On Initial and Final Characterized L-topological Groups

A. S. ABD-ALLAH*

Department of Mathematics, Faculty of Science, El-Mansoura University, El-Mansoura, Egypt

Abstract:
In this research work, new topological notions are proposed and investigated. The notions are named final characterized L-spaces and initial and final characterized L-topological groups. The properties of such notions are deeply studied. We show that all the final lefts and all the final characterized L-spaces are uniquely exist in the category CRL-Sp and hence CRL-Sp is topological category over the category SET of all sets. By the notion of final characterized L-space, the notions of characterized quotient pre L-spaces and characterized sum L-spaces are introduced and studied. The characterized L-subspaces together with their related inclusion mappings and the characterized quotient pre L-spaces together with their related canonical surjections are the equalizers and co-equalizers, respectively in CRL-Sp. Moreover, we show that the initial and final lefts and then the initial and final characterized L-topological groups uniquely exist in the category CRL-TopGrp. Hence, the category CRL-TopGrp is topological category over the category Grp of all groups. By the notion of initial and final characterized L-topological groups, the notions of characterized L-subgroups, characterized product L-topological groups and characterized L-topological quotient groups are introduced and studied. We show that the category CRL-TopGrp is concrete and co-concrete category of the category L-Top. Finally, we show that the special faithful functors $J: CRL-TopGrp \rightarrow L-Top$ and $J^*: L-Top \rightarrow CRL-TopGrp$ are isomorphism, that is, the category CRL-TopGrp is algebraic and co-algebraic category over the category L-Top as in sense of [7].

Keywords: L-filter, topological L-space, operations, characterized L-space, categories L-Top, Grp, CRL-Sp, SCRL-Sp, CR-Sp, CRL-TopGrp and CR-TopGrp, $\varphi_{1,2}$ L-neighborhood filters, $\varphi_{1,2}$ L-continuous, $\varphi_{1,2}$ L-open, $\varphi_{1,2}$ L-homeomorphism, $\varphi_{1,2}$ L-homomorphism, final characterized L-space, characterized quotient pre L-space, characterized sum L-space, characterized L-topological group, characterized L-subgroup, characterized product L-topological group, characterized L-topological quotient group.

1. Introduction

The notion of L-filter has been introduced by Eklund et al. [10]. By means of this notion a point-based approach to L-topology related to the usual points has been developed. More general concept for L-filter introduced by Gündler in [11] and L-filters are classified by types. Because of the specific type of L-filter however the approach of Eklund is related only to L-topologies which are stratified, that is, all constant L-sets are open. The more specific L-filters considered in the former papers are called now homogeneous. The operation on the ordinary topological space $(X, T)$ has been defined by Kasahara ([16]) as a mapping $\varphi$ from $T$ into $2^X$ such that, $A \subseteq A^\varphi$, for all $A \in T$. In [5], Abd El-Monsef's et al. extended Kasahara's operation to the power set $P(X)$ of a set $X$. Kandil et al. ([15]), extended Kasahara's and Abd El-Monsef's operations by introducing an operation on the class of all L-sets endowed with an L-topology $\tau$ as a mapping $\varphi: L^X \rightarrow L^X$ such that $\text{int} \mu \subseteq \mu^\varphi$ for all $\mu \in L^X$, where $\mu^\varphi$ denotes the value of $\varphi$ at $\mu$. The notions of the L-filters and the operations on the class of all L-sets on $X$ endowed with an L-topology $\tau$ are applied in [2,3,4] to introduce a more general theory including all the weaker and stronger forms of the L-topology. By means of these notions the notion of $\varphi_{1,2}$-interior of L-set, $\varphi_{1,2}$ L-convergence and $\varphi_{1,2}$ L-neighborhood filters are defined and applied to introduced many special classes of separation axioms. The notion of $\varphi_{1,2}$-interior operator for L-sets is defined as a mapping $\varphi_{1,2}^\text{int}: L^X \rightarrow L^X$ which fulfill (11) to (15) in [2]. There is a one-to-one correspondence between

* Present address: Department of Mathematics, Faculty of Science and humanities studies, Salman Bin Abdulaziz Univ., P. O. Box 132012, Code No. 11941 Hotat Bani Tamim, Saudi Arabia
E-mail address: asabdallah@hotmail.com and drhmsa1961@yahoo.com

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the class of all \( \varphi_{1,2} \)-open L-subsets of \( X \) and these operators, that is, the class \( \varphi_{1,2} \cdot \text{OF}(X) \) of all \( \varphi_{1,2} \)-open L-subsets of \( X \) can be characterized by these operators. Then the triple \((X, \varphi_{1,2} \cdot \text{OF}(X))\) will be called the characterized L-space of \( \varphi_{1,2} \)-open L-subsets. The characterized L-spaces are characterized by many of characterizing notions in [2,3], for example by: \( \varphi_{1,2} \)-neighborhood filters, \( \varphi_{1,2} \)-interior of the L-filters and by the set of \( \varphi_{1,2} \)-inner points of the L-filters. Moreover, the notions of closeness and compactness in characterized L-spaces are introduced and studied in [4].

This paper is devoted to introduce and study the notions of final characterized L-spaces and initial and final characterized L-topological groups as a generalization of the weaker and stronger forms of the final topological L-space and initial and final L-topological group introduced in [8, 18]. **In section 2**, some definitions and notions related to L-sets, L-topologies, L-filters, operations on L-sets, characterized L-spaces, \( \varphi_{1,2} \)-L-neighborhood filters, \( \varphi_{1,2} \)- continuity and \( \varphi_{1,2} \)-homomorphisms as a morphisms between them are presented. **Section 3**, is devoted to introduce and study the notion of final characterized L-spaces. We show that all the final lefts and all the final characterized L-spaces are uniquely exist in the category \( \text{CRL-Sp} \). Further notions related to the notion of characterized L-spaces are e.g. those of a characterized quotient pre L-spaces and a characterized sum L-spaces are investigated as special cases for the notions of final characterized L-spaces. By the initial and final lefts in \( \text{CRL-Sp} \) we show that the category \( \text{CRL-Sp} \) is topological category over the category \( \text{SET} \) of all sets in sense of [7,19] and it is also complete and co-complete category, that is, all limits and all co-limits in \( \text{CRL-Sp} \) exist, which of course are unique up to isomorphisms. According to general procedure, we show that the characterized L-subspaces together with their related inclusion mappings and the characterized quotient pre L-spaces together with their related canonical surjections are equalizers and co-equalizers in \( \text{CRL-Sp} \), respectively. **Section 4**, is devoted to introduce and study the notion of initial characterized L-topological groups as a generalization of the weaken and stronger forms of the initial L-topological groups which introduced in [8]. It will be shown that the initial lefts and then the initial characterized L-topological groups are uniquely exist in the category \( \text{CRL-TopGrp} \) and therefore, the category \( \text{CRL-TopGrp} \) is topological category over the category \( \text{Grp} \) of all groups. More generally, we show that the category \( \text{CRL-TopGrp} \) is concrete category of the category \( \text{L-Top} \) of all topological spaces and the faithful functor \( \mathcal{F}: \text{CRL-Grp} \rightarrow \text{L-Top} \) is isomorphism. Thus, the category \( \text{CRL-TopGrp} \) is algebraic category over the category \( \text{L-Top} \) in sense of [7]. Finally, by the notion of initial characterized L-topological groups, the notions of characterized L-subgroups and characterized product L-topological groups are introduced and studied. **In section 5**, the notion of final characterized L-topological groups are introduced and studied as a generalization of the weaken and stronger forms of the final L-topological groups introduced in [8]. It will be shown that the final lefts and then the final characterized L-topological groups are uniquely exist in the category \( \text{CRL-TopGrp} \). More generally, we show that the category \( \text{CRL-TopGrp} \) is co-concrete category of the category \( \text{L-Top} \) of all topological L-spaces and the faithful functor \( \mathcal{F}^*: \text{L-Top} \rightarrow \text{CRL-TopGrp} \) is isomorphism. Thus, the category \( \text{CRL-TopGrp} \) is co-algebraic category over the category \( \text{L-Top} \) in sense of [7]. By the notion of final characterized L-topological groups, the notions of characterized L-topological quotient groups is introduced and studied. Finally, we present a relation between the characterized L-topological quotient groups and the characterized product L-topological groups.

2. Preliminaries

In this research work we consider L be a completely distributive complete lattice with different least and last elements 0 and 1, respectively. Consider \( L_0 = L \setminus \{0\} \) and \( L_* = L \setminus \{1\} \). Sometimes we will assume more specially that L is complete chain, that is, L is a complete lattice whose partial ordering is a linear one. For a set \( X \) , let \( L^X \) be the set of all L-subsets of \( X \) , that is, of all mappings \( f : X \rightarrow L \). Assume that an order-reversing involution \( \alpha \mapsto \alpha^* \) of L is fixed. For each L-set \( \mu \subseteq L^X \), let \( \mu^* \) denote the complement of \( \mu \) and it is defined by: \( \mu^*(x) = \mu(x)^* \) for all \( x \in X \). Denote by \( \overline{\alpha} \) the constant L-subset of \( X \) with value \( \alpha \in L \). For all
\( x \in X \), and for all \( \alpha \in L_0 \), the L-subset \( x_\alpha \) of \( X \) whose value \( \alpha \) at \( x \) and 0 otherwise is called an L-point in \( X \). Now, we begin by recalling some facts on the L-filters.

**L-filters.** The L-filter on a set \( X \) ([11]) is a mapping \( \mathcal{M} : L^X \rightarrow L \) such that the following conditions are fulfilled:

(F1) \( \mathcal{M}(\alpha) \leq \alpha \) for all \( \alpha \in L \) and \( \mathcal{M}(1) = 1 \).

(F2) \( \mathcal{M}(\mu \land \rho) = \mathcal{M}(\mu) \land \mathcal{M}(\rho) \) for all \( \mu, \rho \in L^X \).

The L-filter \( \mathcal{M} \) is called homogeneous ([11]) if \( \mathcal{M}(\alpha) = \alpha \) for all \( \alpha \in L \). For each \( x \in X \), the mapping \( \hat{x} : L^X \rightarrow L \) defined by \( \hat{x}(\mu) = \mu(x) \) for all \( \mu \in L^X \) is a homogeneous L-filter on \( X \). For each \( \mu \in L^X \), the mapping \( \hat{\mu} : L^X \rightarrow L \) defined by \( \hat{\mu}(\eta) = \bigwedge_{0 \leq \eta(x)} \eta(x) \) for all \( \eta \in L^X \) is also homogeneous L-filter on \( X \), called homogenous L-filter at the L-subset \( \mu \in L^X \). Let \( \mathcal{F}_LX \) and \( F_LX \) will be denote the sets of all L-filters and of all homogeneous L-filters on a set \( X \), respectively. If \( \mathcal{M} \) and \( \mathcal{N} \) are L-filters on a set \( X \), \( \mathcal{M} \) is said to be finer than \( \mathcal{N} \), denoted by \( \mathcal{M} \leq \mathcal{N} \), provided \( \mathcal{M}(\mu) \geq \mathcal{N}(\mu) \) holds for all \( \mu \in L^X \). Noting that if \( L \) is a complete chain then \( \mathcal{M} \) is not finer than \( \mathcal{N} \), denoted by \( \mathcal{M} \not \leq \mathcal{N} \), provided there exists \( \mu \in L^X \) such that \( \mathcal{M}(\mu) < \mathcal{N}(\mu) \) holds.

For each non-empty set \( \mathcal{A} \) of the L-filters on \( X \) the supremum \( \bigvee_{\mathcal{M} \in \mathcal{A}} \mathcal{M} \) exists ([11]) and given by:

\[
\bigvee_{\mathcal{M} \in \mathcal{A}} \mathcal{M}(\mu) = \bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}(\mu)
\]

for all \( \mu \in L^X \). Whereas the infimum \( \bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M} \) of \( \mathcal{A} \) does not exists in general as an L-filter. If the infimum \( \bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M} \) exists, then we have:

\[
\bigwedge_{\mathcal{M} \in \mathcal{A}} \mathcal{M}(\mu) = \bigvee_{\mu \leq \mu_1 \leq \ldots \leq \mu_n} (\mathcal{M}_1(\mu_1) \land \cdots \land \mathcal{M}_n(\mu_n))
\]

For all \( \mu \in L^X \), where \( n \) is a positive integer, \( \mu_1, \ldots, \mu_n \) is a collection such that \( \mu_1 \land \cdots \land \mu_n \leq \mu \) and \( \mathcal{M}_1, \ldots, \mathcal{M}_n \) are L-filters from \( \mathcal{A} \). Let \( X \) be a set and \( \mu \in L^X \), then the homogeneous L-filter \( \hat{\mu} \) at \( \mu \in L^X \) is the L-filter on \( X \) given by:

\[
\hat{\mu} = \bigvee_{0 \leq \mu(x)} \hat{x}(\mu)
\]

**L-filter bases.** A family \( (\mathcal{B}_\alpha)_{\alpha \in L_0} \) of non-empty subsets of \( L^X \) is called a valued L-filter base ([11]) if the following conditions are fulfilled:

(V1) \( \mu \in \mathcal{B}_\alpha \) implies \( \alpha \leq \sup \mu \).

(V2) For all \( \alpha, \beta \in L_0 \) with \( \alpha \land \beta \in L_0 \) and all \( \mu \in \mathcal{B}_\alpha \) and \( \rho \in \mathcal{B}_\beta \) there are \( \gamma \geq \alpha \land \beta \) and \( \eta \geq \mu \land \sigma \) such that \( \eta \in \mathcal{B}_\gamma \).

Each valued base \( (\mathcal{B}_\alpha)_{\alpha \in L_0} \) defines the L-filter \( \mathcal{M} \) on \( X \) ([11]) by \( \mathcal{M}(\mu) = \bigvee_{\rho \in \mathcal{B}_\alpha, \rho \preceq \mu} \alpha \) for all \( \mu \in L^X \). Conversely, each L-filter \( \mathcal{M} \) can be generated by a valued base, e.g. by \( (\alpha \text{-pr} \mathcal{M})_{\alpha \in L_0} \) with \( \alpha \text{-pr} \mathcal{M} = \{ \mu \in L^X \mid \alpha \leq \mathcal{M}(\mu) \} \). The family \( (\alpha \text{-pr} \mathcal{M})_{\alpha \in L_0} \) is a family of prefilter on \( X \) and is called the large valued base of \( \mathcal{M} \). Recall that a prefilter on \( X \) ([16]) is a non-empty proper subset \( \mathcal{F} \) of \( L^X \) such that:

(1) \( \mu, \rho \in \mathcal{F} \) implies \( \mu \land \rho \in \mathcal{F} \) and (2) from \( \mu \in \mathcal{F} \) and \( \mu \leq \rho \) it follows \( \rho \in \mathcal{F} \).
Topological L-spaces. By an L-topology on a set $X$ ([9, 14]), we mean a subset of $\mu \in L^X$ which is closed with respect to all suprema and all finite infima and contains the constant L-sets $\overline{0}$ and $\overline{1}$. A set $X$ equipped with an L-topology $\tau$ on $X$ is called topological L-space. For each topological L-space $(X, \tau)$, the elements of $\tau$ are called open L-subsets of this space. If $\tau_1$ and $\tau_2$ are L-topologies on a set $X$, $\tau_2$ is said to be finer than $\tau_1$ and $\tau_1$ is said to be coarser than $\tau_2$ provided $\tau_1 \subseteq \tau_2$ holds. For each L-set $\mu \in L^X$, the strong $\alpha$-cut and the weak $\alpha$-cut of $\mu$ are ordinary subsets of $X$ defined by the subsets $S_\alpha (\mu) = \{ x \in X : \mu(x) > \alpha \}$ and $W_\alpha (\mu) = \{ x \in X : \mu(x) \geq \alpha \}$, respectively. For each complete chain $L$, the $\alpha$-level topology and the initial topology ((17)) of an L-topology $\tau$ on $X$ are defined as follows: 
$$\tau_\alpha = \{ S_\alpha (\mu) \in P(X) : \mu \in \tau \} \quad \text{and} \quad i(\tau) = \inf \{ \tau_\alpha : \alpha \in L_1 \},$$ 
respectively, where $\inf$ is the infimum with respect to the finer relation on topologies. On other hand if $(X,\mathcal{T})$ is ordinary topological space, then the induced L-topology on $X$ is denoted by $O_{(L^X,\mathcal{T})}$. Denote by $L$-$Top$ and $Top$ to the categories of all topological L-spaces and all ordinary topological spaces, respectively.

