

Two-point resistance on hypercubic lattices with second nearest neighbor resistors

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Abstract

The electric resistance between two arbitrary lattice points in infinite d- dimensional hypercubic lattices with second nearest neighbor (2nd NN) resistors is calculated by using lattice Green's function. The resistance for square, and simple cubic lattices with 2nd NN resistors is discussed in detail. For large separation between lattice points the finite limiting value of the resistance for a simple cubic lattice with second nearest neighbor resistors is calculated.

Keywords: Green’s function, hypercubic lattices, resistance.

1. Introduction

A problem of fundamental interest in electric circuit theory is the determination of the effective resistance between two nodes in an infinite d - dimensional hypercubic lattice of identical resistances R. Atkinson and van Steenwijk [1] have studied the problem based on the principle of the superposition of current distributions[2]. Also, the resistance of the hypercubic lattice in d dimensions as well as triangular and honeycomb lattices has also been studied by Cserti [3] by using the lattice Green’s function approach. In a special case, it is well-known that the resistance between nearest neighbor lattice points in an infinite d - dimensional hypercubic lattice with resistors connecting first nearest neighbor (1st NN) points is R / d [3].

In Ref.[4], Cserti and co-authors have generalized lattice Green’s function method established in [3] for a resistor lattice that is a uniform tiling of d-dimensional space with electrical resistors. They presented several examples in one, two and three dimensions as applications of the general method. In [5], a further application of the lattice Green’s function method has been presented to calculate the two-point resistance on a perturbed lattice that is obtained by removing one bond from the ideal lattice.

In the physics literature, numerous applications [6-22] of the lattice Green’s function method [3-5] have been presented to determine the resistance and capacitance in several perfect and perturbed lattice structures.

We return to infinite hypercubic lattices, to best our knowledge, the calculations done so far have been performed for hypercubic lattices with resistors connecting only first nearest neighbor lattice sites. The consideration can be extended to lattices with resistors connecting farther lattice points. In this work, we use the lattice Green's function approach [3] to calculate the resistance between any two lattice sites in infinite d - dimensional hypercubic lattices with resistors connecting 2nd NN sites in addition to resistors connecting 1st NN sites.

The paper is arranged as follows. In section 2, following [3] the general formula for the resistance between arbitrary lattice sites for d - dimensional infinite hypercubic lattice up to 2nd NN resistors is obtained. The resistance of square and simple cubic lattices with 2nd NN resistors are calculated. A brief conclusion is drown in section 3.

2.Hypercubic lattices, $d \geq 2$

Let us consider an infinite, d – dimensional hypercubic lattice consisting of equal resistances R. In addition to resistors connecting the 1st NN lattice sites in the hypercube lattice, there are resistors connecting the 2nd NN lattice sites. The unit cell of the lattice is a hypercube consisting of one lattice site at a corner of the hypercube cube. All lattice sites are characterized by the position vector

$$\mathbf{r} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + \cdots + p_d\mathbf{a}_d , \tag{1}$$

where $\mathbf{a}_i (i = 1, \cdots, d)$ are orthogonal primitive translation vectors, and p_i are integers. Here, $|\mathbf{a}_i| = a$,where a is the lattice constant of d – dimensional hypercubic lattices.

Let $V(\mathbf{r})$ and $I(\mathbf{r})$ be the electric potential and current at point $\mathbf{r}$, respectively and are given by their Fourier transforms as the following:

$$V(\mathbf{r}) = \frac{V_c}{(2\pi)^d} \int_{BZ} d^d\mathbf{k}\, V(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2}$$

$$I(\mathbf{r}) = \frac{V_c}{(2\pi)^d} \int_{BZ} d^d\mathbf{k}\, I(\mathbf{k})\, e^{i\mathbf{k}\cdot\mathbf{r}} \tag{3}$$

where $V_c = a^d$ is the volume of the unit cell and $\mathbf{k} = (k_1, \cdots, k_d)$ is the wave vector in the d-dimensional Fourier space (in the reciprocal lattice) and is restriced to the first Brillouin zone (BZ) which is a d-dimensional hypercube with sides $k_i = 2\pi / a_i (i = 1, \cdots, d)$. According to Kirchhoff's and Ohm's laws, the current entering the site $\mathbf{r}$ (from outside the lattice) can be written as

$$I(\mathbf{r}) = \frac{1}{R}\sum_{i=1}^{d\geq 2}\left[V(\mathbf{r}) - V(\mathbf{r}\pm\mathbf{a}_i)\right] + \frac{1}{R}\sum_{i=1}^{d\geq 2}\sum_{j\neq i}\left[V(\mathbf{r}) - V(\mathbf{r}\pm\mathbf{a}_i\pm\mathbf{a}_j)\right] \tag{4}$$

