X \to X$ for $x_0 \in X$, we have the sequences $\{x_n\}$ and $\{y_n\}$ in X in the following manner:

$$y_n = s'_n f(x_n) + (1 - s'_n) x_n$$
$$x_{n+1} = s_n f(y_n) + (1 - s_n) x_n$$

where $0 \leq s'_n \leq 1$, $0 \leq s_n \leq 1$ and $\{s'_n\}$ & $\{s_n\}$ are both convergent to non zero number.

Definition 2.2 (Rana, Chauhan, Negi 2010)**:** The sequences $\{x_n\}$ and $\{y_n\}$ constructed above is called Ishikawa sequences of iterations or Relative Superior sequences of iterates. We denote it by $RSO(x_0, s_n, s'_n, t)$. Notice that $RSO(x_0, s_n, s'_n, t)$ with s'_n=1 is $SO(x_0, s_n, t)$ i.e. Mann's orbit and if we place $s_n = s'_n = 1$ then $RSO(x_0, s_n, s'_n, t)$ reduces to $O(x_0, t)$.

We remark that Ishikawa orbit $RSO(x_0, s_n, s'_n, t)$ with $s'_n = 1/2$ is relative superior orbit. Now we define Mandelbrot sets for function with respect to Ishikawa iterates. We call them as Relative Superior Mandelbrot sets.

Definition 2.3 (Rana, Chauhan, Negi 2010)**:** Relative Superior Mandelbrot set RSM for the function of the form $Q_c(z) = z^n + c$, where n = 1, 2, 3, 4… is defined as the collection of $c \in C$ for which the orbit of 0 is bounded *i.e.* $RSM = \{c \in C : Q_c^k(0) : k = 0,1,2...\}$ is bounded.

In functional dynamics, we have existence of two different types of points. Points that leave the interval after a finite number are in stable set of infinity. Points that never leave the interval after any number of iterations have bounded orbits. So, an orbit is bounded if there exists a positive real number, such that the modulus of every point in the orbit is less than this number.

The collection of points that are bounded, *i.e.* there exists M, such that $|Q^n(z)| \le M$, for all n, is called as a prisoner set while the collection of points that are in the stable set of infinity is called the escape set. Hence, the boundary of the prisoner set is simultaneously the boundary of escape set and that is Julia set for Q.

Definition 2.4 **(**Chauhan, Rana, Negi 2010)**:** The set of points RSK whose orbits are bounded under relative superior iteration of the function Q (z) is called Relative Superior Julia sets. Relative Superior Julia set of Q is boundary of Julia set RSK

3. Generating the fractals:

We have used in this paper escape time criteria of Relative Superior Ishikawa iterates for function $z \to \tan(z^n) + c$.

*Escape Criterion for Quadratics***:** Suppose that $|z| > \max\{|c|, 2/s, 2/s'\}$, then $|z_n| > (1+\lambda)^n |z|$ and $|z_n| \to \infty$ as $n \to \infty$. So, $|z| \ge |c|$ & $|z| > 2/s$ as well as $|z| > 2/s'$ shows the escape criteria for quadratics.

*Escape Criterion for Cubics***:** Suppose $|z| > \max\{|b|, (|a| + 2/s)^{1/2}, (|a| + 2/s')^{1/2}\}$ then $|z_n| \to \infty$ as $n \to \infty$. This gives an escape criterion for cubic polynomials

*General Escape Criterion***:** Consider $|z| > \max\{|c|, (2/s)^{1/n}, (2/s')^{1/n}\}$ then $|z_n| \to \infty$ as $n \to \infty$ is the escape criterion. (Escape Criterion derived in **(**Rana, Chauhan, Negi 2010)).