Operation on L-sets. In the sequel, let a topological L-space $(X, \tau)$ be fixed. By the operation (15) on a set $X$ we mean a mapping $\varphi : L^X \rightarrow L^X$ such that $\int \mu \leq \mu^0$ holds, for all $\mu \in L^X$, where $\mu^0$ denotes the value of $\varphi$ at $\mu$. The class of all operations on $X$ will be denoted by $O_{(L^X,\tau)}$. The constant operation on the class of all ordinary operations $S_{\alpha} (\mu) \in P(X)$ will be denoted by $O_{(P(X),\mathcal{T})}$, and also $\tau$ are said to be stratified provided $\alpha \in \tau$ holds for all $\alpha \in L$, that is, all constant L-sets are open ((17)). By identity operation on $X$, then obviously, $\int \cdot \leq \inf$ with respect to all suprema and all finite infima and contains the constant L-sets $\overline{0}$ and $\overline{1}$.

$$\varphi : L^X \rightarrow L^X$$ 

will be called: 
(i) Isotone if $\mu \leq \rho$ implies $\mu^0 \leq \rho^0$, for all $\mu, \rho \in L^X$.
(ii) Weakly finite intersection preserving (wfip, for short) with respect to $A \subseteq L^X$ if $\rho \land \mu^0 \leq (\rho \land \mu)^0$ holds, for all $\rho \in A$ and $\mu \in L^X$.
(iii) Idempotent if $\mu^0 = (\mu^0)^0$, for all $\mu \in L^X$.

$\varphi$-open L-sets. Let a topological L-space $(X, \tau)$ be fixed and $\varphi \in O_{(L^X,\tau)}$. The L-set $\mu : X \rightarrow L$ will be called $\varphi$-open L-set if $\mu \leq \mu^0$ holds. We will denote the class of all $\varphi$-open L-sets on $X$ by $\varphi O F (X)$. The L-set $\mu$ is called $\varphi$-closed if its complement $\complement \mu$ is $\varphi$-open. The two operations $\varphi, \psi \in O_{(L^X,\tau)}$ are equivalent and written $\varphi \sim \psi$ if $\varphi O F (X) = \psi O F (X)$.

$\varphi_{1,2}$-interior of L-sets. Let a topological L-space $(X, \tau)$ be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the $\varphi_{1,2}$-interior of the L-set $\mu : X \rightarrow L$ is the mapping $\varphi_{1,2} \cdot \int \mu : X \rightarrow L$ defined by:

$$\varphi_{1,2} \cdot \int \mu = \bigvee_{\rho \in \varphi O F (X), \rho^0 \leq \mu} \rho$$

(2.1)
\( \varphi_{1,2}.\text{int} \mu \) is the greatest \( \varphi_{1} \)-open L-set \( \rho \) such that \( \rho^{\varphi_{1}} \) less than or equal to \( \mu \) (\([2]\)). The L-set \( \mu \) is said to be \( \varphi_{1,2} \)-open if \( \mu \leq \varphi_{1,2}.\text{int} \mu \). The class of all \( \varphi_{1,2} \)-open L- sets on \( X \) will be denoted by \( \varphi_{1,2}OF(X) \).

The complement \( \text{co} \mu \) of a \( \varphi_{1,2} \)-open L-subset \( \mu \) will be called \( \varphi_{1,2} \)-closed, the class of all \( \varphi_{1,2} \)-closed L-subsets of \( X \) will be denoted by \( \varphi_{1,2}CF(X) \). In the classical case of \( L = \{0, 1\} \), the topological L-space \( (X, \tau) \) is up to identification by the ordinary topological space \( (X, T) \) and \( \varphi_{1,2}.\text{int} \mu \) is the classical one. Hence, in this case the ordinary subset \( A \) of \( X \) is \( \varphi_{1,2} \)-open if \( A \subseteq \varphi_{1,2}.\text{int} A \). The complement of a \( \varphi_{1,2} \)-open subset \( A \) of \( X \) will be called \( \varphi_{1,2}C(A) \). The class of all \( \varphi_{1,2} \)-open and the class of all \( \varphi_{1,2} \)-closed subsets of \( X \) will be denoted by \( \varphi_{1,2}O(X) \) and \( \varphi_{1,2}C(X) \), respectively. Clearly, \( F \) is \( \varphi_{1,2} \)-closed if and only if \( \varphi_{1,2}.\text{cl} \, F = F \).

**Proposition 2.1** \([2]\) If \( (X, \tau) \) is a topological L-space and \( \varphi_{1}, \varphi_{2} \in O_{(L^{x}, \tau)} \). Then, the mapping \( \varphi_{1,2}.\text{int} \mu : X \rightarrow L \) fulfills the following axioms:

(i) If \( \varphi_{2} \geq 1_{L^{x}} \), then \( \varphi_{1,2}.\text{int} \mu \leq \mu \) holds.

(ii) \( \varphi_{1,2}.\text{int} \mu \) is isotone, i.e., if \( \mu \leq \rho \) then \( \varphi_{1,2}.\text{int} \mu \leq \varphi_{1,2}.\text{int} \rho \) holds for all \( \mu, \rho \in L^{x} \).

(iii) \( \varphi_{1,2}.\text{int} 1_{L^{x}} = 1_{L^{x}} \).

(iv) If \( \varphi_{2} \geq 1_{L^{x}} \) is isotone operation and \( \varphi_{1} \) is wfip with respect to \( \varphi_{1}OF(X) \), then \( \varphi_{1,2}.\text{int} (\mu \wedge \rho) = \varphi_{1,2}.\text{int} \mu \wedge \varphi_{1,2}.\text{int} \rho \) for all \( \mu, \rho \in L^{x} \).

(v) If \( \varphi_{1} \) is isotope and idempotent operation, then \( \varphi_{1,2}.\text{int} \mu \leq \varphi_{1,2}.\text{int} (\varphi_{1,2}.\text{int} \mu) \) holds.

(vi) \( \varphi_{1,2}.\text{int} (\bigvee_{\mu_{1}}) = \bigvee_{\mu_{1}} \varphi_{1,2}.\text{int} \mu_{1} \) for all \( \mu_{1} \in \varphi_{1,2}O(X) \).

**Proposition 2.2** \([2]\) Let \( (X, \tau) \) be a topological L-space and \( \varphi_{1}, \varphi_{2} \in O_{(L^{x}, \tau)} \). Then the following are fulfilled:

(i) If \( \varphi_{2} \geq 1_{L^{x}} \), then the class \( \varphi_{1,2}OF(X) \) of all \( \varphi_{1,2} \)-open L-sets on \( X \) forms an extended L-topology on \( X \), denoted by \( \tau^{\varphi_{1,2}} \) (\([13]\)).

(ii) If \( \varphi_{2} \geq 1_{L^{x}} \), then the class \( \varphi_{1,2}OF(X) \) of all \( \varphi_{1,2} \)-open L-sets on \( X \) forms a supra L-topology on \( X \), denoted by \( \tau^{\varphi_{1,2}} \) (\([13]\)).

(iii) If \( \varphi_{2} \geq 1_{L^{x}} \) is isotone and \( \varphi_{1} \) is wfip with respect to \( \varphi_{1}OF(X) \), then \( \varphi_{1,2}OF(X) \) is a pre L-topology on \( X \), denoted by \( \tau_{\varphi_{1,2}}^{\varphi_{1}} \) (\([13]\)).

(iv) If \( \varphi_{2} \geq 1_{L^{x}} \) is isotone and idempotent operation and \( \varphi_{1} \) is wfip with respect to \( \varphi_{1}OF(X) \), then \( \varphi_{1,2}OF(X) \) forms an L-topology on \( X \), denoted by \( \tau_{\varphi_{1,2}}^{\varphi_{1}} \) (\([9, 14]\)).

From Propositions 2.1 and 2.2, if the topological L-space \( (X, \tau) \) be fixed and \( \varphi_{1}, \varphi_{2} \in O_{(L^{x}, \tau)} \). Then

\[
\varphi_{1,2}OF(X) = \{ \mu \in L^{x} \mid \mu \leq \varphi_{1,2}.\text{int} \mu \}
\]  \hspace{1cm} (2.2)

and the following conditions are fulfilled:

(11) If \( \varphi_{2} \geq 1_{L^{x}} \), then \( \varphi_{1,2}.\text{int} \mu \leq \mu \) holds for all \( \mu \in L^{x} \).

(12) If \( \mu \leq \rho \) then \( \varphi_{1,2}.\text{int} \mu \leq \varphi_{1,2}.\text{int} \rho \) holds for all \( \mu, \rho \in L^{x} \).

(13) \( \varphi_{1,2}.\text{int} 1_{L^{x}} = 1_{L^{x}} \).
If \( \varphi_2 \geq 1_{L^x} \) is isotone and \( \varphi_1 \) is wisp with respect to \( \varphiOF(X) \), then
\[
\varphi_{1,2} \cdot \text{int}(\mu \land \rho) = \varphi_{1,2} \cdot \text{int} \mu \land \varphi_{1,2} \cdot \text{int} \rho
\]
for all \( \mu, \rho \in L^x \).

(15) If \( \varphi_2 \geq 1_{L^x} \) is isotone and idempotent, then
\[
\varphi_{1,2} \cdot \text{int}(\varphi_{1,2} \cdot \text{int} \mu) = \varphi_{1,2} \cdot \text{int} \mu
\]
for all \( \mu \in L^x \).

**Characterized L-spaces.** Independently on the L-topologies, the notion of \( \varphi_{1,2} \)-interior operator for L-sets can be defined as a mapping \( \varphi_{1,2} \cdot \text{int} : L^x \to L^x \) which fulfills (11) to (15). It is well-known that (2.1) and (2.2) give a one-to-one correspondence between the class of all \( \varphi_{1,2} \)-open L-sets and these operators, that is, \( \varphi_{1,2} \cdot OF(X) \) can be characterized by \( \varphi_{1,2} \)-interior operators. In this case the pair \( (X, \varphi_{1,2} \cdot \text{int}) \) as well as the pair \( (X, \varphi_{1,2} \cdot OF(X)) \) will be called characterized L-space ([2]) of \( \varphi_{1,2} \)-open L-sets of \( X \). If \( (X, \varphi_{1,2} \cdot \text{int}) \) and \( (X, \psi_{1,2} \cdot \text{int}) \) are two characterized L-spaces, then \( (X, \varphi_{1,2} \cdot \text{int}) \) is said to be finer than \( (X, \psi_{1,2} \cdot \text{int}) \) and denoted by \( \varphi_{1,2} \cdot \text{int} \leq \psi_{1,2} \cdot \text{int} \) provided \( \varphi_{1,2} \cdot \text{int} \mu \geq \psi_{1,2} \cdot \text{int} \mu \) holds for all \( \mu \in L^x \).

The characterized L-space \( (X, \varphi_{1,2} \cdot \text{int}) \) of all \( \varphi_{1,2} \)-open L-sets is said to be stratified if and only if \( \varphi_{1,2} \cdot \text{int} \alpha = \alpha \) for all \( \alpha \in L \). As shown in [2], the characterized L-space \( (X, \varphi_{1,2} \cdot \text{int}) \) is stratified if the related L-topology is stratified. Moreover, the characterized L-space \( (X, \varphi_{1,2} \cdot \text{int}) \) is said to have the weak infimum property ([13]) provided for all \( \mu \in L^x \) and \( \alpha \in L \). The characterized L-space \( (X, \varphi_{1,2} \cdot \text{int}) \) is said to be strongly stratified ([13]) provided \( \varphi_{1,2} \cdot \text{int} \) is stratified and have the weak infimum property.

If \( \varphi_1 = \text{int} \) and \( \varphi_2 = 1_{L^x} \), then the class \( \varphi_{1,2} \cdot OF(X) \) of all \( \varphi_{1,2} \)-open L-set of \( X \) coincide with \( \tau \) which is defined in [9,14] and hence the characterized L-space \( (X, \varphi_{1,2} \cdot \text{int}) \) coincide with the topological L-space \( (X, \tau) \).

**\( \varphi_{1,2} \)-neighborhood filters.** An important notion in the characterized L-space \( (X, \varphi_{1,2} \cdot \text{int}) \) is that of a \( \varphi_{1,2} \)-neighborhood filter at the point and at the ordinary subset in this space. Let \( (X, \tau) \) be a topological L-space and \( \varphi_1, \varphi_2 \in O_{(L^x, \tau)} \). As follows by (11) to (15) for each \( x \in X \), the mapping
\[
\mathcal{N}_{\varphi_{1,2}}(x) : L^x \to L
\]
which is defined by
\[
\mathcal{N}_{\varphi_{1,2}}(x)(\mu) = (\varphi_{1,2} \cdot \text{int} \mu)(x)
\]
for all \( \mu \in L^x \) is L-filter, called \( \varphi_{1,2} \)-L-neighborhood filter at \( x \) ([2]). If \( \varphi \neq F \subseteq P(X) \), then the \( \varphi_{1,2} \)-L-neighborhood filter at \( F \) will be denoted by \( \mathcal{N}_{\varphi_{1,2}}(F) \) and it will be defined by:
\[
\mathcal{N}_{\varphi_{1,2}}(F) = \bigvee_{x \in F} \mathcal{N}_{\varphi_{1,2}}(x).
\]
Since \( \mathcal{N}_{\varphi_{1,2}}(x) \) is L-filter for all \( x \in X \), then \( \mathcal{N}_{\varphi_{1,2}}(F) \) is also L-filter on \( X \). Moreover, because of \( [\chi_F] = \bigvee_{x \in F} x \), then we have \( \mathcal{N}_{\varphi_{1,2}}(F) \geq [\chi_F] \) holds.

If the related \( \varphi_{1,2} \)-interior operator fulfill the axioms (11) and (12) only, then the mapping \( \mathcal{N}_{\varphi_{1,2}}(x) : L^x \to L \), which is defined by (2.3) is an L-stack ([15]), called \( \varphi_{1,2} \)-L-neighborhood stack at \( x \). Moreover, if the \( \varphi_{1,2} \)-interior operator fulfill the axioms (11), (12) and (14) such that in (14) instead of \( \rho \in L^x \) we choose \( \overline{\alpha} \), then the mapping \( \mathcal{N}_{\varphi_{1,2}}(x) : L^x \to L \), is an L-stack with the cutting property, called here \( \varphi_{1,2} \)-L-neighborhood stack with the cutting property at \( x \). Obviously, the \( \varphi_{1,2} \)-L-neighborhood filters fulfill the following axioms:
(N1) \( x \leq N_{\alpha_2}(x) \) holds for all \( x \in X \).

(N2) \( N_{\alpha_2}(x)(\mu) \leq N_{\alpha_2}(x)(\rho) \) holds for all \( \mu, \rho \in L^X \) and \( \mu \leq \rho \).

(N3) \( N_{\alpha_2}(x)(y \rightarrow N_{\alpha_2}(y)(\mu)) = N_{\alpha_2}(x)(\mu) \), for all \( x \in X \) and \( \mu \in L^X \).

Clearly, \( y \mapsto N_{\alpha_2}(y)(\mu) \) is the L-set \( \varphi_{1,2} \cdot \text{int} \mu \).

The characterized L-space \( (X, \varphi_{1,2} \cdot \text{int}) \) of all \( \varphi_{1,2} \)-open L-subsets of a set \( X \) is characterized as a filter pre L-topology ([2]), that is, as a mapping \( N_{\alpha_2}(x) : X \rightarrow \mathcal{F}(X) \) such that the axioms (N1) to (N3) are fulfilled.

