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Common Fixed Point Theorem for Compatible Mapping of Type (A)

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Abstract

The purpose of this paper is to prove a common fixed point theorem involving two pairs of compatible mappings of type (A) using six maps using a contractive condition. This article represents a useful generalization of several results announced in the literature.

Key Words: Complete metric space, Compatible mapping of type (A), Commuting mapping, Cauchy Sequence, Fixed points.

1. Introduction

The study of common fixed point of mappings satisfying contractive type conditions has been studied by many mathematicians.Seesa(1982) introduce the concept of weakly commuting mapping and proved some theorem of commutativity by useing the condition to weakly commutativity, Jungck(1988) gave more generalized commuting and weakly commuting maps called compatible maps and use it for compatibility of two mappings. After that Jungck Muthy and Cho(1993) made another generalization of weak commuting mapping by defining the concept of compatible map of type (A).

We proposed to re-analysis the theorems of Aage C.T (2009) on common fixed point theorem compatibility of type (A)

2.Preliminaries

Definition 2.1. Self maps S and T of metric space (X,d) are said to be weakly commuting pair

$$\text{iff} \quad d(STx,TSx) \leq d(Sx,Tx) \text{ for all } x \text{ in } X.$$

Definition 2.2. Self maps S and T of a metric space (X,d) are said to be compatible of type (A) if lim $d(TSx_n,SSx_n)=0$ and lim $d(STx_n, TTx_n)=0$ as $n\to\infty$ whenever $\{x_n\}$ is a sequence in X such that lim Sx_n=lim $Tx_n=t$ as $n\to\infty$ for some t in X.

Definition 2.3. A function Φ: $[0,\infty)\to[0,\infty)$ is said to be a contractive modulus if $\Phi(0)=0$ and

$$\Phi(t)<t \quad \text{for } t>0.$$

3.Main Result

Theorem 3.1. Let S, R, T, U, I and J are self mapping of a complete metric space (X,d) into itself satisfying the conditions

(i) $SR(X)\subset J(X)$, $TU(X)\subset I(X)$

(ii) $d(SRx,TUy)\leq \alpha\, d(Ix,Jy)+\beta\,[d(Ix,SRx)+d(Jy,TUy)]+\gamma\,[d(Ix,TUy)+d(Jy,SRx)]$
for all x,y Є X and α,β and γ are non-negative reals such that $\alpha+2\beta+2\gamma<1$

(iii) One of S,R,T,U,I and J is continuous.

(iv) (SR,I) and (TU.J) are compatible of type (A).Then SR,TU,I,J have a unique common fixed point. Further if the pairs (S,R) , (S,I) , (R,I) , T,U) , (T,I) , (U.J) are commuting *pairs then S,R,T,U,I and J have a unique common fixed point.*

Proof: Let x_0 Є X be arbitrary. Choose a point x_1 in X such that $SRx_0 = Jx_1$.
This can be done since $SR(X))\subset J(X)$.
Let x_2 be a point in X such that $TUx_1 = Ix_2$. This can be done since $TU(X)\subset I(X)$.
In general we can choose x_{2n} , x_{2n+1}, x_{2n+2} …, such that $SRx_{2n}=Jx_{2n+1}$ and $TUx_{2n+1}=Ix_{2n+2}$.
So that we obtain a sequence SRx_0, TUx_1, SRx_2, TUx_3 …..
Using condition (ii) we have

$$\begin{aligned} d(SRx_{2n},TUx_{2n+1}) &\leq \alpha\, d(I_{2n},Jx_{2n+1})+\beta\,[d(Ix_{2n}, SRx_{2n})+d(Jx_{2n+1}, TUx_{2n+1})]+\gamma\,[d(Ix_{2n}, TUx_{2n+1})+d(Jx_{2n+1},SRx_{2n})] \\ &= \alpha\, d(TUx_{2n-1}, SRx_{2n})+\beta\,[d(TUx_{2n-1},SRx_{2n})+d(SRx_{2n}, TUx_{2n+1})]+\gamma[d(TUx_{2n-1},TUx_{2n+1})+d(SRx_{2n}, SRx_{2n})] \\ &\leq \alpha\, d(TUx_{2n-1}, SRx_{2n})+\beta\,[d(TUx_{2n-1}, SRx_{2n})+d(SRx_{2n}, TUx_{2n+1})]+\gamma\,[d(TUx_{2n-1},SRx_{2n})+d(SRx_{2n},TUx_{2n+1})] \\ &= (\alpha+\beta+\gamma)\, d(TUx_{2n-1}, SRx_{2n})+(\beta+\gamma)\,(SRx_{2n}, TUx_{2n+1}) \end{aligned}$$

