Common Fixed Point Theorems Using Faintly Compatible Mappings In Fuzzy Metric Spaces

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Abstract: In this paper we prove common fixed point theorems using faintly compatible mappings in fuzzy metric space. Our results extend and generalized the results of A. Jain et.al. [5].

Keywords: Fuzzy Metric Spaces, non compatible mappings, faintly compatible mappings and sub sequentially continuous mappings.

1. Introduction: Weak compatibility is one of the weaker forms of the commuting mappings. Many researchers use this concept to prove the existence of unique common fixed point in fuzzy metric space. Al-Thagafi and Shahzad [2] introduced the concept of occasionally weakly compatible (owc) and weaken the concept of nontrivial weakly compatible maps.

Recently, R.K. Bist and R. P. Pnat [3] criticize the concept of owc as follows "Under contractive conditions the existence of a common fixed point and occasional weak compatibility are equivalent conditions, and consequently, proving existence of fixed points by assuming owc is equivalent to proving the existence of fixed points by assuming the existence of fixed points". Therefore use of owc is a redundancy for fixed pint theorems under contractive conditions.

This redundancy can be also seen in recent result of A. Jain et.al. [5]. To remove this we used faintly compatible mapping in our paper which is weaker than weak compatibility or semi compatibility. Faintly compatible maps introduced by Bisht and Shahzad [4] as an improvement of conditionally compatible maps, Pant and Bisht [8], introduced the concept of conditional compatible maps. This gives the existence of a common fixed point or multiple fixed point or coincidence points under contractive and non-contractive conditions.

The aim of this paper is remove redundancy of results of A. Jain et.al. [5], and prove the existence of common fixed point using faintly compatible maps in fuzzy metric space.

2. Preliminaries:

In this section, we recall some definitions and useful results which are already in the literature.

Definition 2.1[10]: A binary operation *: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is continuous t- norm if * satisfies the following conditions:

(i) * is commutative and associative; (ii) * is continuous; (iii) $a * 1 = a \ \forall a \in [0; 1]$;

(iv) $a * b \leq c*d$ whenever $a \leq c$ and $b \leq d \ \forall a, b, c, d \in [0,1]$.

Example of continuous t-norm 2.2[10]: $a * b = \min \{a, b\}$, minimum t-norm.

George and Veeramani modified the nothing of fuzzy metric space of Kramosil and Michalek as follows:

Definition 2.3: The 3-tuple (X, M, *) is called a fuzzy metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X^2\times(0,\infty)$ satisfying the following conditions: $\forall x, y, z \in X, t, s > 0$;

(GV - 1) $M(x, y, t) > 0$;

(GV - 2) $M(x, y, t) = 1$ iff $x = y$;

(GV - 3) $M(x, y, t) = M(y, x, t)$;

(GV - 4) $M(x, y, t)*M(y, z, s) \leq M(x, z, t + s)$;

(GV - 5) $M(x, y, \cdot): [0,\infty) \to [0, 1]$ is continuous.

Definition 2.4: A pair of self-maps (A, S) on a fuzzy metric space (X, M, *) is said to be

(a) Non-compatible: if (A, S) is not compatible, i.e., if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Sx_n = x$, for some $x \in X$, and $\lim_{n\to\infty}M(ASx_n, SAx_n, t) \neq 1$ or non-existent $\forall t > 0$.

(b) Conditionally compatible [8]: if whenever the set of sequences $\{x_n\}$ satisfying $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Sx_n$, is non-empty, there exists a sequence $\{z_n\}$ in X such that $\lim_{n\to\infty}Az_n = \lim_{n\to\infty}Sz_n = t$, for some $t\in X$ and $\lim_{n\to\infty}M(ASx_n, SAx_n, t) = 1$ for all $t > 0$.

(c) Faintly compatible [4]: if (A, S) is conditionally compatible and A and S commute on a non-empty subset of the set of coincidence points, whenever the set of coincidence points is nonempty.

(d) Satisfy the property (E.A.) [1]: if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Sx_n = x$, for some $x \in X$.