Note that the initial value z_0 should be infinity, since infinity is the critical point of $z \to \tan(z^n) + c$. However instead of starting with z_0 = infinity, it is simpler to start with $z_1 = c$, which yields the same result. (*A critical point of* $z \to F(z) + c$ *is a point where* $F'(z) = 0$*).*

4. Geometry of Relative Superior Mandelbrot Sets and Relative Superior Julia Sets:

The fractals generated from the equation $z \to \tan(z^n) + c$ possesses symmetry along the real axis

Relative Superior Mandelbrot Sets:

- In case of quadratic polynomial, the central body is bifurcated from middle. The body is maintaining symmetry along the real axis. Secondary lobes are very small initially for s = 1, s' =1. As the value of the set changes to s =0.1, s' =0.5, the central body gets more unified along with existence of very small major secondary lobe. But as the value is changed to s =0.5 s' =0.4, the central body is merged into one along with existence of only one major secondary lobe. The fractal generated for s =0.5 s' =0.4appears to be in the form of an umbrella.
- In case of Cubic polynomial, the central body is showing bifurcation into two equal parts, each part containing one major secondary lobe which appears to be similar in size of central body. The symmetry of this body is maintained along both axes. For s =0.5 s' =0.4, the central body merges with the major secondary lobes and the figure thus generated contains two minor secondary lobes

attached on each side. As the value of relative Superior Mandelbrot set changes to s =0.1, s' =0.5, the bifurcation of the central body becomes invisible while the minor secondary bulbs attached to the body on the either side shows increase in their size.

- In case of Biquadratic polynomial, the central body is divided into three parts, each part having one major secondary bulb. The secondary bulbs present on either side of the real axis shows larger extensions. The body is maintaining symmetry along the real axis. . For s =0.5 s' =0.4, the two of the major secondary lobes merges with the central body along the real axis, along with the presence of minor secondary lobe present on each side. As the value of the set changes to s =0.1, s' =0.5, bifurcation of the central body becomes invisible while the minor secondary bulbs appears to be grow up in their size.

Relative Superior Julia Sets:

- Relative Superior Julia Sets for the transcendental function $\tan(z)$ appears to follow law of having $2n$ wings. These sets maintained their symmetry along both the axes *i.e.* along real and imaginary axis.
- The Relative Superior Julia Sets for quadratic function is divided into four wings with central black body. Its symmetry exists along both axes.
- The Relative Superior Julia Sets for Cubic function is divided into six wings having reflectional and rotational symmetry, along with a middle black region, that represents its Mandelbrot Set The zoom in Fig.2 showing s' =0.4 for s=0.5,illustrates this phenomenon.
- The Relative Superior Julia Sets for Biquadractic function is divided into eight wings possessing the reflectional and rotational symmetry, along with a black central escape region, which resembles to its Mandelbrot sets.