Hence $d(SRx_{2n}, TUx_{2n+1})\leq kd(SRx_{2n}, TUx_{2n-1})$ where $k=(\alpha+\beta+\gamma)/1-(\beta+\gamma)<1$,
Similarly we can show $d(SRx_{2n}, TUx_{2n-1})\leq k\, d(SRx_{2n-2}, TUx_{2n-1})$
Therefore $d(SRx_{2n}, TUx_{2n+1})\leq k^2 d(SRx_{2n-2}, TUx_{2n-1})$

$$\leq k^{2n} d(SRx_0, TUx_1)$$

Which implies that the sequence is a Cauchy sequence and since (X,d) is complete so the sequence has a limit point z in X. Hence the subsequences $\{SRx_{2n}\}=\{Jx_{2n-1}\}$ and $\{TUx_{2n-1}\}=\{Ix_{2n}\}$ also converges to the point z in X.
Suppose that the mapping I is continuous. Then $I^2x_{2n}\to Iz$ and $ISRx_{2n}\to Iz$ as $n\to\infty$.
Since the pair (SR,I) is compatible of type (A). we get $SRIx_{2n}\to Iz$ as $n\to\infty$.

Now by (ii)

$d(SRIx_{2n}, TUx_{2n+1}) \le \alpha\, d(I^2x_{2n}, Jx_{2n+1}) + \beta\,[d(I^2x_{2n}, SRIx_{2n}) + d(Jx_{2n+1}, TUx_{2n+1})] + \gamma\,[d(I^2x_{2n}, TUx_{2n+1}) + d(Jx_{2n+1}, SRIx_{2n})]$

letting $n\to\infty$, we get

$d(Iz,z) \le \alpha\, d(Iz,z) + \beta\,[d(Iz,z) + d(z,z)] + \gamma\,[d(Iz,z) + d(z,Iz)]$
$=(\alpha+2\gamma)\, d(Iz,z)$

This gives $d(Iz,z)=0$ since $0\le \alpha+2\gamma<1$, Hence $Iz=z$.

Further $d(SRz,TUx_{2n+1}) \le \alpha d(Iz,Jx_{2n+1}) + \beta[d(Iz,SRz) + d(Jx_{2n+1},TUx_{2n+1})] + \gamma\,[d(Iz,TUx_{2n+1}) + d(Jx_{2n+1}, SRz)]$

Letting $Jx_{2n+1}, TUx_{2n+1} \to z$ as $n\to\infty$ and $Iz=z$ we get

$d(SRz,z) \le \alpha\, d(z,z) + \beta\,[d(z,SRz) + d(z,z)] + \gamma\,[d(z,z) + d(z,SRz)]$
$= (\beta+\gamma)\, d(SRz,z)$

Hence $d(SRz,z) =0$ i.e $SRz=z$, since $0\le \beta+\gamma <1$. Thus $SRz=Iz=z$

Since $SR(X) \subset J(X)$,there is a point z_1 in X such that $z=SRz=-Jz_1$

Now by (ii)

$d(z,TUz_1) = d(SRz,TU z_1)$
$\le \alpha\, d(Iz,J z_1) + \beta\,[d(Iz,SRz) + d(J z_1,TUz_1)] + \gamma\,[d(Iz,TU z_1) + d(Jz_1, SRz)]$
$= \alpha\, d(z,z) + \beta\,[d(z,z) + d(z,TUz_1)] + \gamma\,[d(z,TU z_1) + d(z,z)]$
$=(\beta+\gamma)\, d(z,TU z_1)$

Hence $d(z,TUz_1) =0$ i.e $TU z_1 =z =Jz_1$, since $0\le \beta+\gamma <1$, Take $y_n = z_1$ for $n\ge 1$

Then $TUy_n\to Tz_1 =z$ and $Jy_n\to J z_1=z$ as $n\to\infty$

Since the pair (TU,J) is compatible of type (A) , we get

Lim $d(TUJy_n , JJy_n) =0$ as $n\to\infty$ implies $d(TUz,Jz)=0$ since $Jy_n =z$ for all $n\ge 1$. Hence $TUz=Jz$.

Now $d(z,TUz) = d(SRz,TUz)$
$\le \alpha\, d(Iz,Jz) + \beta\,[d(Iz,SRz) + d(Jz,TUz)] + \gamma\,[d(Iz, TUz) + d(Jz, SRz)$
$= \alpha\, d(z,TUz) + \beta\,[d(z,z) + d(TUz,TUz)] + \gamma\,[d(z,TUz) + d(TUz,z)]$
$=(\alpha+2\gamma)\, d(z, TUz)$

Since $\alpha+2\gamma<1$, we get $TUz=z$, hence $z=TUz=Jz$ therefore z is common fixed point of SR,TU,I,J when the continuity of I is assumed .

Now suppose that SR is continuous then $S^2R\, x_{2n} \to SRz$, $SRIx_{2n} \to SRz$ as $n\to\infty$.