**\( \varphi_{1,2} \)-L-neighborhoods.** Let \( (X, \tau) \) be a topological L-spaces and \( \varphi_1, \varphi_2 \in O_{(L^X, \tau)} \). Then for each \( \alpha \in L_0 \) and each \( x \in X \), the L-set \( \mu \in L^X \) will be called \( \varphi_{1,2} \)-L-neighborhood at \( x \) if \( \alpha \leq (\varphi_{1,2} \cdot \text{int} \mu)(x) \) holds.

Because of Proposition 2.1, the L-set \( \mu \in L^X \) is \( \varphi_{1,2} \)-L-neighborhood at \( x \) if and only if \( \mu \in \alpha\cdot \text{pr} N_{\alpha_2}(x) \), where \( N_{\alpha_2}(x) \) be given by (2.3). For each \( \alpha \in L_0 \) and each \( x \in X \) let \( N_\alpha(x) = \{ \mu \in L^X : \alpha \leq (\varphi_{1,2} \cdot \text{int} \mu)(x) \} \), then the family \( (N_\alpha(x))_{\alpha \in L_0} \) is the large valued L-filter base of \( N_{\alpha_2}(x) \).

**\( \varphi_{1,2} \)-L-convergence.** Let a topological L-spaces \( (X, \tau) \) be fixed and \( \varphi_1, \varphi_2 \in O_{(L^X, \tau)} \). If \( x \) is a point in the characterized L-space \( (X, \varphi_{1,2} \cdot \text{int}) \), \( F \subseteq X \) and \( \mathcal{M} \) is L-filter on \( X \). Then \( \mathcal{M} \) is said to be \( \varphi_{1,2} \)-L-convergence ([2]) to \( x \) and written \( \mathcal{M} \xrightarrow{\varphi_{1,2} \cdot \text{int}} x \), provided \( \mathcal{M} \) is finer than the \( \varphi_{1,2} \) -neighborhood filter \( N_{\alpha_2}(x) \). Moreover, \( \mathcal{M} \) is said to be \( \varphi_{1,2} \)-convergence to \( F \) and written \( \mathcal{M} \xrightarrow{\varphi_{1,2} \cdot \text{int}} F \), provided \( \mathcal{M} \) is finer than the \( \varphi_{1,2} \)-L-neighborhood filter \( N_{\alpha_2}(F) \).

**\( \varphi_{1,2} \)-closure L-sets.** Let a topological L-space \( (X, \tau) \) be fixed and \( \varphi_1, \varphi_2 \in O_{(L^X, \tau)} \). The \( \varphi_{1,2} \)-closure of the L-set \( \mu : X \rightarrow L \) is the mapping \( \varphi_{1,2} \cdot \text{cl} \mu : X \rightarrow L \) defined by:

\[
(\varphi_{1,2} \cdot \text{cl} \mu)(x) = \bigvee_{\mu \in N_{\alpha_2}(x)} \mathcal{M}(\mu)
\]

for all \( x \in X \). The L-filter \( \mathcal{M} \) my have additional properties, e.g, we may assume that is homogeneous or even that is ultra. Obviously, \( \varphi_{1,2} \cdot \text{cl} \mu \geq \mu \) holds for all \( \mu \in L^X \).

**\( \varphi_{1,2} \)-\( \psi_{1,2} \)-L-continuous and \( \varphi_{1,2} \)-\( \psi_{1,2} \)-L-open mappings.** In the following let a topological L-spaces \( (X, \tau_1) \) and \( (Y, \tau_2) \) are fixed, \( \varphi_1, \varphi_2 \in O_{(L^X, \tau_1)} \) and \( \psi_1, \psi_2 \in O_{(L^Y, \tau_2)} \). The mapping \( f : (X, \varphi_{1,2} \cdot \text{int}) \rightarrow (Y, \psi_{1,2} \cdot \text{int}) \) is said to be \( \varphi_{1,2} \)-\( \psi_{1,2} \)-L-continuous ([2]) if and only if

\[
(\psi_{1,2} \cdot \text{cl} \eta) \circ f \leq \varphi_{1,2} \cdot \text{int} (\eta \circ f)
\]

holds for all \( \eta \in L^Y \). If an order reversing involution \( \alpha \mapsto \alpha' \) of \( L \) is given, then we have that \( f \) is \( \varphi_{1,2} \)-\( \psi_{1,2} \)-L-continuous if and only if \( \varphi_{1,2} \cdot \text{cl} (\eta \circ f) \leq (\psi_{1,2} \cdot \text{cl} \eta) \circ f \) for all \( \eta \in L^Y \), where \( \varphi_{1,2} \cdot \text{cl} \) and \( \psi_{1,2} \cdot \text{cl} \) are the closure operators related to \( \varphi_{1,2} \cdot \text{int} \) and \( \psi_{1,2} \cdot \text{int} \), respectively. Obviously if \( f \) is \( \varphi_{1,2} \)-\( \psi_{1,2} \)-L-continuity mapping, then the inverse mapping \( f^{-1} : (Y, \psi_{1,2} \cdot \text{int}) \rightarrow (X, \varphi_{1,2} \cdot \text{int}) \) is
\( \psi_{1,2} \phi_{1,2} \) L-continuous mapping, that is, \((\psi_{1,2} \text{ int} \mu) \circ f^{-1} \leq \psi_{1,2} \text{ int} (\mu \circ f^{-1}) \) holds for all \( \mu \in L^X \). By means of the \( \phi_{1,2} \) L-neighborhood filter \( \mathcal{N}_{\phi_{1,2}} (x) \) of \( \phi_{1,2} \text{ int} \) at \( x \) and the \( \psi_{1,2} \) L-neighborhood filter \( \mathcal{N}_{\psi_{1,2}} (x) \) of \( \psi_{1,2} \text{ int} \) at \( x \), the \( \phi_{1,2} \psi_{1,2} \) L-continuity of \( f \) is also characterized as follows:

A mapping \( f : (X , \phi_{1,2} \text{ int}) \rightarrow (Y , \psi_{1,2} \text{ int}) \) is \( \phi_{1,2} \psi_{1,2} \) L-continuous if for each \( x \in X \) the inequality

\[
\mathcal{N}_{\psi_{1,2}} (f (x)) \supseteq \mathcal{F}_f (\mathcal{N}_{\phi_{1,2}} (x))
\]

holds. Obviously, in the case of \( L = \{0,1\} \), \( \phi_1 = \psi_1 = \text{ int} \), \( \phi_2 = 1_{L^X} \) and \( \psi_2 = 1_{L^X} \), the \( \phi_{1,2} \psi_{1,2} \) L-continuity of \( f \) coincides with the usual L-continuity.

**Proposition 2.3** [2] Let \( f : (X , \phi_{1,2} \text{ int}) \rightarrow (Y , \psi_{1,2} \text{ int}) \) be a mapping between the characterized L-spaces \((X , \phi_{1,2} \text{ int}) \) and \((Y , \psi_{1,2} \text{ int}) \). Then the following are equivalent:

1. \( f \) is \( \phi_{1,2} \psi_{1,2} \) L-continuous.
2. For each \( \phi_{1,2} \) L-filter \( \mathcal{M} \) on \( X \) and each \( x \in X \) such that \( \mathcal{M} \rightarrow_{\phi_{1,2} \text{ int}} x \) we have \( \mathcal{F}_f (\mathcal{M}) \rightarrow_{\psi_{1,2} \text{ int}} f (x) \).
3. For each \( x \in X \), \( \alpha \in L_0 \) and \( \psi_{1,2} \alpha \) L-neighborhood \( \eta \) at \( f (x) \), we have \( \eta \circ f \) is an \( \phi_{1,2} \alpha \) L-neighborhood at \( x \).
4. \( f^{-1}(\eta) \in \beta_{\phi_{1,2} \text{ int}} \) \( \forall \eta \in \beta_{\psi_{1,2} \text{ int}} \), where \( \beta_{\phi_{1,2} \text{ int}} \) and \( \beta_{\psi_{1,2} \text{ int}} \) are the bases of \((X , \phi_{1,2} \text{ int}) \) and \((Y , \psi_{1,2} \text{ int}) \), respectively.

We will denote by **CRL-Sp, SCRL-Sp** and **CR-Sp** to the categories of all characterized L-spaces, stratified characterized L-spaces and the ordinary characterized spaces with the \( \phi_{1,2} \psi_{1,2} \) L-continuity and \( \phi_{1,2} \psi_{1,2} \) \(-\) continuity as a morphisms between them, respectively. The objects in these categories are characterized L-spaces, stratified characterized L-spaces and characterizat spaces will be denoted by \((X , \phi_{1,2} \text{ int}) \), \((X , \phi_{1,2} \text{ int}^S) \) and \((X , \phi_{1,2} \text{ int}_\alpha) \), respectively.

The mapping \( f : (X , \phi_{1,2} \text{ int}) \rightarrow (Y , \psi_{1,2} \text{ int}) \) is said to be \( \phi_{1,2} \psi_{1,2} \) L-open if and only if

\[
f \circ (\phi_{1,2} \text{ int} \mu) \circ f \leq \psi_{1,2} \text{ int} (f \circ \mu)
\]

holds for all \( \mu \in L^X \). If an order reversing involution \( \alpha \mapsto \alpha' \) of \( L \) is given, then we have that \( f \) is \( \phi_{1,2} \psi_{1,2} \) L-open if and only if \( \phi_{1,2} \text{ cl} (f \circ \mu) \leq f \circ (\psi_{1,2} \text{ cl} \mu) \) for all \( \mu \in L^X \). The mapping \( f : (X , \phi_{1,2} \text{ int}) \rightarrow (Y , \psi_{1,2} \text{ int}) \) is said to be \( \phi_{1,2} \psi_{1,2} \) L-homeomorphism if and only if it is bijective \( \phi_{1,2} \psi_{1,2} \) L-continuous and \( \phi_{1,2} \psi_{1,2} \) L-open mapping.

**Proposition 2.4** [1] Let \( f : (X , \phi_{1,2} \text{ int}) \rightarrow (Y , \psi_{1,2} \text{ int}) \) be a mapping between the characterized L-spaces \((X , \phi_{1,2} \text{ int}) \) and \((Y , \psi_{1,2} \text{ int}) \). Then the following are equivalent:

1. \( f \) is \( \phi_{1,2} \psi_{1,2} \) L-open.
2. For each \( \phi_{1,2} \) L-filter \( \mathcal{N} \) on \( Y \) and each \( y \in Y \) such that \( \mathcal{N} \rightarrow_{\phi_{1,2} \text{ int}} y \) we have \( \mathcal{F}_f (\mathcal{N}) \rightarrow_{\psi_{1,2} \text{ int}} f^{-1}(y) \), where \( \mathcal{F}_f (\mathcal{N}) \) is the preimage of \( \mathcal{N} \).
3. For each \( y \in Y \), \( \alpha \in L_0 \) and \( \phi_{1,2} \alpha \) L-neighborhood \( \mu \) at \( f^{-1}(y) \), we have \( \mu \circ f^{-1} \) is an \( \psi_{1,2} \alpha \) L-neighborhood at \( y \).
(4) $f(\mu) \in \psi_{\phi_1,\phi_2}(Y)$ for all $\mu \in \beta_{\phi_1,\phi_2}$, where $\beta_{\phi_1,\phi_2}$ is a base of $(X, \phi_{1,2})$.

**Characterized L-topological groups.** In the following let $G$ be a multiplicative group. We denote, as usual, the identity element of $G$ by $e$ and the inverse of $x$ in $G$ by $x^{-1}$. Consider $\tau$ is an L-topology on $G$ and $\phi_1, \phi_2 \in O_{(e^x, e^y)}$. Then the pair $(G, \phi_{1,2})$ will be called an characterized L-topological group ([1]) if and only if the mappings:

$$\alpha : (G \times G, \phi_{1,2}) \rightarrow (G, \phi_{1,2})$$

and

$$\beta : (G, \phi_{1,2}) \rightarrow (G, \phi_{1,2})$$

that defined by:

$$\alpha((x, y)) = x \cdot y \quad \forall (x, y) \in G \times G$$

and

$$\beta(x) = x^{-1} \quad \forall x \in G$$

are $\phi_{1,2}$ L-continuous, respectively.

If $\phi_1 = \text{int}$ and $\phi_2 = 1$, then the characterized L-topological group $(G, \phi_{1,2})$ is coincide with the L-topological group $(G, \tau)$ which is defined in [6,8]. As shown in [1], the characterized L-topological groups are characterized by an equivalent definition as will as in the following proposition:

**Proposition 2.5** Let $G$ be a multiplicative group, $\tau$ is an L-topology on $G$ and $\phi_1, \phi_2 \in O_{(e^x, e^y)}$. Then,

$$(G, \phi_{1,2})$$

is characterized L-topological group if and only if the mapping

$$\gamma : (G \times G, \phi_{1,2}) \rightarrow (G, \phi_{1,2})$$

which is defined by:

$$\gamma(x, y) = x \cdot y^{-1} \quad \text{for all } (x, y) \in G$$

is $\phi_{1,2}$ L-continuous.

Denote by $\text{CRL-TopGrp}$ and $\text{CR-TopGrp}$ for the categories of all characterized L-topological groups and all characterized topological groups with all the $\phi_{1,2}$ L-continuous homeomorphisms and with all the $\phi_{1,2}$ L-continuous homomorphism as morphisms mappings between them, respectively. As shown in [1], the category $\text{CRL-TopGrp}$ is concrete category over the category $\text{Grp}$ of all groups.

### 3. Initial and final characterized L-spaces

We make at first the relation between the farness on L-sets and the finer relation between characterized spaces to define the $\alpha$-level and initial characterized spaces for an L-topological space $(X, \tau)$ by means of the functors $\omega$ and $\iota$. For an ordinary topological space $(X, T)$, the induced characterized L-space is also introduced by using the functor $\omega$. The functors $\omega$ and $\iota$ are extended for any complete distributive lattice $L$ to the functors $\omega_L$ and $\iota_L$. We further notions related to the notion of characterized L-spaces are e.g. those of characterized L-subspace, characterized product L-space, characterized quotient L-space and characterized sum L-space are investigated as special cases from the notions of initial and final characterized L-spaces. By the initial and final lefts in $\text{CRL-Sp}$ we show that the category $\text{CRL-Sp}$ is topological category in sense of [7,19] and it is also complete and co-complete category, that is, all limits and all co-limits in $\text{CRL-Sp}$ exist, which of course are unique up to isomorphisms. Moreover, the category $\text{SCRL-Sp}$ is bireflective subcategory of the category $\text{CRL-Sp}$ and it is also topological category ([1]). Spacial cases we already described using the standard specifications, namely the characterized product and coproduct L-spaces. The latter type here is called characterized sum L-space. According to general procedure [6,12], the characterized L-subspaces together with their related inclusion mappings and the characterized quotient L-spaces together with their related canonical surjections are the equalizers and co-equalizers, respectively in $\text{CRL-Sp}$. 