(e) Sub Sequentially continuous [11]: iff there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Sx_n = x$, $x\in X$ and satisfy $\lim_{n\to\infty}ASx_n= Ax$, $\lim_{n\to\infty}SAx_n = Sx$.

Note that, compatibility, non- compatibility and faint compatibility are independent concepts. Faintly compatibility is applicable for mappings that satisfy contractive and non contractive conditions.

(f) **Semi-compatible [5]:** if $\lim_{n\to\infty}ASx_n= Sx$, whenever is a sequence such that $\lim_{n\to\infty}Ax_n = \lim_{n\to\infty}Sx_n = x\in X$.

Lemma 2.5[6]: Let (X, M, *) be a fuzzy metric space and for all $x, y\in X$, $t > 0$ and if there exists a constant $k \in(0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ then $x = y$.

A. Jain et.al. [5], proved the following:

Theorem 2.1[5]: Let A, B, S and T be self mappings of a complete fuzzy metric space (X, M, *). Suppose that they satisfy the following conditions:

(2.1.1) $A(X)\subset T(X)$, $B(X)\subset S(X)$;

(2.1.2) the pair (A, S) is semi-compatible and (B, T) is occasionally weakly compatible;

(2.1.3) there exists $k\in(0, 1)$ such that $\forall x, y\in X$ and $t>0$,

$$M(Ax, By, kt) \geq \min\{M(By, Ty, t), M(Sx, Ty, t), M(Ax, Sx, t)\}.$$

Then A, B, S and T have a unique fixed point in X.

Now we prove some common fixed point theorems for pair of faintly compatible mappings.

3. Main Results:

Theorem 3.1: Let (X, M,*) be a fuzzy metric space and let A, B, S, T, P and Q be self mappings of X such that

(3.1.1) the pairs (A, SP) and (B, TQ) are non compatible, sub sequentially continuous faintly compatible;

(3.1.2) Pair (A, P), (S, P), (B, Q), (T, Q) are commuting;

(3.1.3) there exists $k\in(0,1)$ such that $\forall$ x, $y\in X$ and $t > 0$,

$$M(Ax, By, kt)\geq\min\left\{\frac{aM(SPx,Ax,t)+bM(By,TQy,t)}{a+b},\frac{cM(SPx,By,t)+dM(SPx,TQy,t)}{cM(By,TQy,t)+d},\frac{eM(Ax,TQy,t)+fM(SPx,Ax,t)}{eM(SPx,TQy,t)+f}\right\};$$

where $a,b,c,d,e,f\geq 0$ with a&b, c&d and e&f cannot be simultaneously 0.

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: Non compatibility of (A, SP) and (B, TQ) implies that there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}(SP)x_n=t_1$ for some $t_1\in X$, and $M(A(SP)x_n,(SP)Ax_n,t)\neq 1$ or nonexistent $\forall t > 0$; Also $\lim_{n\to\infty}Bx_n=\lim_{n\to\infty}(TQ)x_n=t_2$ for some $t_2\in X$, and $M(B(TQ)x_n,(TQ)Bx_n,t)\neq 1$ or nonexistent $\forall t>0$.

Since pairs (A, SP) and (B, TQ) are faintly compatible therefore conditionally compatibility of (A, SP) and (B, TQ) implies that there exist sequences $\{z_n\}$ and $\{z_n'\}$ in X satisfying

$\lim_{n\to\infty}Az_n = \lim_{n\to\infty}(SP)z_n = u$ for some $u\in X$, such that $M(A(SP)z_n, (SP)Az_n, t)=1$;

Also $\lim_{n\to\infty}Bz_n' = \lim_{n\to\infty}(TQ)z_n' = v$ for some $v\in X$, such that $M(B(TQ)z_n',(TQ)Bz_n', t)=1$.

