Numerical solution of Fuzzy Hybrid Differential Equation by Third order Runge Kutta Nystrom Method

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Abstract
In this paper we study numerical method for hybrid fuzzy differential equations by an application of Runge–Kutta Nystrom method of order three. Here we state a convergence result and give a numerical example to illustrate the theory. This method is discussed in detail and this is followed by a complete error analysis.

Keywords: Hybrid systems; Fuzzy differential equations; Runge–Kutta Nystrom method

1. Introduction
Hybrid systems are devoted to modeling, design, and validation of interactive systems of computer programs and continuous systems. That is, control systems that are capable of controlling complex systems which have discrete event dynamics as well as continuous time dynamics can be modeled by hybrid systems. The differential systems containing fuzzy valued functions and interaction with a discrete time controller are named hybrid fuzzy differential systems.

In this article we develop numerical methods for solving hybrid fuzzy differential equations by an application of the Runge–Kutta Nystrom method [6]. In Section 2 we list some basic definitions for fuzzy valued functions. Section 3 reviews hybrid fuzzy differential systems. Section 4 contains the Runge–Kutta Nystrom method for approaching hybrid fuzzy differential equations. Section 5 contains a numerical example to illustrate the theory.

2. Preliminaries
By \( \mathbb{R} \) we denote the set of all real numbers. A fuzzy number is a mapping \( u: \mathbb{R} \rightarrow [0, 1] \) with the following properties:

(a) \( u \) is upper semicontinuous,
(b) \( u \) is fuzzy convex, i.e., \( u \left( x + (1 - \lambda) y \right) \geq \min \{ u(x), u(y) \} \) for all \( x, y \in \mathbb{R}, \lambda \in [0, 1] \),
(c) \( u \) is normal, i.e., \( \exists x_0 \in \mathbb{R} \) for which \( u(x_0) = 1 \),
(d) \( \text{Supp } u = \{ x \in \mathbb{R} : u(x) > 0 \} \) is the support of \( u \), and its closure \( \text{cl}(\text{supp } u) \) is compact.

Let \( \mathcal{F} \) be the set of all fuzzy number on \( r \).The r-level set of a fuzzy number \( u \in \mathcal{F}, 0 \leq r \leq 1 \), denoted by \( [u]_r \), is defined as
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\[ [u]_r = \begin{cases} \{x \in \mathbb{R} : u(x) \geq r\} & \text{if } 0 < r < 1, \\ \text{cl}(\text{supp } u) & \text{if } r = 0. \end{cases} \]

It is clear that the \( r \)-level set of a fuzzy number is a closed and bounded interval \([u]_r\), where \( u(x) \) denotes the left-hand end point of \([u]_r\) and \( \text{cl}(\text{supp } u) \) denotes the right-hand side end point of \([u]_r\). Since each \( y \in \mathbb{R} \) can be regarded as a fuzzy number \( y \) is defined by

\[
\gamma = \begin{cases} 1, & t = y \\ 0, & t \neq y \end{cases}
\]

Remark 2.1
Let \( X \) be the Cartesian product of universes \( X_1 \times \ldots \times X_n \), and \( A_1, \ldots, A_n \) be \( n \) fuzzy numbers in \( X_1 \times \ldots \times X_n \) respectively. \( f \) is a mapping from \( X \) to a universe \( Y \). Then the extension principle allows us to define a fuzzy set \( B \) in \( Y \) by

\[
B = \{y : y = f(x_1, \ldots, x_n), (x_1, \ldots, x_n) \in X\},
\]

where \( u_{f^{-1}}(y) = \left\{ \sup_{x_1, \ldots, x_n \in X} f^{-1}(y) \mid u_{A_k}(x_k), f^{-1}(y) \neq 0, \text{ otherwise.} \right\} \)

According to Zadeh’s extension principle, operation of addition on \( \mathbb{E} \) is defined by

\[
(u \oplus v)(x) = \frac{\max(u(x), v(x-y)))}{\max(u(x), v(x-y)))}, \quad x \in \mathbb{R}
\]

and scalar multiplication of a fuzzy number is given by

\[
(k \odot u)(x) = \begin{cases} u(x/k), & k > 0, \\ 0, & k = 0. \end{cases}
\]

Where \( \mathbb{E} \). The Hausdorff distance between fuzzy numbers given by

\[
D(u, v) = \sup_{r \in [0, 1]} \max_{x \in \mathbb{R}} \{|u(x) - v(x)|, v(x) - u(x)|\}.
\]

It is easy to see that \( D \) is a metric in \( \mathbb{E} \) and has the following properties

(i) \( D(u \odot w, v \odot w) = D(u, v), \quad \forall u, v, w \in \mathbb{E} \),
(ii) \( D(k \odot u, k \odot v) = |k| D(u, v), \quad \forall u, v \in \mathbb{E} \),
(iii) \( D(u \odot w, v \odot w) = D(u, v) + D(v, e), \quad \forall u, v, w, e \in \mathbb{E} \),
(iv) \( (D, \mathbb{E}) \) is a complete metric space.