Let \((X, \tau)\) be a topological L-space and \(\varphi_1, \varphi_2 \in O_{(L^X, \tau)}\). Then the \(\alpha\)-level and the initial characterized spaces ([1]) of the characterized L-space \((X, \varphi_{1,2}.\text{int}_\alpha)\) will be denoted by \((X, \varphi_{1,2}.\text{int}_\alpha)\) and \((X, \varphi_{1,2}.\text{int}_\alpha)\), respectively where \(\varphi_{1,2}.\text{int}_\alpha\) and \(\varphi_{1,2}.\text{int}_\alpha\) are the \(\varphi_{1,2}\)-interior operators generates the two classes \((\varphi_{1,2}.OF(X))_\alpha\) and \((\varphi_{1,2}.OF(X))_\alpha\) which are given by 
\[
(\varphi_{1,2}.OF(X))_\alpha = \{S_\alpha(\mu) \in P(X) : \mu \in \varphi_{1,2}.OF(X)\}
\]
\[
(\varphi_{1,2}.OF(X))_\alpha = \inf\{(\varphi_{1,2}.OF(X))_\alpha : \alpha \in L_1\},
\]
respectively, where inf is the infimum with respect to the finer relation on characterized spaces. On other hand if \((X, T)\) is ordinary topological space and \(\varphi_1, \varphi_2 \in O_{(P(X), T)}\), then the induced characterized L-space on \(X\) ([1]) will be denoted by \((X, \varphi_{1,2}.\text{int}_\alpha)\), where \(\varphi_{1,2}.\text{int}_\alpha\) is the \(\varphi_{1,2}\)-interior operator generates the class 
\[
\omega(\varphi_{1,2}.O(X)) \quad \text{which is defined as follows:}
\]
\[
\omega(\varphi_{1,2}.O(X)) = \{\mu \in L^X : S_\alpha(\mu) \in \varphi_{1,2}.O(X) \text{ for all } \alpha \in L_1\}.
\]
\(\omega\) and \(i\) are functors in sense of Lowen in [17] in special case of \(L = I\). These functors extended for any completely distributive complete lattice \(L\) in [1] as follows:

Let \((X, \tau)\) be a topological L-space, \(\varphi_1, \varphi_2 \in O_{(L^X, \tau)}\) and \(\psi_1, \psi_2 \in O_{(L^X, T)}\). Then, the characterized spaces \((X, \varphi_{1,2}.\text{int}_i)\) and \((X, \varphi_{1,2}.\text{int}_o)\) are called initial characterized space and induced characterized L-space on \(X\), respectively where \(\varphi_{1,2}.\text{int}_i\) and \(\varphi_{1,2}.\text{int}_o\) are the \(\varphi_{1,2}\)-interior operators generates the classes 
\[
i_L(\varphi_{1,2}.OF(X)) \quad \text{and} \quad \omega_o(\varphi_{1,2}.O(X))
\]
which are defined by the formulas:
\[
i_L(\varphi_{1,2}.OF(X)) = \inf\{\mu^{-1}(UP(\psi_{1,2}.OF(L)) : \mu \in \varphi_{1,2}.OF(X)\}
\]
and
\[
\omega_o(\varphi_{1,2}.O(X)) = \langle C(\langle(X, \varphi_{1,2}.O(X)), (L, UP(\psi_{1,2}.OF(L))\rangle) \rangle
\]
\(C(\langle(X, \varphi_{1,2}.O(X)), (L, UP(\psi_{1,2}.OF(L))\rangle)\) is the set of all \(\varphi_{1,2}\)-continuous mappings between \((X, \varphi_{1,2}.O(X))\) and \((L, UP(\psi_{1,2}.OF(L))\), where \(UP(\psi_{1,2}.OF(L))\) is the upper \(\psi_{1,2}\)-open L-set generated by the set \(L \downarrow \downarrow (a)\) for \(\downarrow (a) = \{x \in L : x \leq a\}\). If \(\varphi_1 = \text{int}\) and \(\varphi_2 = 1_{L^X}\), then the initial characterized space \((X, \varphi_{1,2}.\text{int}_i)\) and the induced characterized L-space \((X, \varphi_{1,2}.\text{int}_o)\) are coincide with the initial topological space \((X, i(\tau))\) and the induced topological L-space \((X, \omega(\tau))\) which are defined in [8]. As shown in [1], the functors \(\omega_o : \text{CR - Sp} \to \text{CRL - Sp}\), \(i_L : \text{CRL - Sp} \to \text{CR - Sp}\) and \(S_2 : \text{CRL - Sp} \to \text{SCRL - Sp}\) are concrete functors. Moreover, the category \(\text{SCRL - Sp}\) is bireflective subcategory of the category \(\text{CRL - Sp}\) and for each object \((X, \varphi_{1,2}.\text{int})\) of \(\text{CRL - Sp}\) the \(\varphi_{1,2}\)-L-continuous mapping \(1_X\) from the stratification \((X, \varphi_{1,2}.\text{int}^S)\) of \((X, \varphi_{1,2}.\text{int})\) into \((X, \varphi_{1,2}.\text{int})\) is bi-coreflection of \((X, \varphi_{1,2}.\text{int}).\)

Initial characterized L-spaces. Consider a family of characterized L-spaces \((\langle X_i, \psi_{1,2}.\text{int}_i \rangle)_{i \in I}\) and for each \(i \in I\), let \(f_i : X_i \to X_i\) be a mapping from \(X_i\) into \(X_i\). By an initial characterized L-space ([1]) of the family \((\langle X_i, \psi_{1,2}.\text{int}_i \rangle)_{i \in I}\) with respect to \((f_i)_{i \in I}\), we mean the characterized L-space \((X, \varphi_{1,2}.\text{int})\) for which the following conditions are fulfilled:

1. All the mappings \(f_i : (X, \varphi_{1,2}.\text{int}) \to (X_i, \psi_{1,2}.\text{int}_i)\) are \(\varphi_{1,2}\)-\(\psi_{1,2}\)-L-continuous.
(2) For an characterized L-space \((Y, \delta_{1,2}, \text{int})\) and a mapping \(f : Y \rightarrow X\), the mapping 
\(f : (Y, \delta_{1,2}, \text{int}) \rightarrow (X, \phi_{1,2}, \text{int})\) is \(L\)-continuous if all the mappings 
\(f_i \circ f : (Y, \delta_{1,2}, \text{int}) \rightarrow (X, \psi_{1,2,\text{int}_i})\) are \(L\)-continuous for all \(i \in I\).

The initial characterized L-space \((X, \phi_{1,2}, \text{int})\) for a family \(((X_i, \psi_{1,2,\text{int}_i}))_{i \in I}\) of characterized L-spaces with respect to the family \((f_i)_{i \in I}\) of mappings exists and will be given by
\[\phi_{1,2,\text{int}} \mu = \bigvee_{\mu_i \leq \mu, s \in \mathcal{C}_l} (\psi_{1,2,\text{int}_i} \circ f_i)\] 
for all \(\mu \in L^X\).

As shown in [1], the initial lefts and then the initial characterized L-spaces are uniquely exist in the category CRL-Sp. Hence, the category CRL-Sp is topological category over the category SET of all sets. Moreover, the initial characterized L-space \((X, \phi_{1,2}, \text{int})\) for a family of characterized L-spaces \(((X_i, \psi_{1,2,\text{int}_i}))_{i \in I}\) with respect to a family of mappings \((f_i)_{i \in I}\) is stratified if and only if \((X_i, \psi_{1,2,\text{int}_i})\) is stratified for some \(i \in I\). In the following we consider some special cases for the initial characterized L-spaces.

**Characterized L-subspaces.** Let \(A\) be non-empty subset of a characterized L-space \((X, \phi_{1,2}, \text{int})\) and 
\(i_A : A \rightarrow X\) be the inclusion mapping of \(A\) into \(X\). Then the mapping \(\phi_{1,2,\text{int}_A} : L^A \rightarrow L^X\) which is defined by:
\[\phi_{1,2,\text{int}_A} \sigma = \bigvee_{\mu \leq \sigma} (\phi_{1,2,\text{int}} \circ i_A)\] 
for all \(\sigma \in L^A\) is initial \(\phi_{1,2,\text{int}}\)-operator of \(\phi_{1,2,\text{int}}\) with respect to the inclusion mapping \(i_A : A \rightarrow X\), called the induced \(\phi_{1,2,\text{int}}\)-operator of \(\phi_{1,2,\text{int}}\) on the subset \(A\) of \(X\) and \((A, \phi_{1,2,\text{int}_A})\) is initial characterized L-space called characterized L-subspace \((1)\) of the characterized L-space \((X, \phi_{1,2,\text{int}})\). As shown in [1], the characterized L-subspaces \((A, \phi_{1,2,\text{int}_A})\) of the characterized L-spaces \((X, \phi_{1,2,\text{int}})\) always exist and the related initial \(\phi_{1,2,\text{int}}\)-operator of them is given by (3.2). Moreover, \((A, \phi_{1,2,\text{int}_A})\) is stratified if \((X, \phi_{1,2,\text{int}})\) is stratified.

**Characterized product L-spaces.** Assume that for each \(i \in I\), \((X_i, \psi_{1,2,\text{int}_i})\) be the characterized L-space of \(\psi_{1,2}\)-open \(L\)-subset of \(X_i\). Let \(X\) be the cartesian product \(\prod_{i \in I} X_i\) of the family \((X_i)_{i \in I}\) and \(P : X \rightarrow X_i\) is the related projection. Then the mapping \(\phi_{1,2,\text{int}} : L^X \rightarrow L^X\) which is defined by:
\[\phi_{1,2,\text{int}} \mu = \bigvee_{\mu \leq \mu_i} (\psi_{1,2,\text{int}_i} \circ P)\] 
for all \(\mu \in L^X\) is initial \(\phi_{1,2,\text{int}}\)-operator of \(\psi_{1,2,\text{int}}\) with respect to the projection mapping \(P : X \rightarrow X_i\), called the \(\phi_{1,2}\)-product operator of the \(\psi_{1,2}\)-interior operators \(\psi_{1,2,\text{int}_i}\) and \((X, \phi_{1,2,\text{int}})\) is initial characterized L-space called characterized product L-space \((1)\) of the characterized L-spaces \((X_i, \psi_{1,2,\text{int}_i})\) with respect to the family \((P : X \rightarrow X_i)_{i \in I}\) of projections and will be denoted by \((\prod_{i \in I} X_i, \prod_{i \in I} \psi_{1,2,\text{int}_i})\).

**Initial lefts in CRL-Sp.** For the general notion of initial left we refer the standard books of category theory which include the categorical topology, e.g. [7,19]. The notion of initial left is meant here with respect to the forgetful functor of CRL-Sp to SET. It can be defined as follows:
The family of one and the same domain \( (f_i : (X, \varphi_{1,2}, \text{int}) \rightarrow (X_{i1}, \psi_{1,2}, \text{int}))_{i \in I} \), where I is any classe in the category CRL-Sp is called initial left ([1]) of the family \( (f_i : X \rightarrow X_{i1}, \psi_{1,2}, \text{int})_{i \in I} \) provided for any characterized L-space \((Y, \sigma_{1,2}, \text{int})\) of the \( \sigma_{1,2} \)-open L-subsets of the set \( Y \), the mapping \( f : (Y, \sigma_{1,2}, \text{int}) \rightarrow (X, \varphi_{1,2}, \text{int}) \) is \( \sigma_{1,2} \varphi_{1,2} \) L-continuous if all the compositions \( f_i \circ f : (Y, \sigma_{1,2}, \text{int}) \rightarrow (X_{i1}, \psi_{1,2}, \text{int}) \) are \( \sigma_{1,2} \psi_{1,2} \) L-continuous. As shown in [1], for each family \( (f_i : X \rightarrow X_{i1}, \psi_{1,2}, \text{int})_{i \in I} \) of the mappings \( f_i : X \rightarrow X_{i1} \) and of \( \psi_{1,2} \)-interior operators \( \psi_{1,2}, \text{int}_i \) defined on the co-domains \( X_{i1} \) of these mappings, the family \( (f_i : (X, \varphi_{1,2}, \text{int}) \rightarrow (X_{i1}, \psi_{1,2}, \text{int}))_{i \in I} \) is initial left, where the initial \( \varphi_{1,2} \)-interior operator \( \varphi_{1,2}, \text{int} \) defined by (3.1).

**Lemma 3.1** [1] Let \((X, \varphi_{1,2}, \text{int})\) and \((Y, \sigma_{1,2}, \text{int})\) are the characterized product L-spaces for the families \(( (X, \varphi_{1,2}, \text{int}) )_{i \in I} \) and \(( (Y, \sigma_{1,2}, \text{int}) )_{i \in I} \) of characterized L-spaces. Then for each \( i \in I \), the mapping \( f_i : (X, \varphi_{1,2}, \text{int}) \rightarrow (Y, \sigma_{1,2}, \text{int}) \) is \( \sigma_{1,2} \psi_{1,2} \) L-continuous (resp. \( \sigma_{1,2} \delta_{1,2} \) L-open) mapping, and then the product mapping \( f = \prod f_i : (X, \varphi_{1,2}, \text{int}) \rightarrow (Y, \sigma_{1,2}, \text{int}) \) is defined by \( f((x_i)_{i \in I}) = (f_i(x_i))_{i \in I} \) for all \((x_i)_{i \in I} \in X = \prod X_i \) is \( \varphi_{1,2} \sigma_{1,2} \) L-continuous (resp. \( \varphi_{1,2} \sigma_{1,2} \) L-open).

**Final characterized L-spaces.** It is well-known (cf.e.g [7,19]) that in a topological category all final lifts uniquely exists and hence also all final structures exist. They are dually defined. In case of the category CRL-Sp the final structures can easily be given, as is shown in the following:

Let \( I \) be a class and for each \( i \in I \), let \((X_i, \psi_{1,2}, \text{int}_i)\) be a characterized L-space of \( \psi_{1,2} \)-open L-subsets of \( X_i \) and \( f_i : X_i \rightarrow X \) be a mapping from \( X_i \) into a set \( X \). By a final characterized L-space of \((X_i, \psi_{1,2}, \text{int}_i)_{i \in I} \) with respect to the family \((f_i)_{i \in I} \) of mappings we mean the characterized L-space \((X, \varphi_{1,2}, \text{int})\) for which the following conditions are fulfilled:

1. All the mappings \( f_i : (X_i, \psi_{1,2}, \text{int}_i) \rightarrow (X, \varphi_{1,2}, \text{int}) \) are \( \psi_{1,2} \varphi_{1,2} \) L-continuous.
2. For an characterized L-space \((Y, \delta_{1,2}, \text{int})\) and a mapping \( f : X \rightarrow Y \), the mapping \( f : (X, \varphi_{1,2}, \text{int}) \rightarrow (Y, \delta_{1,2}, \text{int}) \) is \( \varphi_{1,2} \delta_{1,2} \) L-continuous if all the mappings \( f \circ f_i : (X_i, \psi_{1,2}, \text{int}_i) \rightarrow (Y, \delta_{1,2}, \text{int}) \) are \( \psi_{1,2} \delta_{1,2} \) L-continuous for all \( i \in I \),

\[
\begin{align*}
X &\xrightarrow{f} Y \\
f_i &\uparrow
\end{align*}
\]

(See Fig. 3.1)

In the following proposition we show that the final characterized L-space \((X, \varphi_{1,2}, \text{int})\) for a family \(( (X_i, \psi_{1,2}, \text{int}_i) )_{i \in I} \) of characterized L-spaces with respect to the family \((f_i)_{i \in I} \) of mappings exists and will be defined.

**Proposition 3.1** The final characterized L-space \((X, \varphi_{1,2}, \text{int})\) for the family of characterized L-spaces \(( (X_i, \psi_{1,2}, \text{int}_i) )_{i \in I} \) with respect to the family of mappings \((f_i)_{i \in I} \) always exists and it is given by:

\[
(\varphi_{1,2}, \text{int}) \mu(x) = \bigwedge_{x_i \in f_i^{-1} \{x\}, i \in I} \psi_{1,2}, \text{int}_i (\mu \circ f_i)(x_i) \wedge \mu(x)
\]  

(3.4)
for all \( x \in X \) and \( \mu \in L^X \).