As the pairs (A, SP) and (B, TQ) are sub sequentially continuous, we get

$\lim_{n\to\infty}A(SP)z_n = Au$, $\lim_{n\to\infty}(SP)Az_n = (SP)u$

and so $Au = (SP)u$ i.e. (u is coincidence point of A and (SP));

Also $\lim_{n\to\infty}B(TQ)z_n' = Bv$, $\lim_{n\to\infty}(TQ)Bz_n' = (TQ)v$

and so $Bv = (TQ)v$ i.e. (v is coincidence point of B and (TQ)).

Since pairs (A, SP) and (B, TQ) are faintly compatible, we get

$A(SP)u=(SP)Au$ & so $AAu=A(SP)u=(SP)Au=(SP)(SP)u$;

and Also $B(TQ)v=(TQ)Bv$ & so $BBv=B(TQ)v=(TQ)Bv=(TQ)(TQ)v$.

Now we show that $Au=Bv$, $AAu= Au$, $BBv=Bv$, $PAu=Au$ and $QAu=Au$.

By taking x=u and y=v in (3.1.3),

$$M(Au, Bv, kt)\geq\min\left\{\frac{aM(SPu,Au,t)+bM(Bv,TQv,t)}{a+b},\frac{cM(SPu,Bv,t)+dM(SPu,TQv,t)}{cM(Bv,TQv,t)+d},\frac{eM(Au,TQv,t)+fM(SPu,Au,t)}{eM(SPu,TQv,t)+f}\right\};$$

$$M(Au, Bv, kt)\geq\min\left\{\frac{aM(Au,Au,t)+bM(Bv,Bv,t)}{a+b},\frac{cM(Au,Bv,t)+dM(Au,Bv,t)}{cM(Bv,Bv,t)+d},\frac{eM(Au,Bv,t)+fM(Au,Au,t)}{eM(Au,Bv,t)+f}\right\};$$

$$M(Au, Bv, kt)\geq\min\{1, M(Au, Bv, t), 1\};$$

$M(Au, Bv, kt) \geq M(Au, Bv, t)$, lemma (2.5) $\Rightarrow$ Au=Bv.

By taking x=Au and y=v in (3.1.3),

$$M(AAu, Bv, kt)\geq\min\left\{\begin{matrix}\frac{aM(SPAu,AAu,t)+bM(Bv,TQv,t)}{a+b},\frac{cM(SPAu,Bv,t)+dM(SPAu,TQv,t)}{cM(Bv,TQv,t)+d},\\ \frac{eM(AAu,TQv,t)+fM(SPAu,AAu,t)}{eM(SPAu,TQv,t)+f}\end{matrix}\right\};$$

$$M(AAu, Bv, kt)\geq\min\left\{\begin{matrix}\frac{aM(AAu,AAu,t)+bM(Bv,Bv,t)}{a+b},\frac{cM(AAu,Bv,t)+dM(AAu,Bv,t)}{cM(Bv,Bv,t)+d},\\ \frac{eM(AAu,Bv,t)+fM(AAu,AAu,t)}{eM(AAu,Bv,t)+f}\end{matrix}\right\};$$

$$M(AAu, Bv, kt)\geq\min\{1, M(AAu, Bv, t), 1\};$$

$M(AAu, Bv, kt) \geq M(AAu, Bv, t)$, lemma (2.5) $\Rightarrow$ AAu=Bv=Au.

By taking x=u and y=Bv in (3.1.3),

$$M(Au, BBv, kt) \geq \min\left\{ \frac{aM(SPu,Au,t)+bM(BBv,TQBv,t)}{a+b}, \frac{cM(SPu,BBv,t)+dM(SPu,TQBv,t)}{cM(BBv,TQBv,t)+d}, \frac{eM(Au,TQBv,t)+fM(SPu,Au,t)}{eM(SPu,TQBv,t)+f} \right\};$$

$$M(Au, BBv, kt) \geq \min\left\{ \frac{aM(Au,Au,t)+bM(BBv,BBv,t)}{a+b}, \frac{cM(Au,BBv,t)+dM(Au,BBv,t)}{cM(BBv,BBv,t)+d}, \frac{eM(Au,BBv,t)+fM(Au,Au,t)}{eM(Au,BBv,t)+f} \right\};$$

$M(Au, BBv, kt) \geq \min\{1, M(Au, BBv, t), 1\}$;

$M(Au, BBv, kt) \geq M(Au, BBv, t)$, lemma (2.5) $\Rightarrow$ Au=BBv $\Rightarrow$ BBv=Au=Bv.