Substituting Eqs.(2) and (3) in (4) gives

$$L(\mathbf{k})V(\mathbf{k}) = -RI(\mathbf{k}) \tag{5}$$

where $L(\mathbf{k})$ is the lattice Laplacian of d-dimensional hypercube and is given by

$$\begin{aligned} L(\mathbf{k}) &= -2\sum_{i=1}^{d\geq 2}\left(1 - \cos\mathbf{k}\cdot\mathbf{a}_i\right) - 2\sum_{i=1}^{d\geq 2}\sum_{j\neq i}\left(1 - \cos\mathbf{k}\cdot\mathbf{a}_i \cos\mathbf{k}\cdot\mathbf{a}_j\right) \\ &= -\left(2d - 2\sum_{i=1}^{d\geq 2}\cos\mathbf{k}\cdot\mathbf{a}_i\right) - \left(2d(d-1) - 2\sum_{i=1}^{d\geq 2}\sum_{j\neq i}\cos\mathbf{k}\cdot\mathbf{a}_i \cos\mathbf{k}\cdot\mathbf{a}_j\right) \\ &= -2d^2 + 2\sum_{i=1}^{d\geq 2}\cos\mathbf{k}\cdot\mathbf{a}_i + 2\sum_{i=1}^{d\geq 2}\sum_{j\neq i}\cos\mathbf{k}\cdot\mathbf{a}_i \cos\mathbf{k}\cdot\mathbf{a}_j \end{aligned} \tag{5}$$

where $2d$ and $2d(d-1)$ are numbers of 1st and 2nd NN lattice sites, respectively.

The Fourier transform of the lattice Green's function is defined by

$$G(\mathbf{k}) = -L^{-1}(\mathbf{k}) = \left(2d^2 - 2\sum_{i=1}^{d\geq 2}\cos\mathbf{k}\cdot\mathbf{a}_i - 2\sum_{i=1}^{d\geq 2}\sum_{j\neq i}\cos\mathbf{k}\cdot\mathbf{a}_i \cos\mathbf{k}\cdot\mathbf{a}_j\right) \tag{6}$$

Writing $\mathbf{k}\cdot\mathbf{a}_i = \theta_i$, Eq.(6) becomes

$$G(\theta_1, \cdots, \theta_d) = \left(2d^2 - 2\sum_{i=1}^{d\geq 2}\cos\theta_i - 2\sum_{i=1}^{d\geq 2}\sum_{j\neq i}\cos\theta_i \cos\theta_j\right)^{-1} \tag{7}$$

The lattice Green's function is defined by

$$\begin{aligned} G(p_1, \cdots, p_d) &= \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_d\, G(\theta_1, \cdots, \theta_d)\, e^{-i(p_1\theta_1 + \cdots + p_d\theta_d)} \\ &= \frac{1}{(2\pi)^d}\int_{-\pi}^{\pi} d\theta_1 \cdots \int_{-\pi}^{\pi} d\theta_d\, \frac{e^{-i(p_1\theta_1 + \cdots + p_d\theta_d)}}{2d^2 - 2\sum_{i=1}^{d\geq 2}\cos\theta_i - 2\sum_{i=1}^{d\geq 2}\sum_{j\neq i}\cos\theta_i \cos\theta_j} \end{aligned} \tag{8}$$

The resistance between the origin and lattice site $\mathbf{r} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + \cdots + p_d\mathbf{a}_d$ is given by [3]

$$R(p_1, p_2, \cdots, p_d) = 2R\left[G(0, 0, \cdots, 0) - G(p_1, p_2, \cdots, p_d)\right] \tag{9}$$

Substituting Eqs.(8) into (9), we have

$$R(p_1,\cdots,p_d)=\frac{R}{(2\pi)^d}\int_{-\pi}^{\pi}d\theta_1\cdots\int_{-\pi}^{\pi}d\theta_d\,\frac{1-\cos p_1\theta_1\cdots\cos p_d\theta_d}{d^2-\sum_{i=1}^{d\geq 2}\cos\theta_i-\sum_{i=1}^{d\geq 2}\sum_{j\neq i}\cos\theta_i\cos\theta_j} \tag{10}$$