**Proof.** Let \( \varphi_{1,2}.\text{int} \) be the operator defined (3.4). For each \( x \in X \), \( \mu \in L^X \) and for all \( i \in I \) with \( x_i \in f_i^{-1}(\{x\}) \) we have \( \bigwedge_{x_i, \sigma f_i^{-1}(\{x\}), i, d} \psi_{1,2}.\text{int} \left( \mu \circ f_i \right)(x_i) \wedge \mu(x) \geq \mu(x) \) and therefore \( \varphi_{1,2}.\text{int} \mu \leq \mu \). Hence, \( \varphi_{1,2}.\text{int} \) fulfills condition (1). For condition (12), let \( \mu, \eta \in L^X \) with \( \mu \leq \eta \), then \( (\mu \circ f_i)(x) \geq (\eta \circ f_i)(x) \) and therefore \( \left( \varphi_{1,2}.\text{int} \mu \right)(x) \geq \bigwedge_{x_i, \sigma f_i^{-1}(\{x\}), i, d} \psi_{1,2}.\text{int} \left( \mu \circ f_i \right)(x_i) \wedge \mu(x) \geq \left( \varphi_{1,2}.\text{int} \eta \right)(x) \) holds for all \( x \in X \). Thus, condition (12) is fulfilled. For all \( x \in X \), \( i \in I \) with \( x_i \in f_i^{-1}(\{x\}) \) we have \( \bigwedge_{x_i, \sigma f_i^{-1}(\{x\}), i, d} \psi_{1,2}.\text{int} \left( \bar{T} \circ f_i \right)(x_i) \wedge \bar{T}(x) \leq \bar{T}(x) \) and therefore \( \varphi_{1,2}.\text{int} \bar{T} = \bar{T} \). Hence, \( \varphi_{1,2}.\text{int} \) fulfill condition (13). Now, let \( \mu, \eta \in L^X \) and \( x \in X \), \( i \in I \) such that \( x_i \in f_i^{-1}(\{x\}) \). Then from the distributives of \( \text{L} \), we have that
\[
\left( \varphi_{1,2}.\text{int} \mu \wedge \varphi_{1,2}.\text{int} \eta \right)(x) = \bigwedge_{x_i, \sigma f_i^{-1}(\{x\}), i, d} \left( \psi_{1,2}.\text{int} \left( \mu \circ f_i \right) \wedge \psi_{1,2}.\text{int} \left( \eta \circ f_i \right) \right)(x_i) \wedge \left( \mu \wedge \eta \right)(x) \\
\geq \bigwedge_{x_i, \sigma f_i^{-1}(\{x\}), i, d} \psi_{1,2}.\text{int} \left( \left( \mu \wedge \eta \right) \circ f_i \right)(x_i) \wedge \left( \mu \wedge \eta \right)(x) \\
= \varphi_{1,2}.\text{int} \left( \mu \wedge \eta \right)(x).
\]
Thus, \( \varphi_{1,2}.\text{int} \) fulfills condition (14). Clearly, \( \varphi_{1,2}.\text{int} \) is idempotent, that is, condition (15) is fulfilled. Hence, \( (X, \mathcal{P}_{1,2}.\text{int}) \) is characterized \( \text{L}-\text{space} \). Since for all \( i \in I \) with \( f_i^{-1}(\{x\}) = \emptyset \), we have \( \left( \varphi_{1,2}.\text{int} \right)(x) = \mu(x) \). Then, because of (3.4) for each \( i \in I \) and \( x_i \in X_i \), we have that the inequality \( \left( \varphi_{1,2}.\text{int} \mu \right)(f_i(x_i)) \geq \psi_{1,2}.\text{int} \left( \mu \circ f_i \right)(x_i) \) holds and therefore, the inequality \( \left( \varphi_{1,2}.\text{int} \mu \right) \circ f_i \leq \psi_{1,2}.\text{int} \left( \mu \circ f_i \right) \) is also holds. Hence, for each \( i \in I \) all the mappings \( f_i : (X_i, \psi_{1,2}.\text{int}_i) \rightarrow (X, \varphi_{1,2}.\text{int}) \) are \( \varphi_{1,2}. \mathcal{P}_{1,2}. \text{int} \)-continuous. Thus, condition (1) is fulfilled.

Now, let \( (Y, \delta_{1,2}.\text{int}) \) is a characterized \( \text{L}-\text{space} \) and \( f : X \rightarrow Y \) be a mapping such that the mappings \( f \circ f_i : (X, \psi_{1,2}.\text{int}_i) \rightarrow (Y, \delta_{1,2}.\text{int}) \) are \( \psi_{1,2}. \delta_{1,2}. \text{int} \)-continuous for all \( i \in I \). Then, we have that \( \left( \delta_{1,2}.\text{int} \mu \right) \circ \left( f \circ f_i \right) \leq \psi_{1,2}.\text{int} \left( \mu \circ f \circ f_i \right) \) holds for all \( \mu \in L^Y \) and because of (3.4) we have that \( \left( \delta_{1,2}.\text{int} \mu \right)(f(x)) = \bigwedge_{x_i, \sigma f_i^{-1}(\{x_i\}), i, d} \psi_{1,2}.\text{int} \left( \mu \circ f \circ f_i \right)(x_i) \wedge \mu(f(x)) \geq \bigwedge_{x_i, \sigma f_i^{-1}(\{x_i\}), i, d} \psi_{1,2}.\text{int} \left( \mu \circ f \circ f_i \right)(x_i) \wedge \mu(f(x)) \) is also holds for all \( \mu \in L^Y \). Hence, the mapping \( f : (X, \mathcal{P}_{1,2}.\text{int}) \rightarrow (Y, \delta_{1,2}.\text{int}) \) is \( \mathcal{P}_{1,2}. \delta_{1,2}. \text{L} \)-continuous, that is, condition (2) is also fulfilled.

Consequently, \( (X, \mathcal{P}_{1,2}.\text{int}) \) is final characterized \( \text{L}-\text{space} \) of the family \( \{(X_i, \psi_{1,2}.\text{int}_i)\}_{i \in I} \) of characterized \( \text{L}-\text{spaces} \) with respect to \( (f_i)_{i \in I} \).  

Because of Proposition 3.1, all the final lefts and all the final characterized \( \text{L}-\text{spaces} \) are uniquely exist in the category \( \text{CRL-Sp} \) and hence \( \text{CRL-Sp} \) is a topological category over the category \( \text{SET} \) of all sets.

**Proposition 3.2** The final characterized \( \text{L}-\text{space} \) \( (X, \mathcal{P}_{1,2}.\text{int}) \) for the family of characterized \( \text{L}-\text{spaces} \) \( \{(X_i, \psi_{1,2}.\text{int}_i)\}_{i \in I} \) with respect to the family of mappings \( (f_i)_{i \in I} \) is stratified if and only if \( (X_i, \psi_{1,2}.\text{int}_i) \) is stratified for some \( i \in I \).
Proof. Assume that \((X_j, \psi_{1,2}, \text{int}_{j})\) is stratified for \(j \in I\). Then because of (3.4), we have that 
\[
(\varphi_{1,2}, \text{int}_{\alpha})(x) = \bigwedge_{x \in f^{-1}(\{\alpha\}) \in \mathcal{A}} \psi_{1,2}(\alpha_j \circ f_j)(x_j) \wedge \alpha(x) \leq \alpha(x)
\]
holds for all \(\alpha \in L\), where \(\varphi_{1,2}, \text{int}_{\alpha} = \overline{\alpha}\) are the constant mappings on \(X\) and \(X_j\) have value \(\alpha\) and \(\alpha_j\), respectively. Hence, \(\varphi_{1,2}, \text{int}_{\alpha} = \overline{\alpha}\) for all \(\alpha \in L\) and therefore \((X, \varphi_{1,2}, \text{int}_{\alpha})\) is stratified.

Conversely, let \((X', \varphi_{1,2}, \text{int}_{\alpha})\) is stratified, that is \(\varphi_{1,2}, \text{int}_{\alpha} = \overline{\alpha}\) for all \(\alpha \in L\). Then 
\[
\bigwedge_{x \in f^{-1}(\{\alpha\}) \in \mathcal{A}} \psi_{1,2}(\alpha_j \circ f_j)(x_j) \wedge \alpha(x) = \overline{\alpha}(x)
\]
holds for all \(x \in X\) and \(i \in I\). Hence, there is \(j \in I\) such that 
\[
\psi_{1,2}(\alpha_j)(x_j) \leq \overline{\alpha}(x) \quad \text{and} \quad \alpha(x) \leq (\alpha_j \circ f_j)(x_j) \leq \alpha_j(x_j),
\]
therefore \(\psi_{1,2}(\alpha_j) = \alpha_j\) for some \(j \in I\). Hence, \((X', \varphi_{1,2}, \text{int}_{\alpha})\) is stratified for \(j \in I\). \(\square\)

In the following we consider the notions of a characterized quotient pre L-space and a characterized sum L-space as special cases from the final characterized L-spaces.

Characterized quotient pre L-spaces. Let \(A\) be non-empty L-subset of the characterized L-space \((X, \varphi_{1,2}, \text{int}_{\alpha})\) and \(f : X \to A\) is a surjective mapping of \(X\) into \(A\). Then the mapping 
\[
\varphi_{1,2}, \text{int}_{f} : L^A \to L^A
\]
which is defined by:
\[
(\varphi_{1,2}, \text{int}_{\alpha})(\mu) = \bigwedge_{x \in f^{-1}(\{\alpha\}) \in \mathcal{A}} \varphi_{1,2}(\mu \circ f)(x)
\]
for all \(\alpha \in A\) and \(\mu \in L^A\) is final pre \(\varphi_{1,2}, \text{int}_{\alpha}\) -interior operator of \(\varphi_{1,2}, \text{int}_{\alpha}\) with respect to the mapping \(f : X \to A\) which is not idempotent, called the quotient pre \(\varphi_{1,2}, \text{int}_{\alpha}\) -interior operator of \(\varphi_{1,2}, \text{int}_{\alpha}\) on the L-subset \(A\) and \((A, \varphi_{1,2}, \text{int}_{f})\) is a final characterized L-space which is not idempotent called characterized quotient pre L-space of the characterized L-space \((X, \varphi_{1,2}, \text{int}_{\alpha})\).

Note that in this case \(\varphi_{1,2}, \text{int}_{\alpha}\) is idempotent but \(\varphi_{1,2}, \text{int}_{f}\) need not be. Even in the classical case of \(L = [0,1]\) with choices \(\varphi_{1,2} = \text{int} \text{ and } \varphi_{1,2} = 1_{x^*}\), we have that \(\varphi_{1,2}, \text{int}_{\alpha}\) is up to an identification the usual topology and \(\varphi_{1,2}, \text{int}_{f}\) is up to an identification the usual pretopology which need not be idempotent. An example is given in [12] (p.234).

Proposition 3.3 Let \(A\) be non-empty subset of a characterized L-space \((X, \varphi_{1,2}, \text{int}_{\alpha})\). Then the characterized quotient pre L-space \((A, \varphi_{1,2}, \text{int}_{f})\) of \((X, \varphi_{1,2}, \text{int}_{\alpha})\) always exists and the quotient \(\varphi_{1,2}, \text{int}_{f}\) -interior operator \(\varphi_{1,2}, \text{int}_{f}\) is given by (3.5). If \((X, \varphi_{1,2}, \text{int}_{f})\) is stratified, then \((A, \varphi_{1,2}, \text{int}_{f})\) also is.

Proof. Let \(a \in A\) and \(\mu \in L^A\) such that \(x \in f^{-1}(\{a\})\) holds, then 
\[
\bigwedge_{x \in f^{-1}(\{a\}) \in \mathcal{A}} \varphi_{1,2}(\mu \circ f)(x) \geq \mu(a)
\]
is also holds and therefore \(\varphi_{1,2}, \text{int}_{\alpha}(\mu) \leq \mu\) holds for all \(\mu \in L^A\). Hence, \(\varphi_{1,2}, \text{int}_{f}\) fulfills condition (11).

For condition (12), let \(a \in A\) and \(\mu, \eta \in L^A\) with \(\mu \leq \eta\) and \(x \in f^{-1}(\{a\})\), then because of (3.5) we have 
\[
(\varphi_{1,2}, \text{int}_{f})(\mu) = \bigwedge_{x \in f^{-1}(\{a\}) \in \mathcal{A}} \varphi_{1,2}(\mu \circ f)(x) \geq \bigwedge_{x \in f^{-1}(\{a\}) \in \mathcal{A}} \varphi_{1,2}(\eta \circ f)(x) = (\varphi_{1,2}, \text{int}_{f})(\eta)(a),
\]
Thus, condition (12) is fulfilled. Since \(\varphi_{1,2}, \text{int}_{f}(\mu) \leq \mu\) for all \(\mu \in L^X\), then we have 
\[
(\varphi_{1,2}, \text{int}_{f})(\mu)(a) = \bigwedge_{x \in f^{-1}(\{a\}) \in \mathcal{A}} \varphi_{1,2}(\mu \circ f)(x) \leq \bigwedge_{x \in f^{-1}(\{a\}) \in \mathcal{A}} (\overline{\mu} \circ f)(x) = \overline{\mu}(a).
\]
Hence, \(\varphi_{1,2}, \text{int}_{f}\) fulfills condition (13). Now, let \(\mu, \eta \in L^A\) and \(a \in A\) such that \(x \in f^{-1}(\{a\})\). Then from the distributives of L and (3.5), we have that
(\varphi_{1,2}.\text{int}_f \mu \land \varphi_{1,2}.\text{int}_f \eta)(a) = \bigwedge_{x \in f^{-1}(a)} (\varphi_{1,2}.\text{int}(\mu \circ f)(x) \land (\varphi_{1,2}.\text{int}(\eta \circ f)(x) )) \\
\geq \bigwedge_{x \in f^{-1}(a)} (\varphi_{1,2}.\text{int}(\mu \land \eta \circ f)(x) ) = \varphi_{1,2}.\text{int}_f (\mu \land \eta)(a).

Since \varphi_{1,2}.\text{int}_f is isotone, it follows \varphi_{1,2}.\text{int}_f \mu \land \varphi_{1,2}.\text{int}_f \eta = \varphi_{1,2}.\text{int}_f (\mu \land \eta). Thus, condition (I4) is also fulfilled. Hence, \((A, \varphi_{1,2}.\text{int}_f)\) is characterized pre-L-space. Since for all \(a \in A\) and \(\mu \in L^A\), we have \((\varphi_{1,2}.\text{int}_f \mu \circ f)(a) \geq \varphi_{1,2}.\text{int}(\mu \circ f)(a)\), then the mapping \(f : (X, \varphi_{1,2}.\text{int}) \rightarrow (A, \varphi_{1,2}.\text{int}_f)\) is \(\varphi_{1,2} \varphi_{1,2}\) L-continuous. Hence, condition (1) is fulfilled.

Now, let \((Y, \delta_{1,2}.\text{int})\) is a characterized pre-L-space and \(g : A \rightarrow Y\) is a surjective mapping such that the composition \(f \circ g : (A, \varphi_{1,2}.\text{int}_f) \rightarrow (Y, \delta_{1,2}.\text{int})\) is \(\varphi_{1,2} \delta_{1,2}\) L-continuous mapping. Then, the inequality 
\((\delta_{1,2}.\text{int}_f \mu \circ (f \circ g) \leq \varphi_{1,2}.\text{int}_f (\mu \circ f \circ g)\) holds for all \(\mu \in L^Y\), therefore because of (3.5), the inequality 
\((\varphi_{1,2}.\text{int}_f \sigma f)(a) = \bigwedge_{x \in f^{-1}(a)} (\varphi_{1,2}.\text{int}(\sigma \circ g \circ f)(x) ) \geq \bigwedge_{x \in f^{-1}(a)} (\delta_{1,2}.\text{int}(\mu \circ g \circ f)(x) ) \geq \delta_{1,2}.\text{int}_f (\sigma \circ f)(a)\) is also holds for all \(a \in A\) and \(\sigma \in L^A\). Hence, the mapping 
\(f : (Y, \delta_{1,2}.\text{int}) \rightarrow (A, \varphi_{1,2}.\text{int}_f)\) is \(\delta_{1,2} \varphi_{1,2}\) L-continuous, that is, condition (2) is also fulfilled. Consequently, \((A, \varphi_{1,2}.\text{int}_f)\) is initial characterized pre-L-space.