Now we have AAu=(SP)Au=Au, Au= BBv=BAu and Au= BBv=(TQ)Bv=(TQ)Au since Bv=Au.
Hence AAu=(SP)Au=BAu=(TQ)Au=Au
i.e. Au is a common coincidence point of A, B, SP and TQ.
By taking x=PAu and y=Au in (3.1.3),

$$M(Ax, By, kt) \geq \min\left\{ \frac{aM(SPx,Ax,t)+bM(By,TQy,t)}{a+b}, \frac{cM(SPx,By,t)+dM(SPx,TQy,t)}{cM(By,TQy,t)+d}, \frac{eM(Ax,TQy,t)+fM(SPx,Ax,t)}{eM(SPx,TQy,t)+f} \right\};$$

$$M(APAu, BAu, kt) \geq \min\left\{ \frac{aM(SPPAu,APAu,t)+bM(BAu,TQAu,t)}{a+b}, \frac{cM(SPPAu,BAu,t)+dM(SPPAu,TQAu,t)}{cM(BAu,TQAu,t)+d}, \frac{eM(APAu,TQAu,t)+fM(SPPAu,APAu,t)}{eM(SPPAu,TQAu,t)+f} \right\};$$

Since (A, P) and (S, P) are commuting, therefore

$$M(PAu, BAu, kt) \geq \min\left\{ \frac{aM(PAu,PAu,t)+bM(BAu,Au,t)}{a+b}, \frac{cM(PAu,BAu,t)+dM(PAu,Au,t)}{cM(BAu,Au,t)+d}, \frac{eM(PAu,Au,t)+fM(PAu,PAu,t)}{eM(PAu,Au,t)+f} \right\};$$

$$M(PAu, Au, kt) \geq \min\left\{ \frac{aM(PAu,PAu,t)+bM(Au,Au,t)}{a+b}, \frac{cM(PAu,Au,t)+dM(PAu,Au,t)}{cM(Au,Au,t)+d}, \frac{eM(PAu,Au,t)+fM(PAu,PAu,t)}{eM(PAu,Au,t)+f} \right\};$$

$M(PAu, Au, kt) \geq \min\{1, M(PAu, Au, t), 1\}$;

$M(PAu, Au, kt) \geq M(PAu, Au, t)$, lemma (2.5) $\Rightarrow$ PAu= Au.

By taking x=Au and y=QAu in (3.2.2),

$$M(AAu, BQAu, kt) \geq \min\left\{ \frac{aM(SPAu,AAu,t)+bM(BQAu,TQQAu,t)}{a+b}, \frac{cM(SPAu,BQAu,t)+dM(SPAu,TQQAu,t)}{cM(BQAu,TQQAu,t)+d}, \frac{eM(AAu,TQQAu,t)+fM(SPAu,AAu,t)}{eM(SPAu,TQQAu,t)+f} \right\};$$

Since (B, Q) and (T, Q) are commuting, therefore

$$M(Au, QAu, kt) \geq \min\left\{ \frac{aM(Au,Au,t)+bM(QAu,QAu,t)}{a+b}, \frac{cM(Au,QAu,t)+dM(Au,QAu,t)}{cM(QAu,QAu,t)+d}, \frac{eM(Au,QAu,t)+fM(Au,Au,t)}{eM(Au,QAu,t)+f} \right\};$$

$M(Au, QAu, kt) \geq \min\{1, M(Au, QAu, t), 1\}$;

$M(Au, QAu, kt) \geq M(Au, QAu, t)$, lemma (2.5) $\Rightarrow$ Au=QAu.

Therefore AAu=(SP)Au=BAu=(TQ)Au=Au $\Rightarrow$ AAu=SPAu=SAu and BAu=TQAu=TAu.