Next consider the initial value problem (IVP)

\[
\begin{cases}
\dot{x}(t) = f(t, x(t)), \\
x(0) = x_0.
\end{cases}
\]

Where \( f \) is continuous mapping from \( \mathbb{R}_+ \times \mathbb{R} \) into \( \mathbb{R} \) and \( x_0 \in \mathbb{E} \) with \( r \)-level sets
The extension principle of Zadeh leads to the following definition of \( f(t, x) \) when \( x = x(t) \) is a fuzzy number

\[
f(t, x)(u) = \sup \{x(t) \mid u \in [x(t), x(t)]\}
\]

It follows that

\[
f(t, x) \in [f(t, x), f(t, x)]
\]

Where

\[
f(t, x) = \min \{f(t, u) \mid u \in [x(t), x(t)]\}
\]

Theorem 2.1

Let \( f \) satisfy

\[
|f(t, s) - f(t, \overline{s})| \leq g(s) , s \geq t, \forall s, \forall t \in \mathbb{R}
\]

Where \( g: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a continuous mapping such that \( r \to g(t, r) \) is non-decreasing and the initial value problem

\[
\begin{align*}
    x'(t) &= g(t, x(t)), \\
    x(0) &= x_0
\end{align*}
\]

has a solution on \( \mathbb{R} \) for \( u_0 > 0 \) and that \( u(t) = 0 \) is the only solution of (2.2) for \( u_0 = 0 \). Then the fuzzy initial value problem (2.1) has a unique solution.

3. The hybrid fuzzy differential system

Consider the hybrid fuzzy differential equation

\[
\begin{align*}
    x'_p(t) &= f(t, x(t), \lambda_0(x(t))), \\
    x_{(0)} &= x_0
\end{align*}
\]

Where \( \lambda \in \mathbb{C}[\mathbb{R}^+ \times \mathbb{E} \times \mathbb{E}, \mathbb{E}], \lambda_0 \in \mathbb{C}[\mathbb{E}, \mathbb{E}] \).

To be specific the system will be as follows

\[
x'(t) = \begin{cases}
    x'_p(t) &= f(t, x_p(t), \lambda_0(x_p(t))), & x_0(t) = x_0, & t_0 \leq t \leq t_1, \\
    x'_p(t) &= f(t, x_p(t), \lambda_0(x_p(t))), & x_0(t) = x_0, & t_1 \leq t \leq t_2, \\
    \vdots \\
    x'_p(t) &= f(t, x_p(t), \lambda_0(x_p(t))), & x_0(t) = x_0, & t_n \leq t \leq t_{n+1}.
\end{cases}
\]

With respect to the solution of (3.1), we determine the following function:

\[
x(t) = \begin{cases}
    x_0(t), & t_0 \leq t \leq t_1, \\
    x_1(t), & t_1 \leq t \leq t_2, \\
    \vdots \\
    x_n(t), & t_n \leq t \leq t_{n+1}.
\end{cases}
\]

We note that the solutions of (3.1) are piecewise differentiable in each interval for \( t \in [t_k, t_{k+1}] \) for a fixed \( x \in \mathbb{E} \) and \( k = 0, 1, 2, \ldots \).

Therefore we may replace (3.1) by an equivalent system
\[
\begin{align*}
\begin{cases}
\tilde{y}(t) = \tilde{f}(t, \tilde{x}_n, \lambda_n, \tilde{y}_n), & \tilde{y}_n = \tilde{y}_n,
\end{cases}
\end{align*}
\]

which possesses a unique solution \( \tilde{y}(t) \) which is a fuzzy function. That is for each \( t \), the pair \( (\tilde{y}(t), \tilde{x}(t)) \) is a fuzzy number, where \( \tilde{x}(t) \), \( \tilde{y}(t) \) are respectively the solutions of the parametric form given by

\[
\begin{align*}
\begin{cases}
\tilde{y}'(t) = \tilde{f}_1(t, \tilde{x}(t), \tilde{y}(t)), & \tilde{y}(b) = y_0,
\end{cases}
\end{align*}
\]