Finally, let \((X, \varphi_{1,2}.\text{int}_f)\) is stratified. Then, \(\varphi_{1,2}.\text{int}_f \tilde{\alpha} = \tilde{\alpha}\) for all \(\alpha \in L\) and therefore 
\(\bigwedge_{x \in f^{-1}(\alpha)} (\varphi_{1,2}.\text{int} \tilde{\alpha})(x) = \tilde{\alpha}(a)\), where \(\tilde{\alpha}\) and \(\tilde{\alpha}\) are the constant mappings on \(X\) and \(A\) respectively. Because of (3.5), we have \(\varphi_{1,2}.\text{int}_f \tilde{\alpha} = \tilde{\alpha}\) for all \(\alpha \in L\). Hence, \((A, \varphi_{1,2}.\text{int}_f)\) is stratified. \(\Box\)

Characterized sum L-spaces. Assume that for each \(i \in I\), \((X_i, \psi_{1,2}.\text{int}_i)\) be an characterized L-space of \(\psi_{1,2}\)-open \(L\) -subset of \(X_i\). Let \(X\) be the disjoint union \(\bigcup_{i \in d} (X_i \times \{i\})\) of the family \((X_i)_{i \in d}\) and for each \(i \in I\), let \(e_i : X_i \rightarrow X\) be the canonical injection of \(X_i\) into \(X\) given by \(e_i(x_i) = (x_i, i)\). Then the mapping \(\varphi_{1,2}.\text{int} : L^X \rightarrow L^X\) which is defined by:

\[(\varphi_{1,2}.\text{int} \mu)(a, i) = \psi_{1,2}.\text{int}_i (\mu \circ e_i)(a)\]  
(3.6)

for all \(i \in I\), \(a \in X_i\) and \(\mu \in L^X\) is final \(\varphi_{1,2}\)-interior operator of \((\psi_{1,2}.\text{int}_i)_{i \in d}\) with respect to the canonical injection \((e_i)_{i \in d}\). \(\varphi_{1,2}.\text{int}\) will be called a sum \(\varphi_{1,2}\)-interior operator of the \(\psi_{1,2}\)-interior operators \((\psi_{1,2}.\text{int}_i)_{i \in d}\) and will be denoted by \(\sum_{i \in d} \psi_{1,2}.\text{int}_i\). The pair \((X, \varphi_{1,2}.\text{int})\) is final characterized L-space called characterized sum L-space of the characterized L-spaces \((X_i, \psi_{1,2}.\text{int}_i)\) with respect to the family of canonical injection \((e_i)_{i \in d}\) and will be denoted by \(\sum_{i \in d} (X_i, \psi_{1,2}.\text{int}_i)\) or \((X, \varphi_{1,2}.\text{int})\) for shorts.

Proposition 3.4 For each \(i \in I\), let \((X_i, \psi_{1,2}.\text{int}_i)\) be a characterized L-space of \(\psi_{1,2}\)-open \(L\) -subset of \(X_i\). Then the characterized sum L-prespace \(\sum_{i \in d} (X_i, \psi_{1,2}.\text{int}_i)\) of \((X_i, \psi_{1,2}.\text{int}_i)\) always exists and the sum \(\varphi_{1,2}\)-interior operator \(\varphi_{1,2}.\text{int}\) is given by (3.6). If \((X_i, \psi_{1,2}.\text{int}_i)\) stratified for each \(i \in I\), then the characterized sum L-space \(\sum_{i \in d} (X_i, \psi_{1,2}.\text{int}_i)\) is also stratified.
Proof. The first part is similar to that of Proposition 3.3. For the second part, let $i \in I$, $a \in X_i$ and $\alpha \in L^X$ , where $X$ is the disjoint union $\bigsqcup_{i \in I} (X_i \times \{i\})$ of the family $\{(X_i)_{i \in I}\}$. Because of (3.6) we have
\[(\varphi_{i,2} \cdot \text{int} \overline{\alpha})(a,i) = \psi_{i,1,2} \cdot \text{int}_1 (\overline{\alpha} \cdot e_i)(a) = (\psi_{i,2} \cdot \text{int}_1 \overline{\alpha})(a,i) = \overline{\alpha}(a,i) \text{ and therefore } \varphi_{i,2} \cdot \text{int} \overline{\alpha} = \overline{\alpha}.
\]
Hence, $\sum_{i \in I} \left( (X_i, \psi_{i,1,2}) \right)^{\text{int}}_{i \in I}$ is stratified. □

Final lefts in CRL-Sp. For the general notion of initial and final left we refer the standard books of category theory which include the categoriological topology, e.g. [6,23]. The notion of final left is meant here with respect to the forgetful functor of CRL-Sp to SET. It can be defined as follows:
The family of one and the same co-domain $\left( f_{i,1} : (X_i, \psi_{i,1,2}) \rightarrow (X, \varphi_{i,1,2})\right)_{i \in I}$, where $I$ is any close of morphisms in the category CRL-Sp is called final left of the family $\left( f_{i,1} : (X_i, \psi_{i,1,2}) \rightarrow (Y, \varphi_{i,1,2})\right)_{i \in I}$ provided for any characterized L-space $\left( Y, \sigma_{i,1,2}\right)$ of $\sigma_{i,1,2}$-open subsets of $Y$ the mapping $f : (X, \varphi_{i,1,2}) \rightarrow (Y, \sigma_{i,1,2})$ is $\varphi_{i,1,2} \cdot \sigma_{i,1,2}$ L-continuous if all the compositions mappings $f \circ f_{i,1} : (X_i, \psi_{i,1,2}) \rightarrow (Y, \sigma_{i,1,2})$ are $\psi_{i,1,2} \cdot \sigma_{i,1,2}$ L-continuous.

Proposition 3.7 For each family $\left( f_{i,1} : (X_i, \psi_{i,1,2}) \rightarrow (Y, \sigma_{i,1,2})\right)_{i \in I}$ consisting of the mappings $f_{i,1} : X_i \rightarrow Y$ and of the $\psi_{i,1,2}$-interior operators $\psi_{i,1,2} \cdot \text{int}_{i,1}$. on the domains $X_i$ of these mappings, the family $\left( f_{i,1} : (X_i, \psi_{i,1,2} \cdot \text{int}_{i,1} \rightarrow (X, \varphi_{i,1,2} \cdot \text{int}_{i,1})\right)_{i \in I}$ with the final $\varphi_{i,1,2}$ -interior operator $\varphi_{i,1,2} \cdot \text{int} : L^X \rightarrow L^X$ of $\left( \psi_{i,1,2} \cdot \text{int}_{i,1}\right)_{i \in I}$ with respect to $\left( f_{i,1}\right)_{i \in I}$ defined by (3.4) is a final left.

Proof. Let a characterized L-space $\left( Y, \sigma_{i,1,2}\right)$ of $\sigma_{i,1,2}$-open subsets of $Y$ and a mapping $f : X \rightarrow Y$ be fixed. If all the mappings $f \circ f_{i,1} : (X_i, \psi_{i,1,2}) \rightarrow (Y, \sigma_{i,1,2})$ are $\psi_{i,1,2} \cdot \sigma_{i,1,2}$ L-continuous, that is, if $\left( \psi_{i,1,2} \cdot \text{int} \eta \right) \circ (f \circ f_{i,1}) \leq \psi_{i,1,2} \cdot \text{int}_{i,1}(\eta \circ f \circ f_{i,1})$ holds for all $\eta \in L^Y$, then because of (3.4), we have that $\left( \psi_{i,1,2} \cdot \text{int} \eta \right)(f(x)) = \bigwedge_{x \in f_{i,1}^{-1}(\{x\})} \psi_{i,1,2} \cdot \text{int}_{i,1}(\eta \circ f)(f_{i,1}(x)) \leq \psi_{i,1,2} \cdot \text{int} \eta \circ f(x) \leq \psi_{i,1,2} \cdot \text{int} \eta \circ f \circ f_{i,1}(x) \leq \psi_{i,1,2} \cdot \text{int} \eta \circ f \circ f_{i,1}(x)$ holds for all $x \in X$ and $\eta \in L^Y$. Hence, the mapping $f : (X, \varphi_{i,1,2}) \rightarrow (Y, \sigma_{i,1,2})$ is $\varphi_{i,1,2} \cdot \sigma_{i,1,2}$ L-continuous. Thus, the family $\left( f_{i,1} : (X_i, \varphi_{i,1,2}) \rightarrow (X_i, \psi_{i,1,2} \cdot \text{int}_{i,1})\right)_{i \in I}$ is a final left of $\left( \psi_{i,1,2} \cdot \text{int}_{i,1}\right)_{i \in I}$ with respect to $\left( f_{i,1}\right)_{i \in I}$. □

4. Initial characterized L-topological groups

In this section we show that the category CRL-TopGrp of all characterized L-topological groups is topological category over the category Grp of all groups and hence all initial characterized L-topological groups exist and can be characterized.

Consider a family of characterized L-topological groups $\left( (G_i, \psi_{i,1,2} \cdot \text{int}_{G_i})\right)_{i \in I}$ and for each $i \in I$, let $f_{i,1} : G \rightarrow G_i$ be a homomorphism mapping from a group $G$ into the groups $G_i$. Then for any characterized L-topological group $(G, \varphi_{i,1,2} \cdot \text{int}_G)$, the family $\left( f_{i,1} : (G, \varphi_{i,1,2} \cdot \text{int}_G) \rightarrow (G_i, \psi_{i,1,2} \cdot \text{int}_{G_i})\right)_{i \in I}$ is called initial lifts for the family $\left( f_{i,1} : G \rightarrow G_i, \psi_{i,1,2} \cdot \text{int}_{G_i}\right)_{i \in I}$ in the category CRL-TopGrp provided the following conditions are fulfilled:
(1) All the mappings $f_i : (G, \varphi_{1,2, \text{int}_G}) \to (G_i, \psi_{1,2, \text{int}_{G_i}})$ are $\varphi_{1,2, \text{L-int}}$-continuous homomorphism for all $i \in I$.

(2) For an characterized L-topological group $(H, \delta_{1,2, \text{int}_H})$ and a mapping $f : H \to G$, the mapping $f : (H, \delta_{1,2, \text{int}_H}) \to (G, \varphi_{1,2, \text{int}_G})$ is $\delta_{1,2, \text{L}}$-continuous homomorphism if and only if all the composition mappings $f_i \circ f : (H, \delta_{1,2, \text{int}_H}) \to (G_i, \psi_{1,2, \text{int}_{G_i}})$ are $\delta_{1,2, \text{L}}$-continuous.

Hence, by an initial characterized L-topological group we mean the characterized L-topological group which provides the initial lifts in the category CRL-TopGrp.

To prove that all initial lifts and all initial characterized L-topological groups exist in the category CRL-TopGrp we need to prove at first that in case of $f_i : G \to G_i$ is an injective homomorphism for each $i \in I$, and $\varphi_{1,2, \text{int}_G}$ is $\varphi_{1,2, \text{L-int}}$-operator for an initial characterized L-topology on a group $G$ of $(\psi_{1,2, \text{int}_{G_i}})_{i \in I}$, we get that $(G, \varphi_{1,2, \text{int}_G})$ is also characterized L-topological group. Now, we consider the case of $I$ being a singleton.

**Proposition 4.1** Let $(H, \delta_{1,2, \text{int}_H})$ be a characterized L-topological group and let $f : G \to H$ be an injective homomorphism from a group $G$ into $H$. Then the initial characterized L-space $(G, f^{-1}(\delta_{1,2, \text{int}_H}))$ of $(H, \delta_{1,2, \text{int}_H})$ with respect to $f$ is characterized L-topological group.

**Proof.** Let at first $\gamma_G : (G \times G, f^{-1}(\delta_{1,2, \text{int}_H}) \times f^{-1}(\delta_{1,2, \text{int}_H})) \to (G, f^{-1}(\delta_{1,2, \text{int}_H}))$ and $\gamma_H : (H \times H, \delta_{1,2, \text{int}_H} \times \delta_{1,2, \text{int}_H}) \to (H, \delta_{1,2, \text{int}_H})$ are the mappings defined by (2.8) and let $\eta \in \beta_{f^{-1}(\delta_{1,2, \text{int}_H})}$, where $\beta_{f^{-1}(\delta_{1,2, \text{int}_H})}$ is the base of $(G, f^{-1}(\delta_{1,2, \text{int}_H}))$ that generated by $f^{-1}(\delta_{1,2, \text{int}_H})$. Then, $\eta = f^{-1}(\rho)$ for some $\rho \in \beta_{\delta_{1,2, \text{int}_H}}$. Since $(H, \delta_{1,2, \text{int}_H})$ is characterized L-topological group, then $\gamma_H$ is $\delta_{1,2}$-$\delta_{1,2}$-L-continuous and therefore from Proposition 2.3, we have $\gamma_H^{-1}(\rho) \in \beta_{\delta_{1,2, \text{int}_H} \times \delta_{1,2, \text{int}_H}}$. Because of $f$ is an injective homomorphism, then for all $x, y \in G$ we have

$$
\gamma_G^{-1}(x, y) = (\rho \circ f \circ \gamma_G)(x, y) = (\rho \circ f)(x, y^2) = \rho(f(x)f(y)) = \gamma_H(f(x), f(y))
$$

that is, $\gamma_G^{-1}(x, y) = (f \times f)^{-1}(\gamma_H^{-1}(\rho))$. Since $(G, f^{-1}(\delta_{1,2, \text{int}_H}))$ is initial characterized L-space of $(H, \delta_{1,2, \text{int}_H})$ with respect to the mapping $f$, then $f : (G, f^{-1}(\delta_{1,2, \text{int}_H})) \to (H, \delta_{1,2, \text{int}_H})$ is $\delta_{1,2}$-$\delta_{1,2}$-L-continuous and from Lemma 3.1, it follows that the product mapping $f \times f : G \times G \to H \times H$ is $\delta_{1,2}$-$\delta_{1,2}$-L-continuous. Therefore, $(f \times f)^{-1}(\gamma_H^{-1}(\rho)) \in \beta_{f^{-1}(\delta_{1,2, \text{int}_H}) \times f^{-1}(\delta_{1,2, \text{int}_H})}$ and $\beta_{f^{-1}(\delta_{1,2, \text{int}_H}) \times f^{-1}(\delta_{1,2, \text{int}_H})} \subseteq (f \times f)^{-1}(\gamma_H^{-1}(\rho))$. Hence, $(f \times f)^{-1}(\gamma_H^{-1}(\rho)) \in \beta_{f^{-1}(\delta_{1,2, \text{int}_H}) \times f^{-1}(\delta_{1,2, \text{int}_H})}$, and therefore from Proposition 2.3 it follows that $\gamma_G$ is $\delta_{1,2}$-$\delta_{1,2}$-L-continuous. Hence, because of Proposition 2.5, $(G, f^{-1}(\delta_{1,2, \text{int}_H}))$ is characterized L-topological group. □
Generally we consider the case of $I$ is any class consists of more than one elements.

**Proposition 4.2** Let $\left((G_i, \psi_{1,2}.\text{int}_{G_i})\right)_{i \in I}$ be a family of characterized L-topological groups and for each $i \in I$, let $f_i : G \rightarrow G_i$ be an injective homomorphism from a group $G$ into a group $G_i$. If $(G, \varphi_{1,2}.\text{int}_G)$ is the initial characterized L-space of the family $\left((G_i, \psi_{1,2}.\text{int}_{G_i})\right)_{i \in I}$ with respect to the family $\left(f_i\right)_{i \in I}$, then $(G, \varphi_{1,2}.\text{int}_G)$ is characterized L-topological group.