Hence AAu=BAu=SAu=TAu=PAu=QAu=Au,

i.e. Au is a common fixed point of A, B, S, T, P and Q in X.

The uniqueness follows from (3.3.2).This completes the proof of the theorem.

If we take P=Q=I (the identity map on X) in theorem 3.1 then condition (3.1.2) trivially satisfied and we get the following corollary:

Corollary 3.2: Let (X, M,*) be a fuzzy metric space and let A, B, S, T, P and Q be self mappings of X such that

(3.2.1) the pairs (A, S) and (B, T) are non compatible, sub sequentially continuous faintly compatible;

(3.2.2) there exists $k\in(0,1)$ such that $\forall$ x, $y\in X$ and $t > 0$,

$$M(Ax, By, kt) \geq \min\left\{\frac{aM(Sx,Ax,t)+bM(By,Ty,t)}{a+b}, \frac{cM(Sx,By,t)+dM(Sx,Ty,t)}{cM(By,Ty,t)+d}, \frac{eM(Ax,Ty,t)+fM(Sx,Ax,t)}{eM(Sx,Ty,t)+f}\right\};$$

where a,b,c,d,e,f≥0 with a&b, c&d and e&f cannot be simultaneously 0;

Then A, B, S and T have a unique common fixed point in X.

Proof: The proof is similar to the proof of theorem 3.1 without required condition (3.1.2).

Remark 3.2.1: If we take a=c=e=0 and P= Q=I in theorem 3.1 then we get the result of A. Jain et.al. [5], for faintly compatibility and sequentially continuous map.

Theorem 3.3: Let (X, M,*) be a fuzzy metric space and let A, B, S, T, P and Q be self mappings of X such that

(3.3.1) the pairs (A, SP) and (B, TQ) are non compatible, sub sequentially continuous faintly compatible;

(3.3.2) Pair (A, P), (S, P), (B, Q), (T, Q) are commuting;

(3.3.3) there exists $k\in(0,1)$ such that $\forall$ x, $y\in X$ and $t > 0$,

$$M(Ax, By, kt) \geq \phi\left(\min\left\{\frac{aM(SPx,Ax,t)+bM(By,TQy,t)}{a+b}, \frac{cM(SPx,By,t)+dM(SPx,TQy,t)}{cM(By,TQy,t)+d}, \frac{eM(Ax,TQy,t)+fM(SPx,Ax,t)}{eM(SPx,TQy,t)+f}\right\}\right)$$

where a,b,c,d,e,f≥0 with a&b, c&d and e&f cannot be simultaneously 0 and $\phi:[0,1]\rightarrow[0,1]$ such that $\phi(t) > t$ $\forall$ $0 < t < 1$;

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: The prove follows from theorem 3.1.

Now we are giving more improved form of theorem 3.1 as follows:

Theorem 3.4: Let (X, M,*) be a fuzzy metric space and let A, B, S, T, P and Q be self mappings of X such that

(3.4.1) the pairs (A, SP) and (B, TQ) are non compatible, sub sequentially continuous faintly compatible;

(3.4.2) Pair (A, P), (S, P), (B, Q), (T, Q) are commuting;

(3.4.3) there exists $k\in(0,1)$ such that $\forall$ x, $y\in X$ and $t > 0$,

$$M(Ax, By, kt) \geq \phi\left\{\frac{aM(SPx,Ax,t)+bM(By,TQy,t)}{a+b}, \frac{cM(SPx,By,t)+dM(SPx,TQy,t)}{cM(By,TQy,t)+d}, \frac{eM(Ax,TQy,t)+fM(SPx,Ax,t)}{eM(SPx,TQy,t)+f}\right\};$$

where a, b, c, d, e, f ≥ 0 with a & b, c & d and e & f cannot be simultaneously 0 and

$\phi : [0, 1]^3 \rightarrow [0, 1]$ such that $\phi(1, t, 1) > t$ $\forall$ $0 < t < 1$;

Then A, B, S, T, P and Q have a unique common fixed point in X.