For the chosen grid points on \([b, b_{N_h+1}]\), we allow the \( N_h \)'s to vary over the \([b, b_{N_h+1}]\)'s so that the \( b_{N_h} \)'s may be comparable. To develop the Runge kutta method of order three for (3.1), we follow [6] and define

\[
\begin{align*}
y_{km+1}(t) - y_{km}(t) &= \sum_{j=1}^{\infty} w_j b_j \left( \tilde{y}_{km} \right), \\
y_{km+1}(t) - y_{km}(t) &= \sum_{j=1}^{\infty} w_j b_j \left( \tilde{y}_{km} \right),
\end{align*}
\]

Where \( w_1, w_2, w_3 \) are constants and
Where in Runge Kutta method of order three

\[
\begin{align*}
K_1 &= \left( b_1 \Delta \gamma k_n \right) = m_n \left( (x_n + \Delta \gamma k_n), (y_n + \Delta \gamma k_n) \right) \\
K_2 &= \left( b_2 \Delta \gamma k_n \right) = m_n \left( (x_n + \frac{1}{2} b_2 \Delta \gamma k_n), (y_n + \frac{1}{2} b_2 \Delta \gamma k_n) \right) \\
K_3 &= \left( b_3 \Delta \gamma k_n \right) = m_n \left( (x_n + \beta b_3 \Delta \gamma k_n), (y_n + \beta b_3 \Delta \gamma k_n) \right)
\end{align*}
\]

Next we define
The exact solution at $\Delta h_{k,u}^{-1}$ is given by

$$\begin{align*}
\Delta y_{R,k}(t) &= \frac{1}{2} S_{h} y_{R,k} y_{R,k} + \frac{1}{2} S_{h} y_{R,k} y_{R,k}, \\
\Delta y_{H,k}(t) &= \frac{1}{2} T_{h} y_{R,k} y_{R,k} + \frac{1}{2} T_{h} y_{R,k} y_{R,k}.
\end{align*}$$

The approximate solution is given by

$$\begin{align*}
\Delta y_{R,k}(t) &= \frac{1}{2} S_{h} y_{R,k} y_{R,k} + \frac{1}{2} S_{h} y_{R,k} y_{R,k}, \\
\Delta y_{H,k}(t) &= \frac{1}{2} T_{h} y_{R,k} y_{R,k} + \frac{1}{2} T_{h} y_{R,k} y_{R,k}.
\end{align*}$$

**Theorem 4.1**

Consider the systems (3.2) and (4.1), for a fixed $N \in \mathbb{Z}^{+}$ and $r \in [0,1]$.

5. **Numerical example**

Before illustrating the numerical solution of a hybrid fuzzy IVP, first we recall the fuzzy IVP:

$$\begin{align*}
x(t) &= x(0), \\
x(0, r) &= [0.75 + 0.25r, 1.125 - 0.125r], 0 \leq r \leq 1.
\end{align*}$$

The exact solution is given by

$$x(t, r) = \left[ (0.75 + 0.25r) e^{(1.125 - 0.125r)t}, 0 \leq r \leq 1 \right].$$

We see that
method with \( N = 2 \) in [6], (5.1) gives

\[
y(1.00) = \left[ (0.75 + 0.25r), (1.25 - 0.125r) \right], \quad 0 \leq r \leq 1.
\]

where

\[
q_{01} = 1 + 0.5 + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{6}
\]

Comparing the Euler in [10] and Runge kutta Nystrom method in [6] we see that Runge-kutta is much closer to the true solution.

**Example 1**

Next consider the following hybrid fuzzy IVP

\[
\begin{align*}
\lambda_k(t) &= \lambda_k(t) + m(t)\lambda_k(t(x(t))) \quad \text{for} \quad t \in [0, 2] \quad k = 0, 1, 2, ... \\
\lambda_k(0) &= \lambda_k[1, 2, ...]
\end{align*}
\]