**Proof.** Let at first the mappings $\gamma^G_i : (G \times G \times \varphi_{1,2}.\text{int}_G \times \varphi_{1,2}.\text{int}_G) \rightarrow (G, \varphi_{1,2}.\text{int}_G)$ and $\gamma_{G_i} : (G_i \times G_i \times \psi_{1,2}.\text{int}_{G_i} \times \psi_{1,2}.\text{int}_{G_i}) \rightarrow (G_i, \psi_{1,2}.\text{int}_{G_i})$ are defined by (2.8). Since $f_i \circ \gamma^G_i = \gamma_{G_i} \circ (f_i \times f_i)$, $f_i$ and $\gamma_{G_i}$ are $\varphi_{1,2} \psi_{1,2} L$-continuous and $\varphi_{1,2} \psi_{1,2} L$-continuous, respectively, then $f_i \circ \gamma^G_i$ is $\varphi_{1,2} \psi_{1,2} L$-continuous. Because of condition of the initial lifts in the category CRL-Top, $\gamma^G_i$ is $\varphi_{1,2} \psi_{1,2} L$-continuous and hence $(G, \varphi_{1,2}.\text{int}_G)$ is characterized L-topological group.

In the following proposition we show that the initial lefts and then the initial characterized L-topological groups uniquely exist in the category CRL-TopGrp. Hence, the category CRL-TopGrp is topological category over the category Grp of all groups.

**Proposition 4.3** Let $\left((G_i, \psi_{1,2}.\text{int}_{G_i})\right)_{i \in I}$ be a family of characterized L-topological groups and for each $i \in I$, let $f_i : G \rightarrow G_i$ be an injective homomorphism from a group $G$ into a group $G_i$. If $(G, \varphi_{1,2}.\text{int}_G)$ is the initial characterized L-space of the family $\left((G_i, \psi_{1,2}.\text{int}_{G_i})\right)_{i \in I}$ with respect to the family of injective homomorphism mappings $\left(f_i\right)_{i \in I}$, then the family $\left(f_i : (G, \varphi_{1,2}.\text{int}_G) \rightarrow (G_i, \psi_{1,2}.\text{int}_{G_i})\right)_{i \in I}$ is an initial lift of the category CRL-TopGrp.

**Proof.** Because of Propositions 4.1 and 4.2, $(G, \varphi_{1,2}.\text{int}_G)$ is characterized L-topological group. From the definition of the initial lift in CRL-Sp, we get condition (1) from the definition of the initial lift in CRL-TopGrp is fulfilled, that is, all mappings $f_i : (G, \varphi_{1,2}.\text{int}_G) \rightarrow (G_i, \psi_{1,2}.\text{int}_{G_i})$ are $\varphi_{1,2} \psi_{1,2} L$-continuous homomorphism for all $i \in I$.

Let $(H, \delta_{1,2}.\text{int}_H)$ be a characterized L-topological group and a mapping $f : H \rightarrow G$ be a mapping. Then from the definition of the initial lift in CRL-Sp, we have that the mapping $f : (H, \delta_{1,2}.\text{int}_H) \rightarrow (G, \varphi_{1,2}.\text{int}_G)$ is $\delta_{1,2} \varphi_{1,2} L$-continuous if and only if the composition mappings $f_i \circ f : (H, \delta_{1,2}.\text{int}_H) \rightarrow (G_i, \psi_{1,2}.\text{int}_{G_i})$ are $\delta_{1,2} \psi_{1,2} L$-continuous for all $i \in I$. Now, let $f$ is homomorphism. Since $f_i$ is homomorphism for each $i \in I$, then $f_i \circ f$ is also homomorphism for all $i \in I$. On other hand let $f_i \circ f$ is also homomorphism for all $i \in I$. Since $f_i$ is homomorphism for each $i \in I$, then for all $a, b \in H$ we have

$$f_i (f (a \cdot b)) = (f_i \circ f) (a \cdot b) = f_i (f (a)) \cdot f_i (f (b)) = f_i (f (a) \cdot f (b)).$$

Since $f_i$ is injective for all $i \in I$, it follows that $f (a \cdot b) = f (a) \cdot f (b)$ for all $a, b \in H$, that is, $f$ is homomorphism. Hence, $f : (H, \delta_{1,2}.\text{int}_H) \rightarrow (G, \varphi_{1,2}.\text{int}_G)$ is $\delta_{1,2} \varphi_{1,2} L$-continuous homomorphism if and only if all the composition mappings $f_i \circ f : (H, \delta_{1,2}.\text{int}_H) \rightarrow (G_i, \psi_{1,2}.\text{int}_{G_i})$ are $\delta_{1,2} \psi_{1,2} L$-continuous homomorphism for all $i \in I$. Thus, condition (2) from the definition of the initial lift in CRL-
is an initial lift of \((f_i : G \rightarrow G_i, \psi_{i,1,2} \cdot \text{int}_{G_i})_{i \in I}\) in the category CRL-TopGrp. □

Because of Proposition 4.3, the characterized L-topological groups mentioned in Propositions 4.1 and 4.2 are coincide with the initial characterized L-topological groups, that is, if \(\left((G_i, \psi_{i,1,2} \cdot \text{int}_{G_i})\right)_{i \in I}\) is a family of characterized L-topological groups and for each \(i \in I\), the mapping \(f_i : G \rightarrow G_i\) is an injective homomorphism and \((G, \varphi_{1,2} \cdot \text{int}_G)\) is the initial characterized L-space of the family \(\left((G_i, \psi_{i,1,2} \cdot \text{int}_{G_i})\right)_{i \in I}\) with respect to the family of injective homomorphism mappings \((f_i)_{i \in I}\), then \((G, \varphi_{1,2} \cdot \text{int}_G)\) is initial characterized L-topological groups. Hence, the category CRL-TopGrp is concrete category of the category L-Top of all topological spaces and the faithful functor \(\mathcal{F} : \text{CRL-TopGrp} \rightarrow \text{L-Top}\) is isomorphism. Thus, the category CRL-TopGrp is algebraic category over the category \text{L-Top} in sense of [7].

In the following we consider some special cases for the initial characterized L-topological groups.

**Characterized L-subgroups.** Let \(H\) be non-empty subgroup of a characterized L-topological group \((G, \varphi_{1,2} \cdot \text{int}_G)\) and \(i_H : H \rightarrow G\) be the inclusion injective mapping of \(H\) into \(G\). Then the mapping \(\varphi_{1,2} \cdot \text{int}_H : \text{L}^H \rightarrow \text{L}^H\) which is defined by:

\[
\varphi_{1,2} \cdot \text{int}_H \sigma = \bigvee_{\mu \leq \sigma} (\varphi_{1,2} \cdot \text{int}_G \mu) \circ i_H
\]

for all \(\sigma \in \text{L}^H\) is initial \(\varphi_{1,2}\)-interior operator of \(\varphi_{1,2} \cdot \text{int}_G\) with respect to the inclusion injective mapping \(i_H : H \rightarrow G\), called an induced \(\varphi_{1,2}\)-interior operator of \(\varphi_{1,2} \cdot \text{int}_G\) on the subgroup \(H\) of \(G\) and \((H, \varphi_{1,2} \cdot \text{int}_H)\) is initial characterized L-topological group called a characterized L-subgroup of the characterized L-topological group \((G, \varphi_{1,2} \cdot \text{int}_G)\).

**Proposition 4.4** Let \(H\) be non-empty subgroup of a characterized L-topological group \((G, \varphi_{1,2} \cdot \text{int}_G)\). Then the characterized L-subgroup \((H, \varphi_{1,2} \cdot \text{int}_H)\) of \((G, \varphi_{1,2} \cdot \text{int}_G)\) always exists and the initial \(\varphi_{1,2}\)-interior operator \(\varphi_{1,2} \cdot \text{int}_G\) is given by (4.1).

**Proof.** Immediate from Propositions 4.2 and 4.3. □

**Characterized product L-topological groups.** Assume that for each \(i \in I\), \((G_i, \psi_{i,1,2} \cdot \text{int}_{G_i})\) be a characterized L-topological group and \(G\) be the cartesian product \(\prod_{i \in I} G_i\) of the family \((G_i)_{i \in I}\) of groups. If \(P_i : G \rightarrow G_i\) be the related injective projection, then the mapping \(\psi_{1,2} \cdot \text{int}_G : \text{L}^G \rightarrow \text{L}^G\) defined by:

\[
\psi_{1,2} \cdot \text{int}_G \mu = \bigvee_{\mu \leq \mu' \leq \mu} (\psi_{1,2} \cdot \text{int}_{G_i} \mu') \circ P_i
\]

for all \(\mu \in \text{L}^G\) is initial \(\psi_{1,2}\)-interior operator of \(\psi_{1,2} \cdot \text{int}_G\) with respect to the injective projection mapping \(P_i : G \rightarrow G_i\), called product \(\varphi_{1,2}\)-interior operator of the \(\psi_{1,2}\)-interior operators \(\psi_{1,2} \cdot \text{int}_{G_i}\) and \((G, \varphi_{1,2} \cdot \text{int}_G)\) is initial characterized L-topological group called characterized product L-topological group of the characterized L-topological groups \((G_i, \psi_{1,2} \cdot \text{int}_{G_i})\) with respect to the family \((P_i : G \rightarrow G_i)_{i \in I}\) of injective projections and will be denoted by \((\prod_{i \in I} G_i, \prod_{i \in I} \psi_{1,2} \cdot \text{int}_{G_i})\).
5. Final characterized L-topological groups

In this section we show that the final characterized L-topological group exists and it can be the final characterized L-spaces. Since the concrete category CRL-TopGrp of all characterized L-topological groups is topological category over the category Grp of all groups, then all final lifts also uniquely exist. This, even mean that also all final characterized L-topological groups exist.

Consider $(G_i :\psi_{i,2}.\text{int}_{G_i})_{i\in I}$ be a family of characterized L-topological groups and $(f_i : G_i \rightarrow G_j)_{i\in I}$ be a family of homomorphism mappings from the groups $G_i$ into the group $G_j$, indexed by the class $I$. Then for any characterized L-space $(G, \varphi_{1,2}.\text{int}_G)$, the family $(f_i : (G_i :\psi_{i,2}.\text{int}_{G_i}) \rightarrow (G, \varphi_{1,2}.\text{int}_G))_{i\in I}$ is called final lifts for the family $(f_i : G_i \rightarrow G_j :\psi_{i,2}.\text{int}_{G_i})_{i\in I}$ in the category CRL-TopGrp, provided $(G, \varphi_{1,2}.\text{int}_G)$ is characterized L-topological group which fulfills the following conditions:

1. All the mappings $f_i : (G_i :\psi_{i,2}.\text{int}_{G_i}) \rightarrow (G, \varphi_{1,2}.\text{int}_G)$ are $\psi_{i,2} \varphi_{1,2}$ L-continuous homomorphism for all $i \in I$.

2. For an characterized L-topological group $(H, \delta_{1,2}.\text{int}_H)$ and a mapping $f : G \rightarrow H$, the mapping $f : (G, \varphi_{1,2}.\text{int}_G) \rightarrow (H, \delta_{1,2}.\text{int}_H)$ is $\varphi_{1,2} \delta_{1,2}$ L-continuous homomorphism if and only if all the composition mappings $f \circ f_i : (G_i :\psi_{i,2}.\text{int}_{G_i}) \rightarrow (H, \delta_{1,2}.\text{int}_H)$ are $\psi_{i,2} \delta_{1,2}$ L-continuous homomorphism for all $i \in I$, (See Fig. 5.1)

\[ \begin{array}{c}
G_i \\
\uparrow f_i \\
\downarrow f \\
G_j \\
\end{array} \]

Fig.5.1

Hence, by a final characterized L-topological group we mean the characterized L-topological group which provides the final lifts in the category CRL-TopGrp.

To prove that all final lifts and all final characterized L-topological groups exist in the category CRL-TopGrp we need to prove that in case of $f_i : G_i \rightarrow G$ is an injective homomorphism for each $i \in I$, and $\varphi_{1,2}.\text{int}_G$ is $\varphi_{1,2}$-interior operator for an final characterized L-topology on a group $G$ of $(\psi_{1,2}.\text{int}_{G_i})_{i\in I}$ we get that $(G, \varphi_{1,2}.\text{int}_G)$ is also characterized L-topological group. To prove these results we need at first the following lemma.

**Lemma 5.1** If $f : (G, \varphi_{1,2}.\text{int}_G) \rightarrow (H, f (\varphi_{1,2}.\text{int}_G))$ is surjective homomorphism mapping from the characterized L-topological groups $(G, \varphi_{1,2}.\text{int}_G)$ to the group $H$ equipped with the final characterized L-topology generated by $(\varphi_{1,2}.\text{int}_G)$ as a base with respect to $f$, then $f$ is $\varphi_{1,2} \varphi_{1,2}$ L-open.

**Proof.** Immediate from Proposition 2.4. □

Now, we consider the case of $I$ being a singleton.

**Proposition 5.1** Let $(G, \varphi_{1,2}.\text{int}_G)$ be a characterized L-topological group and let $f : G \rightarrow H$ be a homomorphism from a group $G$ onto a group $H$. Then the final characterized L-space $(H, f (\varphi_{1,2}.\text{int}_G))$ of $(G, \varphi_{1,2}.\text{int}_G)$ with respect to $f$ is characterized L-topological group.

**Proof.** Let at first $\gamma_H : (H \times H, f (\varphi_{1,2}.\text{int}_G) \times f (\varphi_{1,2}.\text{int}_G)) \rightarrow (H, f (\varphi_{1,2}.\text{int}_G))$ and $\gamma_G : (G \times G, \varphi_{1,2}.\text{int}_G \times \varphi_{1,2}.\text{int}_G) \rightarrow (G, \varphi_{1,2}.\text{int}_G)$ are the mappings defined by (2.8) and let
\( \mu \in \beta_{i}((\varphi_{1,2}\int G)) \), where \( \beta_{i}((\varphi_{1,2}\int G)) \) is the base of \((H,f(\varphi_{1,2}\int G))\) which is generated by \( f(\varphi_{1,2}\int G) \), then \( f^{-1}(\mu) \in \beta_{i}((\varphi_{1,2}\int G)) \). Since \((G,\varphi_{1,2}\int G)\) is characterized L-topological group, then \( \gamma_{G} \) is \( \varphi_{1,2} \)-L-continuous for all \( L \)-open and therefore from Proposition 2.3, we have \( \gamma_{G}^{-1}(f^{-1}(\mu)) \in \beta_{i}((\varphi_{1,2}\int G)) \). Because of Lemma 5.1, we have that the mapping \( f:(G,\varphi_{1,2}\int G)) \to (H,f(\varphi_{1,2}\int G)) \) is \( \varphi_{1,2} \)-L-open for all \( L \)-open, then the family \( \gamma_{G}^{-1}(f^{-1}(\mu)) \). Consider \( \gamma_{G}^{-1}(f^{-1}(\mu)) \) is \( \beta_{i}((\varphi_{1,2}\int G)) \) and \( \varphi_{1,2} \)-L-continuous and consequently \( (H,f(\varphi_{1,2}\int G)) \) is characterized L-topological group. □

Generally, we consider the case of \( I \) is any class consists of more than one element. Then we have the following result.

Proposition 5.2 Let \( (G_{i},\varphi_{1,2}\int G_{i}))_{i \in I} \) be a family of characterized L-topological groups and for each \( i \in I \), let \( f_{i}:G_{i} \to G \) be a homomorphism from a group \( G \) onto a group \( G_{i} \). If \((G,\varphi_{1,2}\int G)\) is the initial characterized L-space of the family \( (G_{i},\varphi_{1,2}\int G_{i}))_{i \in I} \) with respect to the family \( (f_{i})_{i \in I} \), then \((G,\varphi_{1,2}\int G)\) is characterized L-topological group.