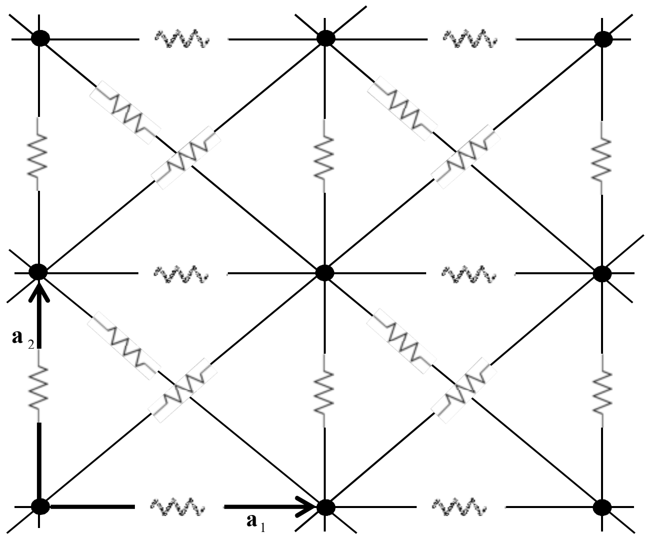

Fig.1. A square lattice with 2nd NN resistors.

2.1. Square lattice with 2nd NN resistors

Figure 1 shows a two-dimensional square lattice with 2nd NN resistors. The resistance between the origin and lattice point $\mathbf{r}=p_1\mathbf{a}_1+p_2\mathbf{a}_2$ can be calculated using Eq.(10), with d=2:

$$R(p_1,p_2)=\frac{R}{4\pi^2}\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\theta_2\,\frac{1-\cos p_1\theta_1\cos p_2\theta_2}{4-\cos\theta_1-\cos\theta_2-2\cos\theta_1\cos\theta_2} \tag{11}$$

We present some analytical examples:

–The resistance between 1st NN lattice sites, $R(1,0)$:

$$R(1,0)=\frac{R}{4\pi^2}\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\theta_2\,\frac{1-\cos\theta_1}{4-\cos\theta_1-\cos\theta_2-2\cos\theta_1\cos\theta_2} \tag{12}$$

Performing the integration over θ_2 using residue theorem and over θ_1 by elementary methods, we find

$$R(1,0)=\frac{2\cosh^{-1}\sqrt{3/2}}{\sqrt{3}\pi}R \tag{13}$$

–The resistance between the origin and lattice point (2, 0):

$$R(2,0)=R\int_{-\pi}^{\pi}\frac{d\theta_1}{2\pi}\int_{-\pi}^{\pi}\frac{d\theta_2}{2\pi}\,\frac{1-\cos 2\theta_1}{4-\cos\theta_1-\cos\theta_2-2\cos\theta_1\cos\theta_2}=\left(\frac{4}{\pi}-\frac{8\cosh^{-1}\sqrt{3/2}}{\sqrt{3}\pi}\right)R \tag{14}$$

–The resistance between 2nd NN lattice sites, $R(1,1)$ can be calculated from the symmetry of the lattice as the following:

$$R(1,0)=R(0,1)\,,\quad R(1,1)=R(-1,-1) \tag{15}$$

and from Eq.(11) we find that

$$R(1,0)+R(0,1)+R(1,1)+R(-1,-1)=R \tag{16}$$

Therefore, the resistance between 2nd NN lattice sites is:

$$R(1,1) = \left(\frac{1}{2} - \frac{2\cosh^{-1}\sqrt{3/2}}{\sqrt{3}\pi} \right) R \tag{17}$$

In Fig.2, the resistances R(p,0) of the square lattice with only 1st NN resistors, and with both 1st and 2nd NN resistors for $0 \le p \le 50$ are plotted as functions of p . One can see that the resistance is always smaller in the square lattice with 1st and 2nd NN resistors than that in the square lattice with 1st NN only [3]. The reason is that the current flowing between the two sites in the square lattice with 1st and 2nd NN resistors has more branches than that in the square lattice with 1st NN only.