5. Generation of Relative Superior Mandelbrot Sets:

5.1 Relative Superior Mandelbrot Set of Quadratic function:

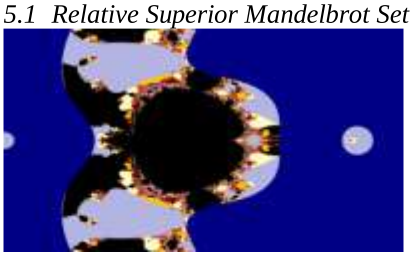

Fig 1: s= s'=1

Fig2: s=0.1, s'=0.5

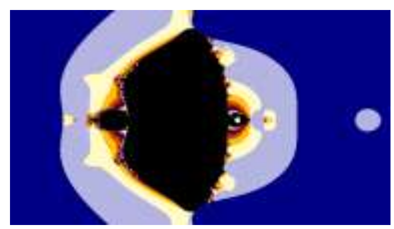

Fig 3: s=0.5, s'=0.4

5.2 Relative Superior Mandelbrot Set of Cubic function:

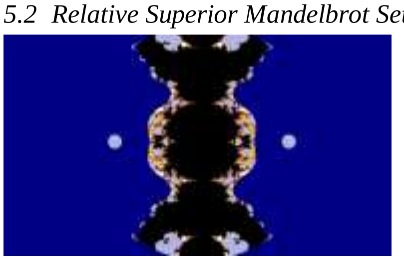

Fig 1: s= s'=1

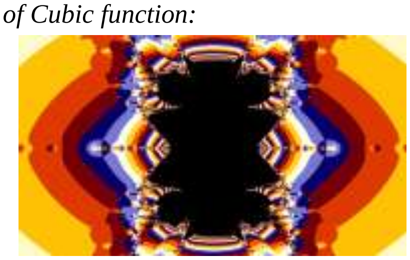

Fig2: s=0.1, s'=0.5

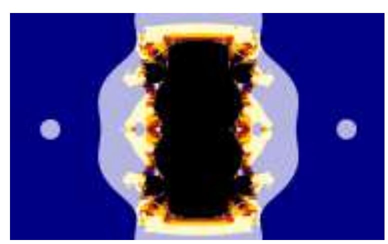

Fig 3: s=0.5, s'=0.4

5.3 Relative Superior Mandelbrot Set of Biquadratic function:

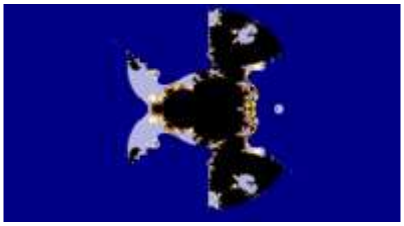
Fig 1: s= s'=1


Fig2: s=0.1, s'=0.5

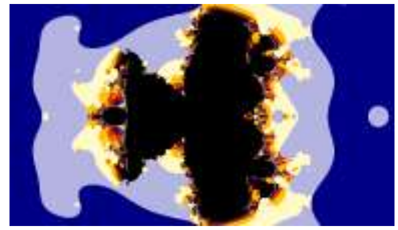
Fig 3: s=0.5, s'=0.4

5.4 Generalization of Relative Superior Mandelbrot Set:


Fig1: Relative Superior Mandelbrot Set for s=0.1, s'=0.5 and n=11

6. Generation of Relative Superior Julia Sets:

6.1 Relative Superior Julia Set of Quadratic function:

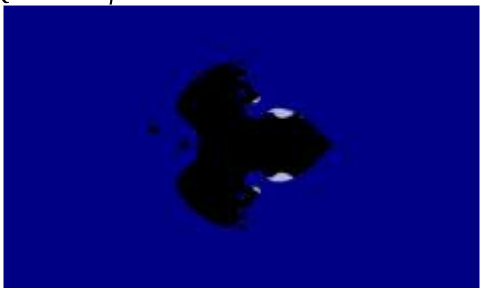

Fig1: Relative Superior Julia Set for s=0.5, s'=0.4, c= -0.4141245468+i0.0186667203

6.2 Relative Superior Julia Set of Cubic function:

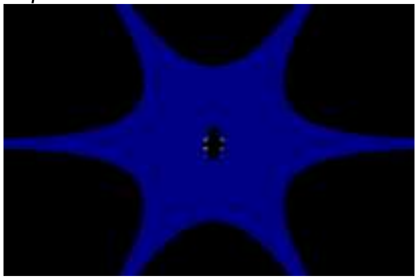

Fig1: Relative Superior Julia Set for s=0.5, s'=0.4, c= -0.09631341431+i0.0695165015

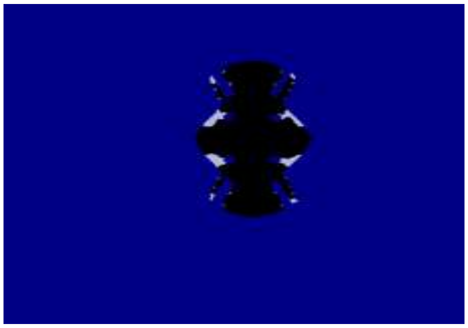

Fig2: Zoom of central part of Relative Superior Julia Set for s=0.5, s'=0.4, c= -0.09631341431+i0.0695165015

6.3 Relative Superior Julia Set of Biquadratic function:

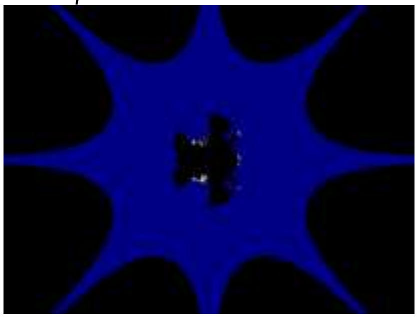

Fig1: Relative Superior Julia Set for s=0.5, s'=0.4, c= 0.08055725605+i0.038502707

7. Fixed points:

7.1 Fixed points of quadratic polynomial

Table 1: Orbit of F(z) at s=0.5 and s'=0.4 for (z_0= - 0.4141245468+i0.0186667203*i*)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1.	0.4145	8.	0.4151
2.	0.3074	9.	0.4151
3.	0.3526	10.	0.4151
4.	0.3901	11.	0.415
5.	0.4072	12.	0.415
6.	0.4131	13.	0.415
7.	0.4148	14.	0.415