Proof. Let \( \gamma_{G_{i}}:(G_{i} \times G,\varphi_{1,2}\int G_{i}\times\varphi_{1,2}\int G_{i}) \to (G,\varphi_{1,2}\int G_{i}) \) is a mapping defined by (2.8) and \( \mu \in \beta_{i}((\varphi_{1,2}\int G_{i})) \). Since \( f_{i}:(G_{i},\varphi_{1,2}\int G_{i}) \to (G,\varphi_{1,2}\int G_{i}) \) is \( \varphi_{1,2} \)-L-continuous for all \( i \in I \), then \( f_{i}^{-1}(\mu) \in \beta_{i}((\varphi_{1,2}\int G_{i})) \) for all \( i \in I \) and because of \( \gamma_{G_{i}} \) is \( \varphi_{1,2} \)-L-continuous for all \( i \in I \), then we have \( \gamma_{G_{i}}^{-1}(f_{i}^{-1}(\mu)) \in \beta_{i}((\varphi_{1,2}\int G_{i})) \). Consider \( \gamma_{G}:(G \times G,\varphi_{1,2}\int G \times \varphi_{1,2}\int G) \to (G,\varphi_{1,2}\int G) \) is a mapping defined by (2.8), then \( \gamma_{G}^{-1}(\mu) = (f_{i} \times f_{i})(\gamma_{G_{i}}^{-1}(f_{i}^{-1}(\mu))) \) and by a similar way to the proof of Proposition 5.1, we have the product mapping \( f_{i} \times f_{i} \) is \( \varphi_{1,2} \)-L-open for all \( i \in I \). Hence, \( \gamma_{G}^{-1}(\mu) \in \beta_{i}((\varphi_{1,2}\int G)) \) and therefore \( \gamma_{G} \) is \( \varphi_{1,2} \)-L-continuous and consequently \((G,\varphi_{1,2}\int G)\) is characterized L-topological group. □

In the following proposition we show that the final lifts and then the final characterized L-topological groups uniquely exist in the concrete category CRL-TopGrp, that is, the characterized L-topological groups mentioned in Propositions 5.1 and 5.2 fulfills the conditions of the final lifts in the category CRL-TopGrp.

Proposition 5.3 Let \( (G_{i},\varphi_{1,2}\int G_{i}))_{i \in I} \) be a family of characterized L-topological groups and for each \( i \in I \), let \( f_{i}:G_{i} \to G \) be a surjective homomorphism from the groups \( G_{i} \) into a group \( G \). If \((G,\varphi_{1,2}\int G)\) is the final characterized L-space of the family \( (G_{i},\varphi_{1,2}\int G_{i}))_{i \in I} \) with respect to the family of surjective homomorphism mappings \( (f_{i})_{i \in I} \), then the family \( (f_{i}:(G_{i},\varphi_{1,2}\int G_{i}) \to (G,\varphi_{1,2}\int G))_{i \in I} \) is a final lift of \((f_{i}:G_{i} \to G,\varphi_{1,2}\int G)_{i \in I} \) in the category CRL-TopGrp.

Proof. The proof goes similarly by using Propositions 5.1 and 5.2 with the properties of the final lifts in the category as in case of Proposition 4.3. □
Because of Proposition 5.3, the characterized L-topological groups mentioned in Propositions 5.1 and 5.2 are coincide with the final characterized L-topological groups, that is, if  \((G_i, \psi_{1,2}, \text{int}_{G_i})\) \(_{i \in I}\) is a family of characterized L-topological groups and for each  \(i \in I\), the mapping  \(f_i : G_i \rightarrow G\) is an surjective homomorphism and  \((G, \varphi_{1,2}, \text{int}_G)\) is the final characterized L-space of the family  \((G_i, \psi_{1,2}, \text{int}_{G_i})\) \(_{i \in I}\) with respect to the family of surjective homomorphism mappings  \((f_i)_{i \in I}\), then  \((G, \varphi_{1,2}, \text{int}_G)\) is final characterized L-topological groups. Hence, the category  \(\text{CRL-TopGrp}\) is co-concrete category of the category  \(\text{L-Top}\) of all topological spaces and the faithful functor  \(\sigma^*: \text{L-Top} \rightarrow \text{CRL-TopGrp}\) is isomorphism.

In the following we consider some special cases for the final characterized L-topological groups.

**Characterized L-topological quotient groups.** The characterized L-topological group is special final characterized L-topological group when the mapping  \(f : G \rightarrow H\) replaced by the canonical mapping  \(h : G \rightarrow G / N\), where  \(N\) is normal subgroup the group  \(G\).

Let  \(N\) be normal subgroup of the characterized L-topological group  \((G, \varphi_{1,2}, \text{int}_G)\) and  \(G / N\) is the corresponding quotient group. If  \(h : G \rightarrow G / N\) is the canonical homomorphism mapping defined by:  \(h(x) = x N\) for all  \(x \in G\), then  \((G / N, h(\varphi_{1,2}, \text{int}_G))\) is final characterized L-topological group called characterized L-topological quotient group of the characterized L-topological group  \((G, \varphi_{1,2}, \text{int}_G)\).

**Proposition 5.4** Let  \((G, \varphi_{1,2}, \text{int}_G)\) be a characterized L-topological group and  \(N\) is a normal subgroup of  \(G\). If  \(G / N\) is the corresponding quotient group, then the canonical surjective homomorphism  \(h : (G, \varphi_{1,2}, \text{int}_G) \rightarrow (G / N, h(\varphi_{1,2}, \text{int}_G))\) which is defined as  \(h(x) = x N\) for all  \(x \in G\) is  \(\varphi_{1,2}\) L-open.  

**Proof.** Follows directly from Lemma 5.1. \(\Box\)

In the following proposition we give the relation between characterized L-topological quotient groups and the characterized product L-topological groups.

**Proposition 5.5** Let  \(I\) be a class and for each  \(i \in I\), let  \((G_i, \psi_{1,2}, \text{int}_{G_i})\) be a characterized L-topological group and  \(N_i\) be a normal subgroup of  \(G_i\). If  \(G = \prod_{i \in I} G_i\) and  \(N = \prod_{i \in I} N_i\) are the related products of the least two families  \((G_i)_{i \in I}\) and  \((N_i)_{i \in I}\), respectively, then the isomorphism mapping  \(f : (G / N, h(\prod_{i \in I} \psi_{1,2}, \text{int}_{G_i})) \rightarrow (\prod_{i \in I} (G_i / N_i), (\prod_{i \in I} h(\psi_{1,2}, \text{int}_{G_i})))\) is  \(\psi_{1,2}\) L-homeomorphism, where  \(h : (G_i, \prod_{i \in I} \psi_{1,2}, \text{int}_{G_i}) \rightarrow (G_i / N_i, h(\psi_{1,2}, \text{int}_{G_i}))\) and  \(h_i : (G_i, \psi_{1,2}, \text{int}_{G_i}) \rightarrow (G_i / N_i, h(\psi_{1,2}, \text{int}_{G_i}))\) are the related canonical surjective homomorphism’s.

**Proof.** Because of the definition of characterized product L-topological groups and the characterized L-topological quotient groups we have that  \((G / N, h(\prod_{i \in I} \psi_{1,2}, \text{int}_{G_i}))\) and  \((\prod_{i \in I} (G_i / N_i), (\prod_{i \in I} h(\psi_{1,2}, \text{int}_{G_i})))\) are characterized L-topological groups. Since  \(h_i\) is  \(\psi_{1,2}\) L-continuous for all  \(i \in I\), then from Lemma 3.1 it follows that the product mapping  \(\prod_{i \in I} h_i : (G_i, h(\prod_{i \in I} \psi_{1,2}, \text{int}_{G_i})) \rightarrow (\prod_{i \in I} (G_i / N_i), (\prod_{i \in I} h(\psi_{1,2}, \text{int}_{G_i})))\) is  \(\psi_{1,2}\) L-continuous. Hence,
f(μ) ∈ \( \prod_{i \in I} (h_i \cdot (\nu_{1,2} \cdot \text{int}_{\nu_i})) \) implies \( h^{-1}(f^{-1}(\mu)) = (\prod_{i \in I} h_i)^{-1}(\mu) \in \prod_{i \in I} \nu_{1,2} \cdot \text{int}_{\nu_i} \). Because of Proposition 5.3, \( h \) is \( \psi_{1,2} \psi_{1,2} \) L-open and surjective mapping, therefore \( f^{-1}(\mu) \in \prod_{i \in I} \nu_{1,2} \cdot \text{int}_{\nu_i} \). Then, \( f \) is \( \psi_{1,2} \psi_{1,2} \) L-continuous isomorphism, that is, \( f \) is bijective \( \psi_{1,2} \psi_{1,2} \) L-continuous.

Now, let \( \eta \in \prod_{i \in I} \nu_{1,2} \cdot \text{int}_{\nu_i} \). Since \( h \) is \( \psi_{1,2} \psi_{1,2} \) L-continuous, then \( h^{-1}(\eta) \in \prod_{i \in I} \nu_{1,2} \cdot \text{int}_{\nu_i} \). Because of \( \prod_{i \in I} h_i \) is the product of \( \psi_{1,2} \psi_{1,2} \) L-open mappings, then Lemma 3.1 implies that \( \prod_{i \in I} h_i \) is \( \psi_{1,2} \psi_{1,2} \) L-open mapping. Therefore, \( f(\eta) = (\prod_{i \in I} h_i)(h^{-1}(\eta)) \in \prod_{i \in I} (h_i \cdot (\nu_{1,2} \cdot \text{int}_{\nu_i})) \), that is, \( f \) is \( \psi_{1,2} \psi_{1,2} \) L-open. Consequently, \( f \) is \( \psi_{1,2} \psi_{1,2} \) L-homeomorphism. □

6. Conclusion

In this paper, we introduced and studied the notions of final characterized L-spaces and initial and final characterized L-topological groups. The properties of such notions are deeply studied. By the notion of final characterized L-spaces, the notions of characterized quotient pre L-spaces and characterized sum L-spaces are introduced and studied. We show that all the final lefts and all the final characterized L-spaces are uniquely exist in the category CRL-Sp and hence CRL-Sp is topological category over the category SET of all sets. The characterized L-subspaces together with their related inclusion mappings and the characterized quotient pre L-spaces together with their related canonical surjection are the equalizers and co-equalizers, respectively in CRL-Sp. Moreover, we show that the initial and final lefts and then the initial and final characterized L-topological groups uniquely exist in the category CRL-TopGrp. Hence, the category CRL-TopGrp is topological category over the category Grp of all groups. By the notion of initial and final characterized L-topological groups, the notions of characterized L-subgroups, characterized product L-topological groups and characterized L-topological quotient groups are introduced and studied. However, we show that the category CRL-TopGrp is concrete and co-concrete category of the category L-Top of all topological L-spaces and that the faithful functors \( \mathcal{F} : \text{CRL} \rightarrow \text{L-Top} \) and \( \mathcal{F}^*: \text{L-Top} \rightarrow \text{CRL} \) are isomorphism’s. Thus, the category CRL-TopGrp is algebraic and co-algebraic category over the category L-Top in sense of [7]. Many new special classes for the final characterized L-spaces, initial characterized L-topological groups, final characterized L-topological groups, characterized product L-topological groups and characterized L-topological quotient groups are listed in Table (1).
<table>
<thead>
<tr>
<th></th>
<th>Operations</th>
<th>Final characterized L-spaces</th>
<th>Initial characterized L-topol. Groups</th>
<th>Final characterized L-topol. Groups</th>
<th>Characterized L-topol. groups</th>
<th>Characterized L-topol. Quotient groups</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\varphi_1 = \text{int}$ $\varphi_2 = 1_{L^t}$</td>
<td>Final L-top. space [18]</td>
<td>Initial L-topol. Group [6,8]</td>
<td>Final L-topol. Group [6,8]</td>
<td>Product L-topol. Group [6,8]</td>
<td>L-topol. Quotient group [6,8]</td>
</tr>
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<td>2</td>
<td>$\varphi_1 = \text{int}$ $\varphi_2 = \text{cl}$</td>
<td>Final $\theta$ L-space</td>
<td>Initial $\theta$ L-topol. Group</td>
<td>Final $\theta$ L-topol. Group</td>
<td>$\theta$ - product L-topol. Group</td>
<td>$\theta$ L-topol. Quotient group</td>
</tr>
<tr>
<td>3</td>
<td>$\varphi_1 = \text{int}$ $\varphi_2 = \text{int} \circ \text{cl}$</td>
<td>Final $\delta$ L-space</td>
<td>Initial $\delta$ L-topol. Group</td>
<td>Final $\delta$ L-topol. Group</td>
<td>$\delta$ - product L-topol. Group</td>
<td>$\delta$ L-topol. Quotient group</td>
</tr>
<tr>
<td>4</td>
<td>$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = 1_{L^t}$</td>
<td>Final semi L-space</td>
<td>Initial semi L-topol. Group</td>
<td>Final semi L-topol. Group</td>
<td>Semi-product L-topol. Group</td>
<td>Semi L-topol. Quotient group</td>
</tr>
<tr>
<td>5</td>
<td>$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = \text{cl}$</td>
<td>Final $(\theta S)$ L-space</td>
<td>Initial $(\theta S)$ L-topol. Group</td>
<td>Final $(\theta S)$ L-topol. Group</td>
<td>$(\theta S)$ - product L-topol. Group</td>
<td>$(\theta S)$ L-topol. Quotient group</td>
</tr>
<tr>
<td>6</td>
<td>$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = \text{int} \circ \text{cl}$</td>
<td>Final $(\delta S)$ L-space</td>
<td>Initial $(\delta S)$ L-topol. Group</td>
<td>Final $(\delta S)$ L-topol. Group</td>
<td>$(\delta S)$ - product L-topol. Group</td>
<td>$(\delta S)$ L-topol. Quotient group</td>
</tr>
<tr>
<td>7</td>
<td>$\varphi_1 = \text{int} \circ \text{cl}$ $\varphi_2 = 1_{L^t}$</td>
<td>Final pre L-space</td>
<td>Initial pre L-topol. Group</td>
<td>Final pre L-topol. Group</td>
<td>Pre-product L-topol. Group</td>
<td>Pre L-topol. Quotient group</td>
</tr>
<tr>
<td>8</td>
<td>$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = S \circ \text{cl}$</td>
<td>Final $(S \theta)$ L-space</td>
<td>Initial $(S \theta)$ L-topol. Group</td>
<td>Final $(S \theta)$ L-topol. Group</td>
<td>$(S \theta)$ - product L-topol. Group</td>
<td>$(S \theta)$ L-topol. Quotient group</td>
</tr>
<tr>
<td>9</td>
<td>$\varphi_1 = \text{cl} \circ \text{int}$ $\varphi_2 = S \circ \text{int} \circ S \circ \text{cl}$</td>
<td>Final $(S \delta)$ L-space</td>
<td>Initial $(S \delta)$ L-topol. Group</td>
<td>Final $(S \delta)$ L-topol. Group</td>
<td>$(S \delta)$ - product L-topol. Group</td>
<td>$(S \delta)$ L-topol. Quotient group</td>
</tr>
<tr>
<td>10</td>
<td>$\varphi_1 = \text{cl} \circ \text{int} \circ \text{cl}$ $\varphi_2 = 1_{L^t}$</td>
<td>Final $\beta$ L-space</td>
<td>Initial $\beta$ L-topol. Group</td>
<td>Final $\beta$ L-topol. Group</td>
<td>$\beta$ - product L-topol. Group</td>
<td>$\beta$ L-topol. Quotient group</td>
</tr>
<tr>
<td>11</td>
<td>$\varphi_1 = \text{int} \circ \text{cl} \circ \text{int}$ $\varphi_2 = 1_{L^t}$</td>
<td>Final $\lambda$ L-space</td>
<td>Initial $\lambda$ L-topol. Group</td>
<td>Final $\lambda$ L-topol. Group</td>
<td>$\lambda$ - product L-topol. Group</td>
<td>$\lambda$ L-topol. Quotient group</td>
</tr>
<tr>
<td>12</td>
<td>$\varphi_1 = S \circ \text{cl} \circ \text{int}$ $\varphi_2 = 1_{L^t}$</td>
<td>Final feebly L-space</td>
<td>Initial feebly L-topol. Group</td>
<td>Final feebly L-topol. Group</td>
<td>Feebly product L-topol. Group</td>
<td>Feebly L-topol. Quotient group</td>
</tr>
</tbody>
</table>

Table (1): Some special classes of final characterized L-spaces; initial characterized L-topological groups, final characterized L-topological groups characterized product L-topological groups and characterized L-topological quotient groups.

References