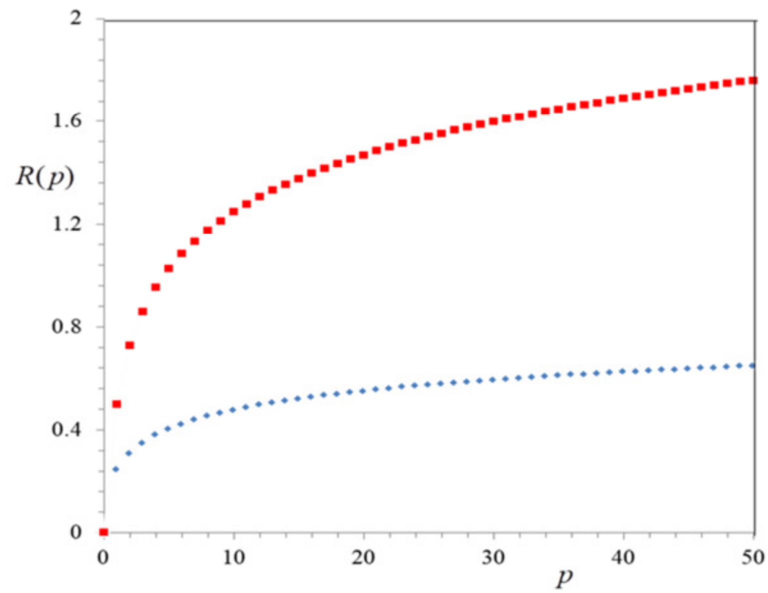

Figure. 2.The resistances $R(p,0)$ in units of R along the p-axis for the square lattice with1st NN resistors only (upper curve) and with both 1st and 2nd NN resistors (lower curve).

2.2. Simple cubic lattice with 2nd NN resistors

In a three-dimensional simple cubic lattice with 2nd NN resistors(see Fig.3), the resistance between the origin and a lattice site $\mathbf{r} = p_1\mathbf{a}_1 + p_2\mathbf{a}_2 + p_3\mathbf{a}_3$ can be determined from Eq.(10):

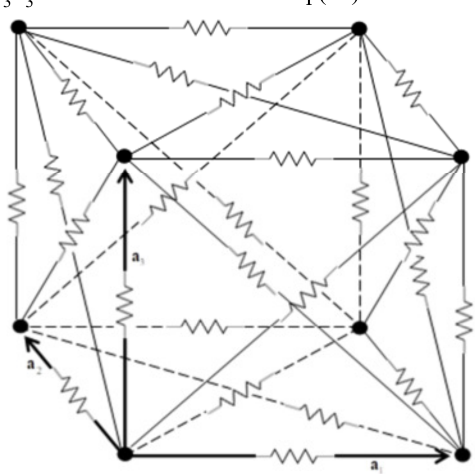

Figure.3. A simple cubic lattice with 2nd NN resistors.

$$R(p_1,p_2,p_3)=\frac{R}{8\pi^3}\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\theta_2\int_{-\pi}^{\pi}d\theta_3\times \frac{1-\cos p_1\theta_1\cos p_2\theta_2\cos p_3\theta_3}{9-\cos\theta_1-\cos\theta_2-\cos\theta_3-2\cos\theta_1\cos\theta_2-2\cos\theta_1\cos\theta_3-2\cos\theta_2\cos\theta_3} \quad (18)$$

Since the integrands in Eq. (18) are very complicated functions in their variables (θ_1, θ_2, θ_3), it is very difficult to obtain analytical results for the resistance. However, we present a few numerical examples:

– The resistance between the 1st NN lattice sites, the resistance along the side of a cube, is

$$R(1,0,0)=\frac{R}{8\pi^3}\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\theta_2\int_{-\pi}^{\pi}d\theta_3\times \frac{1-\cos\theta_1}{9-\cos\theta_1-\cos\theta_2-\cos\theta_3-2\cos\theta_1\cos\theta_2-2\cos\theta_1\cos\theta_3-2\cos\theta_2\cos\theta_3}\approx 0.109766R \quad (19)$$

– The resistance between the 2nd NN lattice sites, the resistance along the diagonal of a cube face, $R(1,1,0)$ can be calculated from the symmetry of the lattice as the following:

$$R(1,0,0)=R(0,1,0)=R(0,0,1),\ R(1,1,0)=R(-1,-1,0)=R(1,0,1)=R(-1,0,-1)=R(0,1,1)=R(0,-1,-1) \quad (20)$$

and from Eq.(18), we find that

$$R(1,1,0)=\frac{1}{6}R-\frac{1}{2}R(1,0,0)\approx 0.111784R \quad (21)$$

It is easy to show much in the same way that in a *d*- dimensional hypercube with 2nd NN resistors, the relation between the resistance $R(1,0,\cdots,0)$ and the resistance $R(1,1,\cdots,0)$ is given by

$$2d\,\underbrace{R(1,0,\cdots,0)}_{d\text{ dimensions}}+2d(d-1)\underbrace{R(1,1,\cdots,0)}_{d\text{ dimensions}}=2R \quad (22)$$