Proof: Non compatibility of (A, SP) and (B, TQ) implies that there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty}Ax_n=\lim_{n\to\infty}(SP)x_n=t_1$ for some $t_1\in X$, and $M(A(SP)x_n,(SP)Ax_n,t)\neq1$ or nonexistent $\forall t > 0$; Also

$\lim_{n\to\infty}Bx_n=\lim_{n\to\infty}(TQ)x_n=t_2$ for some $t_2\in X$, and $M(B(TQ)x_n,(TQ)Bx_n,t)\neq 1$ or nonexistent $\forall t>0$.

Since pairs (A, SP) and (B, TQ) are faintly compatible therefore conditionally compatibility of (A, SP) and (B, TQ) implies that there exist sequences $\{z_n\}$ and $\{z_n'\}$ in X satisfying
$\lim_{n\to\infty}Az_n = \lim_{n\to\infty}(SP)z_n = u$ for some $u\in X$, such that $M(A(SP)z_n, (SP)Az_n, t)=1$;
Also $\lim_{n\to\infty}Bz_n' = \lim_{n\to\infty}(TQ)z_n' = v$ for some $v\in X$, such that $M(B(TQ)z_n',(TQ)Bz_n', t)=1$.

As the pairs (A, SP) and (B, TQ) are sub sequentially continuous, we get
$\lim_{n\to\infty}A(SP)z_n = Au$, $\lim_{n\to\infty}(SP)Az_n = (SP)u$
and so $Au = (SP)u$ i.e. (u is coincidence point of A and (SP));
Also $\lim_{n\to\infty}B(TQ)z_n' = Bv$, $\lim_{n\to\infty}(TQ)Bz_n' = (TQ)v$
and so $Bv = (TQ)v$ i.e. (v is coincidence point of B and (TQ)).

Since pairs (A, SP) and (B, TQ) are faintly compatible, we get
$A(SP)u=(SP)Au$ & so $AAu=A(SP)u=(SP)Au=(SP)(SP)u$;
and Also $B(TQ)v=(TQ)Bv$ & so $BBv=B(TQ)v=(TQ)Bv=(TQ)(TQ)v$.

Now we show that Au=Bv, AAu= Au., PAu=Au and QAu=Au.
By taking x=u and y=v in (3.4.3),

$$M(Au,Bv,kt)\geq\phi\left\{\frac{aM(SPu,Au,t)+bM(Bv,TQv,t)}{a+b},\frac{cM(SPu,Bv,t)+dM(SPu,TQv,t)}{cM(Bv,TQv,t)+d},\frac{eM(Au,TQv,t)+fM(SPu,Au,t)}{eM(SPu,TQv,t)+f}\right\};$$

$$M(Au,Bv,kt)\geq\phi\left\{\frac{aM(Au,Au,t)+bM(Bv,Bv,t)}{a+b},\frac{cM(Au,Bv,t)+dM(Au,Bv,t)}{cM(Bv,Bv,t)+d},\frac{eM(Au,Bv,t)+fM(Au,Au,t)}{eM(Au,Bv,t)+f}\right\};$$

$M(Au,Bv,kt)\geq\phi\{1,M(Au,Bv,t),1\}$;
$M(Au,Bv,kt)\geq M(Au,Bv,t)$, lemma (2.5) $\Rightarrow$ Au=Bv.

By taking x=Au and y=v in (3.4.3),

$$M(AAu,Bv,kt)\geq\phi\left\{\begin{array}{c}\frac{aM(SPAu,AAu,t)+bM(Bv,TQv,t)}{a+b},\frac{cM(SPAu,Bv,t)+dM(SPAu,TQv,t)}{cM(Bv,TQv,t)+d},\\ \frac{eM(AAu,TQv,t)+fM(SPAu,AAu,t)}{eM(SPAu,TQv,t)+f}\end{array}\right\};$$

$$M(AAu,Bv,kt)\geq\phi\left\{\begin{array}{c}\frac{aM(AAu,AAu,t)+bM(Bv,Bv,t)}{a+b},\frac{cM(AAu,Bv,t)+dM(AAu,Bv,t)}{cM(Bv,Bv,t)+d},\\ \frac{eM(AAu,Bv,t)+fM(AAu,AAu,t)}{eM(AAu,Bv,t)+f}\end{array}\right\};$$