Here we observed that the value converges to a fixed point after 11 iterations

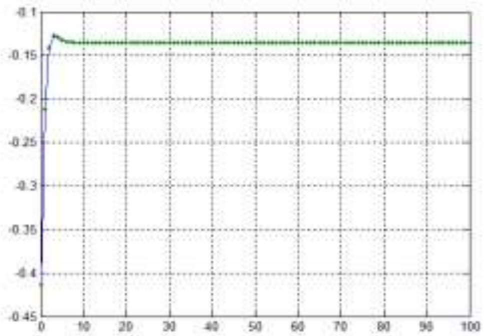

Fig1. Orbit of $F(z)$ at at s=0.5 and s'=0.4 for (z_0= - 0.4141245468+i0.0186667203i)

Table 2: Orbit of F(z) at s=0.1 and s'=0.5 for (z_0=-1.634519296 -0.01947061561 i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
57.	0.4233	67.	0.4237
58.	0.4234	68.	0.4237
59.	0.4234	69.	0.4237
60.	0.4235	70.	0.4237
61.	0.4235	71.	0.4237
62.	0.4236	72.	0.4237
63.	0.4236	73.	0.4237
64.	0.4236	74.	0.4237
65.	0.4236	75.	0.4238
66.	0.4236	76.	0.4238

Here we skipped 56 iteration and observed that the value converges to a fixed point after 74 iterations

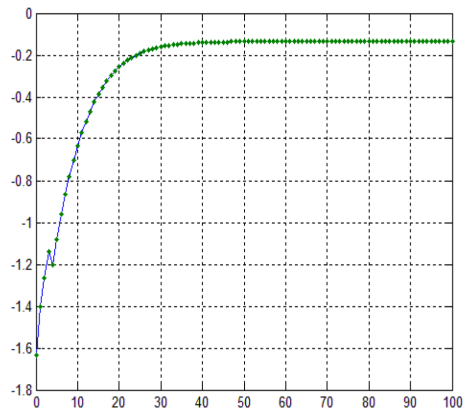

Fig2. : Orbit of F(z) at s=0.1 and s'=0.5 for (z_0=-1.634519296 -0.01947061561 i)

7.2 Fixed points of cubic polynomial

Table 1: Orbit of F(z) at s=0.1 and s'=0.5 for (z_0= 0.005386148102+0.005954274996i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
53.	0.4232	63.	0.4237
54.	0.4232	64.	0.4237
55.	0.4233	65.	0.4238
56.	0.4234	66.	0.4238
57.	0.4235	67.	0.4238
58.	0.4235	68.	0.4238
59.	0.4236	69.	0.4238
60.	0.4236	70.	0.4238
61.	0.4236	71.	0.4239
62.	0.4237	72.	0.4239

*Here we skipped 52 iteration and observed that the value converges to a fixed point after 70 iterat**ions***

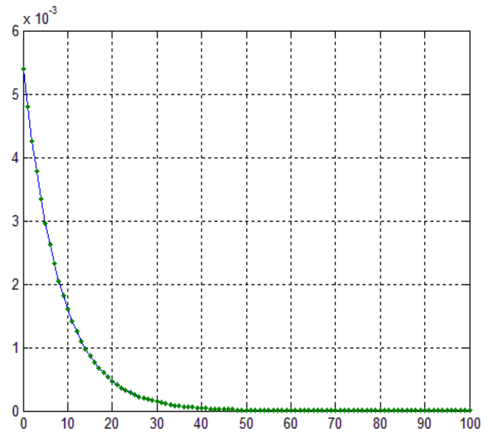

Fig 1. Orbit of F(z) at s=0.1 and s'=0.5 for (z_0= 0.005386148102+0.005954274996i)

Table 2: Orbit of F(z) at s=0.5 and s'=0.4 for (z_0 = -0.09631341431+0.0695165015 i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1.	0.3072	11.	0.4161
2.	0.3484	12.	0.4162
3.	0.3763	13.	0.4163
4.	0.3935	14.	0.4163
5.	0.4036	15.	0.4164
6.	0.4093	16.	0.4164
7.	0.4125	17.	0.4164
8.	0.4143	18.	0.4164
9.	0.4152	19.	0.4164
10.	0.4158	20.	0.4164