In (5.2), \( x(t+m(t)\lambda_k(t(x(t))) \) is a continuous function of \( t, x, \) and \( \lambda_k(t(x(t))) \). Therefore by Example 6.1 of Kaleva [5] and Theorem 4.2 of Buckley and Feuring [2] for each \( k=0,1,2, ... \), the fuzzy IVP

\[
\begin{align*}
\lambda_k(t) &= \lambda_k(t) + m(t)\lambda_k(t(x(t))) \\
\lambda_k(0) &= \lambda_k[1, 2, ...]
\end{align*}
\]

has a unique solution on \([t_0, t_{n+1}] \). To numerically solve the hybrid fuzzy IVP (5.2) we will apply the Runge–Kutta method for hybrid fuzzy differential equations from Section 4 with \( N=2 \) to obtain \( y_{1,2}(r) \) approximating \( x(2.0;r) \). Let \( f: [0, \infty) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) be given by

\[
f(t, x, \lambda_k(t(x(t)))) = x(t) + m(t)\lambda_k(t(x(t))) \quad \text{for} \quad t \in [0, 2] \quad k = 0, 1, 2, ... \\
\lambda_k: \mathbb{R} \to \mathbb{R}
\]

is given by \( \lambda_k(x) = \begin{cases} 0, & \text{if } k = 0 \\ x, & \text{if } k \in [1, 2, ...] \end{cases} \)

Since the exact solution of (5.4) for \( t \in [1, 1.5] \) is

\[
x(t) = x(1.5) \left( 3e^{2t} - 2t \right), \quad 0 \leq r \leq 1.
\]
Then $x(1.5;1)$ is approximately 5.248. Since the exact solution of (5.4) for $t \in [1.5, 2]$ is

$$x(t) = x(1.5) + \int_{1.5}^{t} f(s)\, ds,$$

Then $x(2.0;1)$ is approximately 9.68 and $y_{1,2}(1)$ is approximately 9.65. These observations are summarized in Table 5.1 For additional comparison, Fig 5.1 shows the graphs of $x(2.0)$, $y_{1,2}$, and the corresponding Euler approximation.

References


Figure 5.1. Comparison of Euler and Runge Kutta Nystrom method with the Exact Solution

Table 5.1:
Comparison of Exact and Approximate Solution

At t=1.5

<table>
<thead>
<tr>
<th>r</th>
<th>Exact solution</th>
<th>Approximate solution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Y</td>
<td>vp</td>
</tr>
<tr>
<td>1</td>
<td>5.290221725</td>
<td>5.290221725</td>
</tr>
<tr>
<td>0.8</td>
<td>5.025710639</td>
<td>5.422477268</td>
</tr>
<tr>
<td>0.6</td>
<td>4.761199553</td>
<td>5.554732811</td>
</tr>
<tr>
<td>0.4</td>
<td>4.496688467</td>
<td>5.686988354</td>
</tr>
<tr>
<td>0.2</td>
<td>4.23217738</td>
<td>5.819243898</td>
</tr>
<tr>
<td>0</td>
<td>3.967666294</td>
<td>5.95149941</td>
</tr>
</tbody>
</table>
### At $t=2$

<table>
<thead>
<tr>
<th>$r$</th>
<th>Exact solution</th>
<th>Approximate solution</th>
</tr>
</thead>
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<tr>
<td></td>
<td>$y$</td>
<td>$\tilde{y}$</td>
</tr>
<tr>
<td>0.8</td>
<td>9.193126888</td>
<td>9.18900064</td>
</tr>
<tr>
<td>0.6</td>
<td>8.709278105</td>
<td>10.16082446</td>
</tr>
<tr>
<td>0.4</td>
<td>8.225429321</td>
<td>10.40274885</td>
</tr>
<tr>
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<td>7.741580538</td>
<td>10.64467324</td>
</tr>
<tr>
<td>0</td>
<td>7.257731754</td>
<td>10.88659763</td>
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</table>

### Error for different values of $t$

<table>
<thead>
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<th>$r$</th>
<th>$t=1$</th>
<th>$t=1.5$</th>
<th>$t=2$</th>
</tr>
</thead>
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<tr>
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<td>0.009514468</td>
<td>0.041984965</td>
<td>0.023464911</td>
</tr>
<tr>
<td>0.8</td>
<td>0.009038745</td>
<td>0.039885717</td>
<td>0.022291665</td>
</tr>
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<td>0.6</td>
<td>0.008563022</td>
<td>0.037786469</td>
<td>0.021118420</td>
</tr>
<tr>
<td>0.4</td>
<td>0.008087298</td>
<td>0.035687221</td>
<td>0.019945174</td>
</tr>
<tr>
<td>0.2</td>
<td>0.007611575</td>
<td>0.033587972</td>
<td>0.018771929</td>
</tr>
<tr>
<td>0</td>
<td>0.007135851</td>
<td>0.031488724</td>
<td>0.017598683</td>
</tr>
</tbody>
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