$M(AAu,Bv,kt)\geq\phi\{1,M(AAu,Bv,t),1\}$;
$M(AAu,Bv,kt)\geq M(AAu,Bv,t)$, lemma (2.5) $\Rightarrow$ AAu=Bv=Au.
Similarly we can show BBv=Bv By taking x=u and y=Bv in (3.4.3).
Now we have AAu=(SP)Au=Au, Au= BBv=BAu and Au= BBv=(TQ)Bv=(TQ)Au since Bv=Au.
Hence AAu=(SP)Au=BAu=(TQ)Au=Au
i.e. Au is a common coincidence point of A, B, SP and TQ.
By taking x=PAu and y=Au in (3.4.3),

$$M(Ax,By,kt)\geq\phi\left\{\frac{aM(SPx,Ax,t)+bM(By,TQy,t)}{a+b},\frac{cM(SPx,By,t)+dM(SPx,TQy,t)}{cM(By,TQy,t)+d},\frac{eM(Ax,TQy,t)+fM(SPx,Ax,t)}{eM(SPx,TQy,t)+f}\right\};$$

$$M(APAu, BAu, kt) \geq \phi \left\{ \begin{array}{c} \frac{aM(SPPAu,APAu,t)+bM(BAu,TQAu,t)}{a+b}, \frac{cM(SPPAu,BAu,t)+dM(SPPAu,TQAu,t)}{cM(BAu,TQAu,t)+d}, \\ \frac{eM(APAu,TQAu,t)+fM(SPPAu,APAu,t)}{eM(SPPAu,TQAu,t)+f} \end{array} \right\};$$

Since (A, P) and (S, P) are commuting, therefore

$$M(PAu, BAu, kt) \geq \phi \left\{ \begin{array}{c} \frac{aM(PAu,PAu,t)+bM(BAu,Au,t)}{a+b}, \frac{cM(PAu,BAu,t)+dM(PAu,Au,t)}{cM(BAu,Au,t)+d}, \\ \frac{eM(PAu,Au,t)+fM(PAu,PAu,t)}{eM(PAu,Au,t)+f} \end{array} \right\};$$

$$M(PAu, Au, kt) \geq \phi \left\{ \begin{array}{c} \frac{aM(PAu,PAu,t)+bM(Au,Au,t)}{a+b}, \frac{cM(PAu,Au,t)+dM(PAu,Au,t)}{cM(Au,Au,t)+d}, \\ \frac{eM(PAu,Au,t)+fM(PAu,PAu,t)}{eM(PAu,Au,t)+f} \end{array} \right\};$$

$M(PAu, Au, kt) \geq \phi\{1, M(PAu, Au, t), 1\}$;

$M(PAu, Au, kt) \geq M(PAu, Au, t)$, lemma (2.5) $\Rightarrow$ PAu= Au.

Similarly we can show Au=QAu, by taking x=Au and y=QAu in (3.4.3).

Therefore AAu=(SP)Au=BAu=(TQ)Au=Au $\Rightarrow$ AAu=SPAu=SAu and BAu=TQAu=TAu.

Hence AAu=BAu=SAu=TAu=PAu=QAu=Au,

i.e. Au is a common fixed point of A, B, S, T, P and Q in X.

The uniqueness follows from (3.4.3).This completes the proof of the theorem.

Conclusion: Our theorem 3.1 is an improvement and generalization of theorem 3.1 of A. Jain et.al. [5], in the following way:

(i) Requirement of the semi-compatibility replaced by weaker form faintly compatibility.

(ii) Completeness of the space has been removed completely.

(iii) Our results never require the containment of the ranges.

(iv) In the light of [3], owc mappings have been replaced by faintly compatible mappings.

Open Problem: In this paper, we used weaker form of reciprocal continuity, namely sub-sequentially continuity. Are the results true without any continuity condition?

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