Here observe that the value converges to a fixed point after 14 iterations

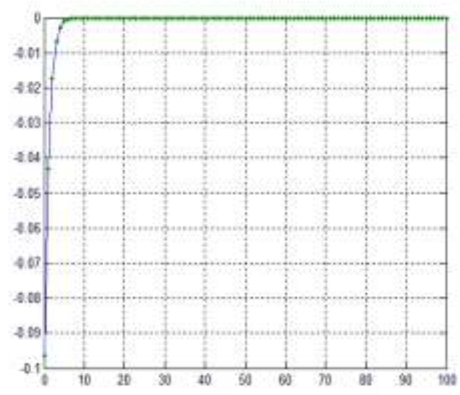

Fig2. Orbit of F(z) at s=0.5 and s'=0.4 for (z_0 = -0.09631341431+0.0695165015 i)

7.3 Fixed points of Biquadratic polynomial

Table 1: Orbit of F(z) at s=0.1 and s'=0.5 for (z_0= 0.3000244068+0.1862906688 i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
54.	0.4804	64.	0.4809
55.	0.4805	65.	0.4809
56.	0.4805	66.	0.481
57.	0.4806	67.	0.481
58.	0.4807	68.	0.481
59.	0.4807	69.	0.481
60.	0.4808	70.	0.481
61.	0.4808	71.	0.481
62.	0.4808	72.	0.4811
63.	0.4809	73.	0.4811

Here we skipped 53 iteration and observed that the value converges to a fixed point after 71 iterations

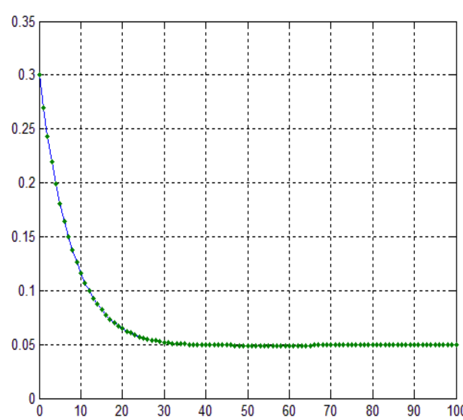

Fig1. Orbit of F(z) at s=0.1 and s'=0.5 for (z_0= 0.3000244068+0.1862906688 i)

Table 2: Orbit of F(z) at s=0.5 and s'=0.4 for (z_0= 0.08055725605+0.038502707i)

Number of iteration i	|F(z)|	Number of iteration i	|F(z)|
1.	0.0893	10.	0.4812
2.	0.2714	11.	0.4812

3.	0.3827	12.	0.4812
4.	0.438	13.	0.4811
5.	0.4637	14.	0.4811
6.	0.4747	15.	0.4811
7.	0.4791	16.	0.4811
8.	0.4806	17.	0.4811
9.	0.4811	18.	0.4811
10.	0.4812	19.	0.4811

Here observe that the value converges to a fixed point after 12 iterations

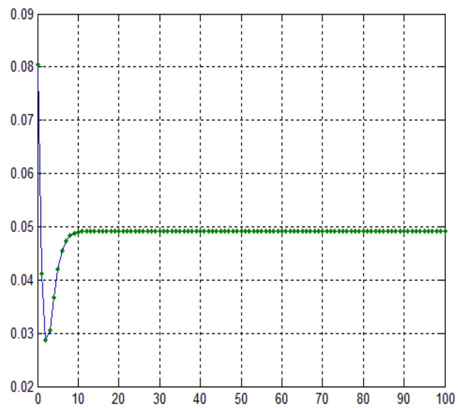

Fig2. Orbit of F(z) at s=0.5 and s'=0.4 for (z_0= 0.08055725605+0.038502707i)

8. Conclusion:

In this paper we studied the tangent function which is one of the members of transcendental family. Relative Superior Julia sets possess 2*n* wings. Besides this, these Julia sets explores the presence of central black region, which are the Mandelbrot images of respective sets. For even powers, Relative Superior Mandelbrot sets show symmetry only along the real axis while on the other hand, for odd terms, body maintains its symmetry along both axes. The fractal images thus developed undertakes the properties of both sine and cosine functions. The results thus obtained are innovative. Our study is unique in sense that we have used escape time criteria for transcendental function to generate fractals using Relative Superior Ishikawa iterates, otherwise results according to past literature would have shown Julia sets to be disconnected.

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