By condition (ii) ,we have

$d(S^2Rx_{2n}, TUx_{2n+1}) \le \alpha\, d(ISRx_{2n}\, Jx_{2n+1}) + \beta\,[d(ISRx_{2n} , S^2Rx_{2n}) + d(Jx_{2n+1} , TUx_{2n+1}) + \gamma[d(ISRx_{2n},TUx_{2n+1}) + d(Jx_{2n+1}, S^2Rx_{2n})]$

letting $n\to\infty$ and using the compatibility of type (A) of the pair (SR,I), we get

$d(SRz,z) \le \alpha d(SRz,z) + \beta\,[d(SRz,SRz) + d(z,z)] + \gamma\,[d(SRz,z) + d(z,SRz)]$
$=(\alpha+2\gamma)\, d(SRz,z)$

Since $\alpha+2\gamma<1$ we get $SRz=z$. But $SR(X) \subset J(X)$ there is a point p in X such that $z=SRz=Jp$,Now by (ii)

$d(S^2Rx_{2n}, TUp) \le \alpha\, d(ISRx_{2n} , Jp) + \beta\,[d(ISRx_{2n}, S^2Rx_{2n}) + d(Jp,TUp)] + \gamma[d(ISRx_{2n} ,TUp) + d(Jp,S^2Rx_{2n})$

letting $n\to\infty$ we have

$d(z,TUp) =d(SRz,TUp)$
$\le \alpha d(z,z) + \beta[d(z,z) + d(z,TUp)] + \gamma[d(z,TUp) + d(z,z)]$
$=(\beta+\gamma)\, d(z,TUp)$

Since $\beta+\gamma<1$, we get $TUp=z$.Thus $z=Jp=TUp$.

Let $y_n=p$ then $TUy_n \to TUp=z$ and $Jy_n\to TUp =z$

Since (TU,J) is compatible of type (A), we have

Lim $d(TUJy_n ,JJy_n) =0$ as $n\to\infty$

This gives $TUJp=JTUp$ or $TUz=Jz$

Further

$d(SRx_{2n}, TUz) \leq \alpha d(Ix_{2n},Jz) + \beta[d(Ix_{2n},SRz) + d(Jz,TUz)] + \gamma [d(Ix_{2n}, TUz)+d(Jz,SRx_{2n})]$

Letting $n\to\infty$, we get

$d(z, TUz) \leq \alpha d(z,TUz) + \beta [d(z,z) + d(TUz,TUz)] + \gamma[d(z,TUz) + d(TUz,z)]$

$= (\alpha+2\gamma)\, d(z,TUz)$

Since $0\leq \alpha+2\gamma <1$ we get $z=TUz$

Again we have $TU(X) \subset I(X)$ there is a point q in X such that $z=TUz=Iq$

Now $d(SRq,z) = d(SRq,TUz) \leq \alpha\, d(Iq,Jz) + \beta\, [d(Iq,SRq) + d(Jz,TUz)] + \gamma[d(Iq,TUz)+d(Jz,SRq)]$

$= \alpha d(z,z) + \beta [d(z,SRq) + d(z,z)] + \gamma [d(z,TUz) + d(z,SRq)]$

$= (\beta+\gamma)\, d(z,SRq)$

Since $0\leq \beta+\gamma<1$ we get $SRq=z$, take $y_n = q$ then $SRy_n \to SRq = z$, $Iy_n \to Iq = z$

Since (SR,I) is compatible of type (A) , we get

$\lim d(ISRy_n , IIy_n) = 0$ as $n\to\infty$

This implies that $SRIq=ISRq$ or $SRz=Iz$.

Thus we have $z=SRz=Iz=Jz=TUz$ Hence z is a common fixed point of SR,TU,I and J, when S is continuous

The proof is similar that z is common fixed point of SR,TU,I and J when I is continuous,R and U is continuous.

For uniqueness let z and w be two common fixed point os SR,TU,I and J , then by condition (ii)

$d(z,w)= d(SRz,TUw) \leq \alpha d(Iz,Jw) + \beta[d(Iz,SRz) + d(Jw,TUw)] + \gamma [d(Iz,TUw) + d(Jw,SRz)]$

$= \alpha d(z,w) + \beta [d(z,z) + d(w,w)] + \gamma [d(z,w) + d(w,z)]$

$= (\alpha+2\gamma)\, d(z,w)$

Since $\alpha+2\gamma<1$ we have $z=w$.

Again let z be the unique common fixed point of both the pairs (SR,I) , (TU,J) then

$Sz=S(SRz) = S(RSz) =SR (Sz)$

$Sz=S(Iz)=I(Sz)$

$Rz=R(SRz)=(RS)(RS) =(SR)(Rz)$

$Rz=R(Iz)=I(Rz)$

Which shows that Sz and Rz is the common fixed point of (SR,I) yielding thereby

$Sz=z=Rz=Iz=SRz$

In view of uniqueness of the common fixed point of the pair (SR,I).

Similarly using the commutativity of (T,U), (T,J), (U,J) it can be shown that

$Tz=z=Uz=Jz=TUz$.Thus z is the unique common fixed point of S,R,T,U,I and J.

Hence the proof.

4. References

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