– The resistance between the third nearest neighbor lattice sites, the resistance along the diagonal of a cube body, is

$$R(1,1,1)=\frac{R}{8\pi^3}\int_{-\pi}^{\pi}d\theta_1\int_{-\pi}^{\pi}d\theta_2\int_{-\pi}^{\pi}d\theta_3\times \frac{1-\cos\theta_1\cos\theta_2\cos\theta_3}{9-\cos\theta_1-\cos\theta_2-\cos\theta_3-2\cos\theta_1\cos\theta_2-2\cos\theta_1\cos\theta_3-2\cos\theta_2\cos\theta_3}\approx 0.119454R \quad (23)$$

Finally, it is interesting to see that the resistance will tend to a finite value as the distance between lattice sites goes to infinity. This can be shown by using Riemann-Lebesque lemma which states: If $f(x)$ is a square integrable function on the interval [a, b], then

$$\int_a^b f(x)\cos px\,dx\to 0\ \ as\ p\to\infty \quad (24)$$

Hence, when the separation of the two lattice points goes to infinity, i.e., $p_1,p_2,p_3\to\infty$ from Eq. (13) $G(p_1,p_2,p_3)\to 0$, and thus, the resistance $R(p_1,p_2,p_3)$ can be written as

$$
\begin{aligned}
&R(p_1,p_2,p_3)\to 2RG(0,0,0)\\
&=\frac{R}{8\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{d\theta_1 d\theta_2 d\theta_3}{9-\cos\theta_1-\cos\theta_2-\cos\theta_3-2\cos\theta_1\cos\theta_2-2\cos\theta_1\cos\theta_3-2\cos\theta_2\cos\theta_3} \qquad (25)\\
&\approx 0.136721R
\end{aligned}
$$

In Fig. 4 we plotted the numerical values of the resistance $R(p,0,0)$ using Eq.(18) for $1\le p\le 100$. It is seen from the figure that the resistance tends rapidly to its asymptotic value given in Eq. (25).

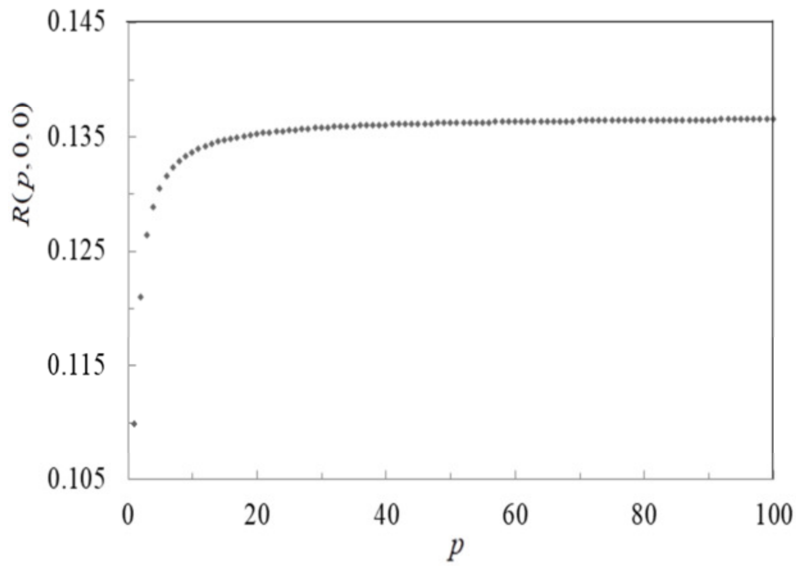

Fig. 4. The resistances $R(p,0,0)$ in units of R along the p-axis for the simple cubic lattice with 2nd NN resistors.

3. Conclusion

In this work, we have used the Green's function technique to determine the two-point resistance on an infinite d-dimensional hypercubic lattice with 2nd NN resistors. Some analytical results for the infinite square network with 2nd NN resistors are presented. The resistances for the square lattice with and without 2nd NN resistors are compared and its found the resistances with 2nd NN are always smaller than the resistances without. Numerical results and the asymptotic value of the resistance for the simple cubic lattice 2nd NN resistors are calculated.

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